Exercises: Green's Theorem

Problem 1. Calculate

$$\oint_C \boldsymbol{f}(\boldsymbol{r}) d\boldsymbol{r}$$

where $\boldsymbol{f} = [y, -x]$, and C is the circle $x^2 + y^2 = 1$ in the positive direction.

Remark: The sign $\oint$ has the same meaning as $\int$ except that the former emphasizes that C is a *closed* curve.

Problem 2. Define Q as the square in $\mathbb{R}^2$ enclosing all the points (x, y) satisfying $0 \le x \le 1$ and $0 \le y \le 1$. Calculate $\oint_C \boldsymbol{f}(\boldsymbol{r}) d\boldsymbol{r}$, where $\boldsymbol{f} = [6y^2, 2x - 2y^4]$, and C is the boundary of Q in the positive direction.

Problem 3. Calculate

$$\oint_C x^2 e^y \, dx + y^2 e^x \, dy$$

where C is the same as in the previous problem.

Problem 4. Define Q as the square in $\mathbb{R}^2$ enclosing all the points (x, y) satisfying $-1 \le x \le 1$ and $-1 \le y \le 1$. Calculate

$$\oint_C \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy$$

where C is the boundary of Q in the positive direction. You can use the fact that

$$\int_{-1}^{1} \frac{2}{x^2 + 1} \, dx = \pi.$$

Problem 5. Prof. Goofy applies the following argument to "show" that the integral in Problem 4 equals 0. But his argument is wrong. Point out the place where he makes a mistake.

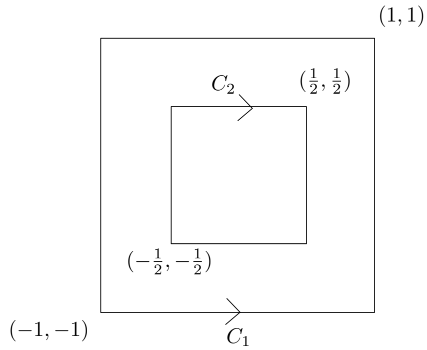
Prof. Goofy's solution: Set $f_1 = \frac{-y}{x^2+y^2}$ and $f_2 = \frac{x}{x^2+y^2}$. Thus:

$$\begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ \frac{\partial f_2}{\partial x} &= \frac{y^2 - x^2}{(x^2 + y^2)^2}. \end{aligned}$$

Let D be the area enclosed by Q. By Green's theorem, we have:

$$\oint_C \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy \;=\; \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} dx dy = \iint_D 0 \, dx dy = 0.$$

Problem 6. Suppose that C is the union of the two arcs C_1 and C_2 as shown in the following figure:



Calculate

$$\int_C (-y)\,dx + x\,dy.$$