Set discrimination of quantum states

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(Received 20 January 2002; published 19 June 2002)

We introduce a notion of set discrimination, which is an interesting extension of quantum state discrimination. A state is secretly chosen from a number of quantum states, which are partitioned into some disjoint sets. A set discrimination is required to identify which set the given state belongs to. Several essential problems are addressed in this paper, including the condition of perfect set discrimination, unambiguous set discrimination, and in the latter case, the efficiency of the discrimination. This generalizes some important results on quantum state discrimination in the literature. A combination of state and set discrimination and the efficiency are also studied.

DOI: 10.1103/PhysRevA.65.062322 PACS number(s): 03.67.−a, 03.65.Wj

Quantum state discrimination is a basic and interesting problem in quantum information. The scenario is that a quantum system is prepared in a state secretly chosen from a number of known quantum states, and one is required to obtain as much information about the identification of the state as possible [1]. Previous work mainly aims at determining the state. While perfect state discrimination turns out to be impossible unless the states to be distinguished are orthogonal with each other, several strategies exist for nonorthogonal states. The first one, advanced by Helstrom and historically known as *quantum hypothesis testing* [2], requires one to make a decision in any event, telling which one the state is. Unambiguous discrimination, which goes to another extreme by demanding the identification error-free but leaving an inconclusive possibility, receives more attention recently. It was first considered by Ivanovic [3], and then by Dieks [4] and Peres [5], all of whom focused on the two-state case. They found the optimal efficiency, i.e., the success probability, to be $1-|\langle p|q\rangle|$, conventionally called IDP limit, where p and q are the two states to be distinguished. Later, Jaeger and Shimony [6] extended the result by augmenting unequal *a priori* probabilities to the two states, and Peres and Terno [7] dealt with the three-state case. The general n-state case has also been considered. In Ref. [8] Chefles showed the important fact that the linear independence of the states is a sufficient and necessary condition for them to be unambiguously discriminated. He and Barnett also got the optimal efficiency in a special case [9], known as equally probable symmetrical states. In Ref. [10], Duan and Guo gave a neat, sufficient and necessary condition for n numbers being the efficiency of the unambiguous discrimination of n given states. In a recent paper [11], Zhang *et al.* gave an upper bound for the optimal average efficiency of the n-state case, and in Ref. [12], Sun *et al.* pointed out the mathematical nature of and gave a family of lower bounds for the optimal average efficiency.

All the previous work is mostly concerned with state discrimination, where one needs to report which *state* the system is in. However, there are some limitations and disadvantages in state discrimination. For example, sometimes it may be very difficult to decide the exact state of a system, and the success probability will be very low. On the other hand, it may not be necessary to find the exact state, and we are only required to know a certain range of states. This leads us to consider an alternative. It is a natural extension of state discrimination thoroughly discussed in the previous literature, and it is to allow one to tell that the state belongs to some set of states. This kind of identification up to set granularity, rather than up to state granularity, is called the set discrimination. There are at least three motivations for considering the set discrimination besides the pure mathematical extension. One is that when a set S of quantum states are linearly dependent, they cannot be unambiguously distinguished. But the linear dependence may be *local*, i.e., only caused by a few of states, say states in S_1 ($S_1 \subset S$). And other states ($S-S_1$) are highly independent. In this case, we naturally hope to be able to at least unambiguously identify the states in $S-S_1$. Even if a state in S_1 is prepared, when it is impossible to be unambiguously identified, we hope to unambiguously know it is from S_1 but not from $S-S_1$. Set discrimination is for these hopes. Later we can see that by set discrimination, when one fails in the state discrimination, sometimes he may have the chance to unambiguously know that the state is in some set. Second, in some other cases, some states, again say states in S_1, are rather close to each other, though linearly independent. At this time unambiguous discrimination does exist, but the success probabilities for these near states are awfully low. Since generally the task of discrimination is to obtain the information of a prepared state as much as possible, and chances of knowing the exact state are quite slim, one may like to compromise by accepting a report with a relatively high success probability that the state is from S_1. This can be also fulfilled by set discriminations. Finally, sometimes we just need to identify an unknown state at set granularity level. For example, in Ref. [1], Chefles designed a thought experiment in which Alice and Bob share an entangled pair and Alice can make measurement in different basis U or V. This will cause Bob's particle collapse to $|u\rangle$ or $|u'\rangle$, if Alice chooses U, and $|v\rangle$ or $|v'\rangle$, if Alice chooses V. If Bob can distinguish states $|u\rangle$, $|u'\rangle$, $|v\rangle$, $|v'\rangle$ perfectly,

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then Alice can transmit one-bit information by choosing basis U or V, and this transmission is instantaneous, thus superluminal, because the collapse needs no time. Of course, perfect state discrimination here is impossible because four states in a two-dimensional Hilbert space are definitely linearly dependent. But in fact, to receive the bit, Bob needs only to know whether his particle is in $\{|u\rangle,|u'\rangle\}$ or in $\{|v\rangle,|v'\rangle\}$. So this is a typical set discrimination. Later we will show that even with the requirement loosened in this way, Bob could not receive the bit perfectly (or even unambiguously) either. Another example, which is less fictive, is the error correction in communication. Sometimes Alice transmits a state in $\{|\psi_1\rangle,\ldots,|\psi_n\rangle\}$ to Bob. The state may be disturbed in the transference and the received state is with large probability a state near the original one. Hoping to correct the possible error, Bob partitions the received-state space into several subsets according to the distances of the received state and the original ones. The error correction is not to identify the received state, but to identify the original state by determining which subset the received state belongs to. In this kind of context, the goal of discrimination is not to identify the state itself, which is the goal of state discrimination, but to tell which set it is in.

It should be pointed that this idea of set discrimination of quantum states are not absolutely new. In Ref. [7], the authors paid some attention to the information carried by the inclusive answer in the state discrimination. A similar idea is also mentioned in Ref. [1]. However, they did not study the set discrimination formally and systematically, which is just the aim of this paper.

In what follows, we denote $S=\cup_{i=1}^{n} S_i$ and $S_i=\{|\psi_{ik}\rangle:k=1,2,...,k_i\}$, where $S_i\cap S_j\neq 0$ for all $i\neq j$. Assume that a quantum system is secretly prepared in one of the states in S. We are to tell which set S_i, the unknown state, is in. Obviously, when each set S_i is a singleton, the set discrimination is reduced to a classical problem of quantum state discrimination. Another extreme is the special case of $n=1$. We exclude this trivial case, and assume $n\geqslant 2$ below.

A discrimination can be represented in two equivalent ways. One is by a generalized measurement, which is a set of linear operators $\{M_1,M_2,\ldots,M_k\}$ such that $\Sigma_{i=1}^{k}M_i^\dagger M_i=I$. In this paper we sometimes use $\{A_0,A_1,\ldots,A_n\}$ to represent a state or set discrimination, where the outcome i ($i=1,...,n$) indicates the state is the ith state (in state discrimination) or the state is in S_i (in set discrimination). A_0 is for the inconclusive report, and may be absent sometimes, meaning that the undecidability is forbidden. Another way for describing a discrimination is to think of it as an arbitrary operation in quantum mechanics, which can be represented by an ancillary system introduced, a unitary evolution, and a von Neumann measurement [13].

We shall begin with considering the condition for perfect set discrimination. A state or set discrimination is said to be perfectly performed if we can always get the correct result without error. It is known that state discrimination is possible if and only if the states are orthogonal. In Ref. [7], it is mentioned that in a three-state case if one state is orthogonal to the other two, then the first state can be detected with certainty. For the general case, the following result holds.

Theorem 1. The sets $S_1,\ldots,S_n$ can be perfectly discriminated if and only if the subspaces they span are orthogonal, i.e., $\langle\psi_i|\psi_j\rangle=0$ *for all* $|\psi_i\rangle\in S_i$ *and* $|\psi_j\rangle\in S_j$, $i\neq j$.

Proof. If the subspaces that $S_1,\ldots,S_n$ span are orthogonal, we can choose an projective measurement $P_1,\ldots,P_n$, where P_i is the projector onto span(S_i). Obviously if the state $|\psi\rangle$ is in S_i, then the probability of occurring outcome j is $\langle\psi|P_i|\psi\rangle=\delta_{ij}$. So the outcome i is sure to occur, indicating that the state $|\psi\rangle$ is in S_i.

Now we assume sets $S_1,\ldots,S_n$ can be perfectly discriminated. Then there exist $A_1,\ldots,A_n$, $\Sigma_{i=1}^{n}A_i^\dagger A_i=I$, such that $\langle\psi_j|A_i^\dagger A_i|\psi_j\rangle=\delta_{ij}$. Thus for all $|\psi_i\rangle\in S_i$ and $|\psi_j\rangle\in S_j$, $i\neq j$,

$$\langle\psi_i|\psi_j\rangle=\langle\psi_i|I|\psi_j\rangle=\left\langle\psi_i\left|\sum_{k=1}^{n}A_k^\dagger A_k\right|\psi_j\right\rangle=0,$$

which concludes the proof. ∎

Note that the condition of $S_1,\ldots,S_n$ being orthogonal is much looser than the condition of all the states in S being orthogonal. In fact, $S_1,\ldots,S_n$ may be orthogonal even when the states in S are linearly dependent. Of course, the linear dependence lies in some set S_i. One of the advantages of set discrimination is that it can put the dependent states together to form a set. Note that after such a measurement as in the proof , we know which S_i the state is in, and the state is unchanged for further discrimination or other kind of processing.

If $S_1,\ldots,S_n$ are not orthogonal, one important strategy is, as in state discrimination, unambiguous discrimination. While a number of states can be unambiguously discriminated if and only if they are linearly independent [8], the unambiguous set discrimination has a similar result.

Theorem 2. For sets $S_1,\ldots,S_n$, the following two statements are equivalent.

(1) They can be unambiguously discriminated, i.e., there exist linear operators $A_0,\ldots,A_n$, *such that* $\Sigma_{i=0}^{n}A_i^\dagger A_i=I$, and for all i, $j=1,...,n$ and all $|\psi_{ik}\rangle\in S_i$, $\langle\psi_{ik}|A_j^\dagger A_j|\psi_{ik}\rangle=\delta_{ij}\gamma_{ik}$, where $\gamma_{ik}>0$ is the success probability of state $|\psi_{ik}\rangle$ being unambiguously discriminated;

(2) $S_1,\ldots,S_n$ are linearly independent, i.e., each state cannot be linearly generated by states in all other sets. To be more precise, for each $i=1,...,n$, if $|\psi_i\rangle\in S_i$, there are no such c_{jk}'s such that $|\psi_i\rangle=\Sigma_{j\neq i;k=1,...,k_j}c_{jk}|\psi_{jk}\rangle$, where $|\psi_{jk}\rangle\in S_j$.

Proof. (1)⇒(2). Suppose $|\psi_{il}\rangle\in S_i$, and $|\psi_{il}\rangle=\Sigma_{j\neq i,k=1,...,k_j}c_{jk}|\psi_{jk}\rangle$. Then

$$\begin{aligned}0<\gamma_{il}&=\langle 0|A_m^\dagger A_m|0\rangle\\&=\sum_{j,j'\neq i;\ k,k'=1,...,k_j}c_{jk}^*c_{j'k'}\langle\psi_{jk}|A_i^\dagger A_i|\psi_{j'k'}\rangle\\&=0\end{aligned}$$

a contradiction. The last equality is due to the fact that $A_i|\psi_j\rangle=0$ whenever $|\psi_j\rangle\in S_j$ and $i\neq j$.

(2)⇒(1). It is sufficient to show that there are $A_1,\dots,A_n$, such that (i) $I-\Sigma_{i=1}^n A_i^\dagger A_i \geqslant 0$ (a matrix $A\geqslant 0$ means it is positive), (ii) $A_i|\psi_{jl}\rangle = 0$ for $i\neq j$, and $A_i|\psi_{ik}\rangle \neq 0$ for all $i=1,...,n$. In fact, requirement (i) can be easily satisfied if we have found $A_1,\dots,A_n$ satisfying requirement (ii), because if $I-\Sigma_{i=1}^n A_i^\dagger A_i$ is not positive, we can let $A_i' = \varepsilon A_i$ with ε being a positive number small enough. So below we aim at finding $A_1,\dots,A_n$ such that requirement (ii) holds. We shall first find A_1, then $A_2,\dots,A_n$ can be found in the same way.

We denote span$(\cup_{i=2}^n S_i)$ by V_{2-n}, and span$(\cup_{i=1}^n S_i)$ by V, and further denote by W_1 the orthogonal complementary space of V_{2-n} in V. Now we need to find a vector $|\omega\rangle$ in W_1, such that $\langle\omega|\psi_{1k}\rangle\neq 0$ for all $k=1,...,k_1$, then let $A_1 = |\omega\rangle\langle\omega|$, which obviously satisfies requirement (ii). Let the projection of $|\psi_{1k}\rangle$ on W_1 be $|\varphi_k\rangle$, then it holds that $\langle\omega|\psi_{1k}\rangle = \langle\omega|\varphi_k\rangle$. Because $S_1,\dots,S_n$ are linearly independent, each $|\varphi_k\rangle$ is not the zero vector. So now the problem is reduced to finding a vector $|\omega\rangle$ in W_1 so that it is not orthogonal to each of finite known states $|\varphi_1\rangle,\dots,|\varphi_{k_1}\rangle$, all of which are in W_1. But such vector does exist because for each $|\varphi_k\rangle$, the vectors orthogonal to $|\varphi_k\rangle$ in W_1 form a (r_1-1)-dimensional subspace. The union (note: not span) of finite such (r_1-1)-dimensional subspaces cannot be the whole r_1 dimensional space W_1. Therefore, there is a state $|\omega\rangle$ in W_1 such that $\langle\omega|\varphi_k\rangle\neq 0$ for all $k=1,...,k_1$, as desired. ■

This theorem also implies that in Chefles's thought experiment, Bob cannot make unambiguous discrimination between sets $\{|u\rangle,|u'\rangle\}$ and $\{|v\rangle,|v'\rangle\}$ because the spaces spanned by $\{|u\rangle,|u'\rangle\}$ and $\{|v\rangle,|v'\rangle\}$ are linearly dependent (in fact the same). This means that even if Bob hopes to receive one-bit always-correct information with some positive probability, he cannot succeed. In other words, probabilistic error-free superluminal communication cannot be implemented in this way. This is consistent with the theory of relativity, because even if Bob can make error-free superluminal communication with little positive success probability, he then can make it with near 100% success probability by repeating the course until success outcome occurs.

Having known what kind of sets of quantum states can be unambiguously discriminated, another question is the discrimination efficiency, i.e., the success probability. In state unambiguous discrimination, linearly independent states $|\psi_1\rangle,\dots,|\psi_n\rangle$ can be unambiguously discriminated with the success probability of $|\psi_i\rangle$ being γ_i if and only if $X-\Gamma \geqslant 0$, where $X=|\langle\psi_i|\psi_j\rangle|_{i,j=1,...,n}$ and $\Gamma=\mathrm{diag}(\gamma_1,...,\gamma_n)$ [10,12]. Now we consider the problem for set discrimination.

Suppose the sets $S_1,\dots,S_n$ can be unambiguously discriminated. We can represent the discrimination as a unitary evolution

$$U(|\psi_{ik}\rangle|\Psi_{BP}\rangle) = \sqrt{\gamma_{ik}}|\Psi_{AB}^{(ik)}\rangle|P_i\rangle + \sqrt{1-\gamma_{ik}}|\Phi_{AB}^{(ik)}\rangle|P_{n+1}\rangle$$

together with a von Newmann measurement on the probe P [10]. Here $|P_1\rangle,\dots,|P_{n+1}\rangle$ are $n+1$ orthonormal states, and B is an auxiliary system. We perform a measurement on P after the evolution. Outcome i indicates the original state is from S_i, and the $(n+1)$th outcome means failure. γ_{ik} here is the efficiency for $|\psi_{ik}\rangle$. Now such a unitary operator U exists if and only if there are $\{\Psi_{AB}^{(ik)}\}$ and $\{\Phi_{AB}^{(ik)}\}$ so that the following inter-inner-product equation holds for all $i,j=1,...,n;\ k=1,...,k_i;\ l=1,...,k_j$ [10]:

$$\langle\psi_{ik}|\psi_{jl}\rangle = \delta_{ij}\sqrt{\gamma_{ik}\gamma_{jl}}\langle\Psi_{AB}^{(ik)}|\Psi_{AB}^{(jl)}\rangle + \sqrt{(1-\gamma_{ik})(1-\gamma_{jl})} \times\langle\Phi_{AB}^{(ik)}|\Phi_{AB}^{(jl)}\rangle. \tag{1}$$

Note that there exists $\{\Phi_{AB}^{(ik)}\}$ such that $[\sqrt{(1-\gamma_{ik})(1-\gamma_{jl})}\langle\Phi_{AB}^{(ik)}|\Phi_{AB}^{(jl)}\rangle]$ is equal to some matrix A if and only if $A\geqslant 0$ and the (ik)th entry on the diagonal of A is $1-\gamma_{ik}$. (Recall that any positive matrix A can be decomposed as $A=C^\dagger C$. Viewing every column of C as a vector will yield the above result.) So there exist $\{\Psi_{AB}^{(ik)}\}$ and $\{\Phi_{AB}^{(ik)}\}$ such that Eq. (1) holds if and only if there exists $\{\Psi_{AB}^{(ik)}\}$ such that $X-\Gamma_{AB}\geqslant 0$, where $X=[\langle\psi_{ik}|\psi_{jl}\rangle]$ and Γ_{AB} is a partitioned diagonal matrix with the ith diagonal block being $[\sqrt{\gamma_{ik}\gamma_{il}}\langle\Psi_{AB}^{(ik)}|\Psi_{AB}^{(il)}\rangle]_{k,l=1,...,k_i}$. We write the sufficient and necessary condition formally as the following theorem.

Theorem 3. The sets $S_1,\dots,S_n$ can be unambiguously discriminated with efficiency $\{\gamma_{ik}: i=1,...,n; k=1,...,k_i\}$ if and only if the matrix $X-\Gamma_{AB}\geqslant 0$.

We may further remove the implicit $\{|\Psi_{AB}^{(ik)}\rangle\}$ in the theorem above by noting that $[\langle\Psi_{AB}^{(ik)}|\Psi_{AB}^{(il)}\rangle]$ is just a positive matrix with all diagonal entries being 1.

Theorem 4. The sets $S_1,\dots,S_n$ can be unambiguously discriminated with efficiency $\{\gamma_{ik}: i=1,...,n; k=1,...,k_i\}$ if and only if there are matrices $\Gamma_1,\dots,\Gamma_n$ such that Γ_i is a $k_i\times k_i$, positive matrix with kth diagonal entry being γ_{ik}, and the matrix $X-\mathrm{diag}\{\Gamma_1,...,\Gamma_n\}\geqslant 0$.

There are two special cases of $\{|\Psi_{AB}^{(ik)}\rangle\}$ in Theorem 3. One is that $|\Psi_{AB}^{(il)}\rangle,\dots,|\Psi_{AB}^{(ik_i)}\rangle$ are orthogonal with each other, in which case Γ_{AB} is a diagonal matrix diag$\{\gamma_{ik}: i=1,...,n; k=1,...,k_i\}$. This means a clear fact that if all the states in $S=\cup_{i=1}^n S_i$ can be state unambiguously discriminated with efficiency $\{\gamma_{ik}: i=1,...,n; k=1,...,k_i\}$ then $S_1,\dots,S_n$ can be set unambiguously discriminated with the same efficiency. It is obvious because if we know exactly what the originally unknown state is, we certainly know which set it belongs to. Another special case is that we let all the $|\Psi_{AB}^{(ik)}\rangle$'s be the same. Then every $\langle\Psi_{AB}^{(ik)}|\Psi_{AB}^{(il)}\rangle = 1$, and we get a proposition as follows.

Proposition 1. If $X-\mathrm{diag}\{\Gamma_1^*,...,\Gamma_n^*\}\geqslant 0$, where $\Gamma_i^* = [\sqrt{\gamma_{ik}\gamma_{il}}]_{k,l=1,...,k_i}$, then sets $S_1,\dots,S_n$ can be unambiguously discriminated with efficiency $\{\gamma_{ik}\}$.

In the rest of the paper, we are mainly concerned with the combination of state and set discrimination. As mentioned in the beginning of the paper, one of the main benefits of set discrimination is for distinguishing linearly dependent states. One of the meaningful situations is about state discrimination. In Ref. [8], Chefles pointed out that unambiguous state discrimination with maximal average efficiency results in the inconclusive states $|\varphi_1\rangle,\dots,|\varphi_n\rangle$ being linearly dependent and thus making any attempt to further unambiguous discrimination impossible. But generally, the linear dependence of the inconclusive states may be local. In this case, further set discrimination may be possible. Even if the inconclusive states are globally dependent, there may be a partition among these states such that the subsets are linearly independent according to the definition in Theorem 2. This means that after the state discrimination fails, we may still have chance to extract some information about the unknown state. Here the information is that we decrease the nondetermination by excluding some impossible candidates.

The two-step measurements can be represented by one measurement, which is actually the combination of state discrimination and set discrimination. The situation is that for some states $S=\cup_{i=1}^{n}S_i$, we design to get some information about the identification of an unknown state $|\psi\rangle\in S$ by telling which state $|\psi\rangle$ is or telling which set S_i the state $|\psi\rangle$ belongs to. Generally, for $S=\cup_{i=1}^{n}S_i$, $S_i=\{|\psi_{ik}\rangle:k=1,2,...,k_i\}$, a state and set combined discrimination can be represented as a measurement

$$\{A_0;A_1,\dots,A_n;A_{11},\dots,A_{1k_1},\dots,A_{n1},\dots A_{nk_n}\},$$

where A_0 is for reporting the inconclusive result, A_i for reporting $|\psi\rangle$ in set S_i, and A_{ik} for reporting $|\psi\rangle$ being $|\psi_{ik}\rangle$. Sometimes certain A_{ik} may be missing with A_i still remaining, meaning that for $|\psi_{ik}\rangle$ we only need or hope to know it is from S_i. Similarly, if some A_j is absent, we mean that for $|\psi_{j1}\rangle,\dots,|\psi_{jk_j}\rangle$, we only want state discrimination.

We consider the efficiency problem in this context. We denote the success probabilities of the unambiguous state and set identification for state $|\psi_{ik'}\rangle$ by γ_{ik} and γ'_{ik}, respectively. Similarly, as the analysis of Theorem 3, we represent the unambiguous discrimination by

$$U(|\psi_{ik}\rangle|\Psi_{BP}\rangle)=\sqrt{\gamma_{ik}}|\Psi_{AB}^{(ik)}\rangle|P_{ik}\rangle+\sqrt{\gamma'_{ik}}|\Psi_{AB}'^{(ik)}\rangle|P_i\rangle + \sqrt{1-\gamma_{ik}-\gamma'_{ik}}|\Phi_{AB}^{(ik)}\rangle|P_{n+1}\rangle,$$

where $i=1,...,n$ and $k=1,...,k_i$. The states $\{|P_{ik}\rangle:i=1,...,n,k=1,...,k_i\}$, $\{|P_i\rangle:i=1,...,n\}$, and $|P_{n+1}\rangle$ are orthogonal with each other. An inter-inner-product yields the following equation:

$$\langle\psi_{ik}|\psi_{jl}\rangle=\delta_{ij}\delta_{kl}\sqrt{\gamma_{ik}\gamma_{jl}}+\delta_{ij}\sqrt{\gamma'_{ik}\gamma'_{jl}}\langle\Psi_{AB}'^{(ik)}|\Psi_{AB}'^{(jl)}\rangle + \sqrt{(1-\gamma_{ik})(1-\gamma_{jl})}\langle\Phi_{AB}^{(ik)}|\Phi_{AB}^{(jl)}\rangle. \quad (2)$$

A further analysis similar to that of Theorem 3 gives the following result.

Theorem 5. The states in $S=\cup_{i=1}^{n}S_i$ can be unambiguously discriminated with state efficiency $\{\gamma_{ik}\}$ and set efficiency $\{\gamma'_{ik}\}$ if and only if $X-\Gamma-\Gamma'_{AB}\geqslant 0$.

We give some remarks about the theorem. If $\Gamma=0$, it is just set discrimination, and Theorem 5 is reduced to Theorem 3. If $\Gamma'_{AB}=0$, it is state discrimination. In fact, the two-step discriminations mentioned above can also be reflected in Theorem 5. When an unambiguous state discrimination is performed, $X-\Gamma$ is positive; if the discrimination is optimal, any increase of γ_i will make $X-\Gamma$ not positive. But we may find Γ'_{AB} such that $X-\Gamma-\Gamma'_{AB}$ is still positive, which means that a further set discrimination is possible.

Of course, in more cases Theorem 5 implies that the greater the Γ is, the lesser the Γ'_{AB} is, and vice versa. So a combination of unambiguous state and set discrimination is a tradeoff between pure state discrimination and pure set discrimination.

We may also get the counterparts of Theorem 4 and Proposition 1 in the given context but we omit them here because of the similarity. We conclude the paper with an example showing the meaning and advantage of the combination of state and set discrimination. Below is a matrix, each row of which is a vector in a nine-dimensional Hilbert space.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Now we are to identify a state $|\psi\rangle$ chosen from the nine. Classical state discrimination will fail to give unambiguous discrimination because the nine states are linearly dependent. But after observation we may find that only the first three states are linearly dependent; the middle three (4–6) are linearly independent; the last three are orthogonal. And the third cluster is orthogonal to the first two. So we divide the nine states into three sets, i.e., $S_1=\{|\psi_1\rangle,|\psi_2\rangle,|\psi_3\rangle\}$, $S_2=\{|\psi_4\rangle,|\psi_5\rangle,|\psi_6\rangle\}$, and $S_3=\{|\psi_7\rangle,|\psi_8\rangle,|\psi_9\rangle\}$. Then we can construct a measurement $\{A_0,A_1,A_2,A_{21},A_{22},A_{23},A_{31},A_{32},A_{33}\}$. A_0 is the inconclusive result; A_1, A_2 report that the state is in S_1, S_2, respectively; others give the exact state identification. Thus the states in S_3 can be perfectly identified; the states in S_2 can be unambiguously identified and can also be unambiguously told to be in S_2; the states in S_1 can be unambiguously told to be in S_1. Therefore, by set discrimination together with state discrimination, we can generally get more information about the unknown state $|\psi\rangle$. Further analysis by Theorem 5 shows that because $|\psi_4\rangle$ and $|\psi_5\rangle$ are close to each other, the efficiency of state discrimination of $|\psi_4\rangle$ (or $|\psi_5\rangle$) is low. But the efficiency of the state discrimination of $|\psi_6\rangle$ can be relatively high because it is far from $|\psi_4\rangle$ and $|\psi_5\rangle$. However, just because of this, the efficiency of the set discrimination of $|\psi_6\rangle$, i.e., the probability of reporting $|\psi\rangle$ is in S_2 if $|\psi\rangle$ is $|\psi_6\rangle$, is less than the efficiency of set discrimination of $|\psi_4\rangle$ or $|\psi_5\rangle$.

In summary, this paper introduces a notion of set discrimination, and makes some discussions about several basic problems in it. We find that perfect set discrimination can be performed if and only if the sets are orthogonal; unambiguous set discrimination can be performed if and only if the sets are linearly independent. State discrimination and set discrimination can be combined. The best efficiency of discrimination is also derived, both in set discrimination and in combined case.

There is some future work to be done. For instance, it is known that there is an intimate connection between quantum cloning and the efficiency of the state discrimination (see [1] and [10], for example). Are there some interesting results about cloning and efficiency of the set discrimination? Questions of this kind remain for future study.

ACKNOWLEDGMENTS

We thank Xiaoming Sun for reading the manuscript and pointing out a minor error in Theorem 2 in its first version. This work was supported by the National Foundation for Distinguished Young Scholars (Grant No. 69725004), the National Key Project for Basic Research (Grant No. 6982300), and the National Foundation of Natural Sciences (Grant No. 69823001).

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