Exercises: Line Integral by Coordinate

Problem 1. Let $C$ be the curve from point $p = (0, 0)$ to $q = (2, 4)$ on the parabola $y = x^2$. Calculate $\int_C (x^2 - y^2) dx$.

Solution: First, write $C$ into its parametric form: $r(t) = [x(t), y(t)]$ where $x(t) = t$, and $y(t) = t^2$. Points $p$ and $q$ are given by $t = 0$ and 2, respectively. Thus:

$$\int_C (x^2 - y^2) dx = \int_0^2 (t^2 - t^4) \frac{dx}{dt} \, dt$$
$$= \int_0^2 (t^2 - t^4) dt$$
$$= \frac{8}{3} - \frac{32}{5}.$$

Problem 2. Let $r(t) = [t, t^2, t^3]$ and $f(x, y, z) = [x - y, y - z, z - x]$. Let $C$ be the curve from the point of $t = 0$ to the point of $t = 1$. Calculate $\int_C f(r) \cdot dr$.

Solution:

$$\int_C f(r) \cdot dr = \int_0^1 f(r) \cdot r'(t) \, dt$$
$$= \int_0^1 [t - t^2, t^2 - t^3, t^3 - t] \cdot [1, 2t, 3t^2] \, dt$$
$$= \int_0^1 t - t^2 + 2t^3 - 2t^4 + 3t^5 - 3t^3 \, dt$$
$$= \int_0^1 t - t^2 - t^3 - 2t^4 + 3t^5 \, dt$$
$$= \frac{1}{60}.$$

Problem 3. Same as in Problem 2, except that $C$ is defined by decreasing $t$ from 1 to 0 (i.e., reversing the direction as in Problem 2).

Solution: When the direction of the arc is reversed, the value of the integer integral (by coordinate) is reversed. Hence, the answer is $-\frac{1}{60}$.

Solution:

$$\int_C f(r) \cdot dr = \int_0^1 f(r) \cdot r'(t) \, dt$$
$$= \int_0^1 [t - t^2, t^2 - t^3, t^3 - t] \cdot [1, 2t, 3t^2] \, dt$$
$$= \int_0^1 t - t^2 + 2t^3 - 2t^4 + 3t^5 - 3t^3 \, dt$$
$$= \int_0^1 t - t^2 - t^3 - 2t^4 + 3t^5 \, dt$$
$$= \frac{1}{60}.$$
Problem 4. Calculate $\int_C \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r}$ where $\mathbf{f}(x, y) = [y^2, -x^2]$, and $C$ is the arc from $(0, 0)$ to $(1, 4)$ on the curve $y = 4x^2$.

Solution. Let us first represent the curve $y = x^2$ in its parametric form: $\mathbf{r}(t) = [t, 4t^2]$. $C$ is defined by increasing $t$ from 0 to 1. Hence:

\[
\int_C \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 \mathbf{f}(\mathbf{r}) \cdot \mathbf{r}'(t) \, dt
= \int_0^1 [y(t)^2, -x(t)^2] \cdot [1, 8t] \, dt
= \int_0^1 y(t)^2 - 8t \cdot x(t)^2 \, dt
= \int_0^1 (4t^2)^2 - 8t \cdot t^2 \, dt
= \int_0^1 16t^4 - 8t^3 \, dt
= \frac{6}{5}.
\]

Problem 5. Calculate

\[
\int_C xy \, dx + x^2 y^2 \, dy
\]

where $C$ is the quarter-arc from $(1, 0)$ to $(0, 1)$ on the circle $x^2 + y^2 = 1$.

Solution. Let us first represent the circle in its parametric form: $\mathbf{r}(t) = [\cos t, \sin t]$. $C$ is defined by increasing $t$ from 0 to $\pi/2$. Hence:

\[
\int_C xy \, dx + x^2 y^2 \, dy = \int_0^{\pi/2} \left( xy \frac{dx}{dt} + x^2 y^2 \frac{dy}{dt} \right) \, dt
= \int_0^{\pi/2} \left( \cos t \sin t \cdot (- \sin t) + (\cos^2 t)(\sin^2 t) \cos t \right) \, dt
= - \int_0^{\pi/2} \sin^4 t \cos t \, dt
= - \int_0^{\pi/2} \sin^4 t \, d(\sin t)
= -\frac{1}{5}.
\]

Problem 6. Let $\mathbf{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Let $p$ be the point given by $t = \pi/4$. Calculate $\frac{dx}{ds}$ at $p$. 

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Solution:

\[
\frac{dx}{ds} = \frac{dx/ \ dt}{ds/ \ dt} = \frac{dx/ \ dt}{\sqrt{(dx/ \ dt)^2 + (dy/ \ dt)^2}} = \frac{x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} = -\frac{\sin(t)}{\sqrt{(-\sin(t))^2 + (\cos(t))^2}} = -\sin(t).
\]

Hence, the value of \(\frac{dx}{ds}\) at \(p\) is \(-\sin(\pi/4) = -1/\sqrt{2}\).

**Problem 7.** Let \(\mathbf{r}(t) = [x(t), y(t), z(t)]\). Let \(p\) be the point given by \(t = t_0\). Prove that \([\frac{dx}{ds}(t_0), \frac{dy}{ds}(t_0), \frac{dz}{ds}(t_0)]\) is a unit tangent vector at \(p\).

**Proof:**

\[
\frac{dx}{ds} = \frac{dx/ \ dt}{ds/ \ dt} = \frac{dx/ \ dt}{{}\sqrt{(dx/ \ dt)^2 + (dy/ \ dt)^2 + (dz/ \ dt)^2}}
\]

Similarly:

\[
\frac{dy}{ds} = \frac{dy/ \ dt}{ds/ \ dt} = \frac{dy/ \ dt}{\sqrt{(dx/ \ dt)^2 + (dy/ \ dt)^2 + (dz/ \ dt)^2}}
\]

\[
\frac{dz}{ds} = \frac{dz/ \ dt}{ds/ \ dt} = \frac{dz/ \ dt}{\sqrt{(dx/ \ dt)^2 + (dy/ \ dt)^2 + (dz/ \ dt)^2}}.
\]

Therefore:

\[
\begin{bmatrix}
\frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds}
\end{bmatrix} = \frac{[x'(t), y'(t), z'(t)]}{\sqrt{(dx/ \ dt)^2 + (dy/ \ dt)^2 + (dz/ \ dt)^2}}
\]

which proves that \([\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}]\) is a tangent vector. Furthermore:

\[
\left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right]^2 = \frac{(dx/ \ dt)^2 + (dy/ \ dt)^2 + (dz/ \ dt)^2}{(dx/ \ dt)^2 + (dy/ \ dt)^2 + (dz/ \ dt)^2} = 1
\]

which means that \([\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}]\) is a unit vector.

**Problem 8.** This problem allows you to see the equivalence of line integral by arc length and line integral by coordinate. Let \(\mathbf{r}(t) = [x(t), y(t)]\) where \(x(t) = \cos(t)\) and \(y(t) = \sin(t)\). Convert \(\int_C x \, dx + \int_C y^2 \, dy\) to line integral by arc length.

**Solution:**

\[
\int_C x \, dx + \int_C y^2 \, dy = \int_C x \frac{dx}{ds} \, ds + \int_C y^2 \frac{dy}{ds} \, ds = \int_C x \frac{dx}{ds} + y^2 \frac{dy}{ds} \, ds \tag{1}
\]
In Problem 4, we have shown that $\frac{dx}{ds} = -\sin(t) = -y(t)$. Similarly:

$$\frac{dy}{ds} = \frac{dy/dt}{ds/dt} = \frac{dy/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2}} = \frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} = \frac{\cos(t)}{\sqrt{(-\sin(t))^2 + \cos(t)^2}} = x(t).$$

Hence:

$$(1) = \int_C -xy + y^2x \, ds.$$