Lower bound on inconclusive probability of unambiguous discrimination

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(Received 7 July 2002; published 20 December 2002)

We derive a lower bound on the inconclusive probability of unambiguous discrimination among n linearly independent quantum states by using the constraint of no signaling. It improves the bound presented in the paper of Zhang, Feng, Sun, and Ying [Phys. Rev. A **64**, 062103 (2001)], and when the optimal discrimination can be reached, these two bounds coincide with each other. An alternative method of constructing an appropriate measurement to prove the lower bound is also presented.

DOI: 10.1103/PhysRevA.66.062313 PACS number(s): 03.67.−a, 03.65.Ta

I. INTRODUCTION

As we all know, precise quantum discrimination among nonorthogonal quantum states is forbidden by the laws of quantum mechanics. However, if a nonzero probability of inconclusive answer is allowed, we can distinguish with certainty linearly independent quantum states. This strategy is usually called *unambiguous discrimination*. Unambiguous discrimination among two equally probable nonorthogonal quantum states was originally addressed by Ivanovic [1], and then by Dieks [2] and Peres [3]. Jaeger and Shimony [4] extended their result to the case of two nonorthogonal states with unequal prior probabilities. Chefles [5] showed that n quantum states can be unambiguously discriminated if and only if they are independent. In another paper, Chefles and Barnett [6] proposed an optimal unambiguous discrimination for equally probable symmetrical states. For the general multistate cases, Zhang *et al.* [7] gave an upper bound for success probability of unambiguous discrimination, but the condition under which the upper bound can be reached was not presented. In fact, it was shown in Ref. [8] that the problem of success probability of unambiguous discrimination is the semidefinite programming problem, which is well known but at present has only numerical solution in mathematics.

The methods to cope with optimal unambiguous discrimination presented in the above literature are the same, namely, first consider the unitary interaction between the system of interest and an ancilla, then measure both systems. However, Barnett and Andersson [9] proposed an interesting alternative viewpoint by using the no-signaling condition to deal with unambiguous discrimination among two quantum states. Here naturally arises a question: can we extend the idea to the most general case of unambiguous discrimination among n independent quantum states? In this paper, we give a ‘‘yes’’ answer to this question by deriving a lower bound on the probability of inconclusive answer when n independent quantum states are discriminated. It coincides with the known bound when the case of $n=2$ is considered.

We organize this paper as follows. In Sec. II, we derive a lower bound on failure probability of unambiguous discrimination by using the constraint of no signaling and discuss the condition under which the lower bound can be reached. The comparison between our bound and the one presented in Ref. [7] is also drawn. Section III aims at proposing an alternative method by constructing an appropriate measurement to prove the bound stated in Sec. II. Section IV concludes this paper and points out a topic for further studies.

II. LOWER BOUND ON PROBABILITY OF INCONCLUSIVE ANSWER

Suppose a quantum system is prepared in one of the n states $|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_n\rangle$ in a k-dimensional Hilbert space with probabilities $p_1,p_2,\ldots,p_n$, respectively, where $k\geqslant n$ is an arbitrary positive integer. What we wish to do is to identify which state the system is prepared in with no errors. Obviously, we cannot hope to give an answer at any time because of the constraint of the laws of quantum mechanics. That is, there will be a nonzero probability that we get an inconclusive result. The optimal strategies to unambiguously discriminate independent quantum states are the ones which minimize the probability of inconclusive result. In the previous literature, e.g., Refs. [1–8], unambiguous discrimination is usually carried by constructing a generalized measurement composed of a set of linear transformation operators $\{M_m,m=0,1,\ldots,n\}$ satisfying the properties

$$\sum_{m=0}^{n} M_m^\dagger M_m=I,$$

$$\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle=P_m\delta_{m,i}\quad(m>0). \tag{1}$$

Intuitively, the first property makes the set $\{M_m\}$ a generalized measurement and the second one ensures that if the outcome of the measurement is m, we can definitely say that the system is in the state $|\psi_m\rangle$. Here P_m is the success probability of $|\psi_m\rangle$ being identified. By using the measurement above, we can easily transform the optimal unambiguous dis-

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crimination problem to the problem of solving the following semidefinite programming:

$$\text{maximize} \quad \sum_{i=1}^{n} p_i P_i$$

$$\text{subject to} \quad X-\Gamma \geqslant 0, \;\; \Gamma \geqslant 0, \tag{2}$$

where $X=[\langle \psi_i|\psi_j\rangle]_{n\times n}$ is the matrix with the (i,j)th entry being $\langle \psi_i|\psi_j\rangle$ and $\Gamma=\mathrm{diag}\{P_1,P_2,\ldots,P_n\}$ [8]. Unfortunately, it has been shown that the semidefinite programming problem above seems hard to find analytic solutions, and only numerical methods are known up to now [10].

Although the optimal strategies for unambiguous discrimination among n quantum states are hard or, be more pessimistic, impossible to obtain, we can simplify the most general problem to consider two easier questions instead. One question is for some special quantum states and prior probabilities, how to derive the optimal discrimination strategies. Chefles considered the case of n symmetric independent quantum states with equal prior probabilities and obtained the maximum probability to unambiguously discriminate them. The other question is that we can derive upper bounds on the optimal success probability of discrimination. One example is, by using a series of proper inequalities, Zhang *et al.* [7] derived an upper bound on success probability P_s of unambiguous discrimination as follows:

$$P_s \leqslant 1-\frac{1}{n-1}\sum_{i\neq j}\sqrt{p_i p_j}|\langle \psi_i|\psi_j\rangle|. \tag{3}$$

Since the upper bound is for the most general case of unambiguous discrimination, it is important and interesting to improve it. Fortunately, by using the constraint of no signaling, which was originally presented by Barnett and Andersson [9], we can derive a better lower bound on inconclusive probability of discrimination.

Suppose Alice and Bob shared a quantum system composed of two separated particles which is prepared in the entangled state

$$|\Phi\rangle=\sum_{i=1}^{n}\sqrt{p_i}|i\rangle_A|\psi_i\rangle_B, \tag{4}$$

where the subscripts A and B denote the particles held by Alice and Bob, respectively, and $|i\rangle, i=1,2,\ldots,n$ are orthonormal basis states for Alice's system.

The reduced density matrix of Bob's system is

$$\rho_B=\mathrm{tr}_A|\Phi\rangle\langle\Phi|=\sum_{i=1}^{n}p_i|\psi_i\rangle\langle\psi_i|. \tag{5}$$

That is, Bob's system is in a mixture of $|\psi_i\rangle$ with respective prior probabilities p_i. Suppose Bob tries to unambiguously distinguish between these states. The probabilities that he correctly identifies $|\psi_i\rangle$ and gets an inconclusive result are P_i and P_0, respectively, where $i=1,2,\ldots,n$. Then after the discrimination, the density matrix of Alice's system is

$$\rho_A=\sum_{i=1}^{n}P_i|i\rangle\langle i|+P_0\rho, \tag{6}$$

where $\rho=\Sigma_{i,j}e_{ij}|i\rangle\langle j|$ is the density matrix of Alice's system corresponding to the inconclusive result of Bob's discrimination. The ith summand of the first term on the right-hand side above shows that if Bob correctly identifies the state $|\psi_i\rangle$, then Alice's system is definitely in the state $|i\rangle$.

According to the no-signaling constraint, any operators performed on Bob's system will not change the density matrix of Alice's system, namely,

$$\rho_A=\mathrm{tr}_B|\Phi\rangle\langle\Phi|=\sum_{i,j=1}^{n}\sqrt{p_i p_j}\langle\psi_j|\psi_i\rangle|i\rangle\langle j|. \tag{7}$$

Taking Eqs. (6) and (7) together, we have

$$\sum_{i=1}^{n}P_i|i\rangle\langle i|+P_0\sum_{i,j=1}^{n}e_{ij}|i\rangle\langle j|=\sum_{i,j=1}^{n}\sqrt{p_i p_j}\langle\psi_j|\psi_i\rangle|i\rangle\langle j|. \tag{8}$$

Comparing the corresponding terms yields the equations as follows:

$$P_i+P_0e_{ii}=p_i, \quad i=1,2,\ldots,n, \tag{9}$$

$$P_0e_{ij}=\sqrt{p_i p_j}\langle\psi_j|\psi_i\rangle, \quad i\neq j. \tag{10}$$

By the equations in Eq. (10), it holds that

$$P_0^2\sum_{i\neq j}e_{ij}e_{ji}=\sum_{i\neq j}p_ip_j|\langle\psi_i|\psi_j\rangle|^2. \tag{11}$$

Now, we derive the global minimum of P_0 from Eq. (11). Since $\rho=\Sigma_{i,j}e_{ij}|i\rangle\langle j|$ is a density matrix, we have $\mathrm{tr}(\rho)=1$ and $\mathrm{tr}(\rho^2)\leqslant 1$, that is,

$$\sum_i e_{ii}=1,$$

$$\sum_i e_{ii}^2+\sum_{i\neq j}e_{ij}e_{ji}\leqslant 1. \tag{12}$$

Using Cauchy inequality, we have

$$\sum_{i\neq j}e_{ij}e_{ji}\leqslant 1-\sum_i e_{ii}^2\leqslant 1-\left(\sum_i e_{ii}\right)^2\Big/ n=(n-1)/n. \tag{13}$$

Then from Eq. (11), we get the lower bound of inconclusive probability as follows:

$$P_0\geqslant P_0^{opt}=\sqrt{\frac{n}{n-1}\sum_{i\neq j}p_ip_j|\langle\psi_i|\psi_j\rangle|^2}. \tag{14}$$

Now we turn to find the condition under which the lower bound of inconclusive probability in Eq. (14) can be reached. From the procedure we derive the global minimum of P_0, we can easily see that it is reached when ρ is a pure state and

$e_{ii}=1/n$. Suppose pure state $\rho=|\phi\rangle\langle\phi|$ for some $|\phi\rangle=\Sigma_i\alpha_i|i\rangle$, then $e_{ij}=\alpha_i\alpha_j^*$. Putting the condition that $e_{ii}=1/n$, we have $|\alpha_i|=1/\sqrt{n}$ for any i, so

$$|e_{ij}|=|\alpha_i\alpha_j^*|=1/n. \tag{15}$$

From the above equation and Eq. (10), we can get

$$\sqrt{p_ip_j}|\langle\psi_i|\psi_j\rangle|\equiv C \quad \text{when} \quad i\neq j, \tag{16}$$

for some constant C. Once a discrimination problem satisfying the constraints (16) is given, we can see from Eq. (14) that

$$P_0^{opt}=nC. \tag{17}$$

Furthermore, taking the second condition, say $e_{ii}=1/n$, back to Eq. (9), and noticing that $P_i\geqslant 0$ we have, for any i and j,

$$|\langle\psi_i|\psi_j\rangle|^2\leqslant p_i/p_j. \tag{18}$$

That is, if the conditions (16) and (18) hold, P_0^{opt} can be reached, which corresponds obviously to the best discrimination strategies.

It is shown from constraints in Eqs. (16) and (18) that whether the optimal discrimination can be reached is determined by the prior probabilities and the inner product between each pair of the states. If the probability of one state is very small, then to reach the optimal discrimination, the norm of the inner product (in other words, the cosine of the angle between the two states) of this state and each of the rest states must be close enough to 0, that is, they must be "almost" orthogonal. On the other hand, if some two of these states have very large norm of the inner product, that is, they are "almost" parallel, then the prior probabilities of these two states must be close enough. What we would like to point out still is that since $|\langle\psi_i|\psi_j\rangle|\leqslant 1$ for any i and j, the constraints in Eq. (18) always hold for the case in which the prior probabilities of all states are equal, i.e., $p_i=1/n$, which occurs rather frequently in practice. Furthermore, if all states are equally distant to each other, the constraints in Eq. (16) also hold. For example, in three-dimensional Hilbert state space, $|\psi_1\rangle=|0\rangle$, $|\psi_2\rangle=\frac{1}{2}|0\rangle+\sqrt{3}/2|1\rangle$, and $|\psi_3\rangle=\frac{1}{2}|0\rangle+\sqrt{3}/6|1\rangle+\sqrt{6}/3|2\rangle$ with the same prior probability $\frac{1}{3}$ satisfy the constraints described in Eqs. (16) and (18). It is easy to construct the optimal measurement strategy by the method introduced in Ref. [5] and the inconclusive probability is $\frac{1}{2}$, which coincides with Eq. (14).

To conclude this section, we draw a comparison between the bound we present and the one in Ref. [7]. Using Cauchy inequality again, we can easily find that

$$\sqrt{\frac{n}{n-1}\sum_{i\neq j} p_ip_j|\langle\psi_i|\psi_j\rangle|^2}\geqslant\frac{1}{n-1}\sum_{i\neq j}\sqrt{p_ip_j}|\langle\psi_i|\psi_j\rangle|, \tag{19}$$

which shows that we derive a more precise bound on the probability of inconclusive answer when an unambiguous discrimination is taken among n independent states. We may also note that if the conditions in Eqs. (16) and (18) are satisfied, these two bounds coincide with each other, otherwise the bound obtained in this paper is strictly better than that given in Ref. [7].

III. AN ALTERNATIVE METHOD TO DERIVE THE BOUND

To discriminate quantum states, we must obtain some information from the system of interest. But the only way for us to get information from a system is to apply a measurement on it. So, in principle, any discrimination must be realized by measurements and, furthermore, the optimal discrimination must also be achieved by a proper measurement. The philosophical consideration leads us to find an alternative derivation of the inconclusive probability bound in terms of quantum measurement. This is exactly the purpose of this section.

In what follows, we present the details of such a derivation of the bound stated in Sec. II by constructing an appropriate measurement. First, from Eq. (1),

$$\begin{aligned}\left(\sum_{i=1}^n p_i|\langle\psi_i|M_0^\dagger M_0|\psi_i\rangle|\right)^2 &=\left(\sum_i p_i|M_0|\psi_i\rangle|^2\right)^2\\ &=\sum_i p_i^2|M_0|\psi_i\rangle|^4\\ &\quad+\sum_{i\neq j} p_ip_j|M_0|\psi_i\rangle|^2|M_0|\psi_j\rangle|^2.\end{aligned} \tag{20}$$

By Cauchy inequality, we have

$$\sum_i p_i^2|M_0|\psi_i\rangle|^4 \geqslant\left(\sum_{i\neq j} p_ip_j|M_0|\psi_i\rangle|^2|M_0|\psi_j\rangle|^2\right)\Big/(n-1) \tag{21}$$

and

$$\sum_{i\neq j} p_ip_j|M_0|\psi_i\rangle|^2|M_0|\psi_j\rangle|^2\geqslant\sum_{i\neq j} p_ip_j|\langle\psi_i|M_0^\dagger M_0|\psi_j\rangle|^2. \tag{22}$$

So, the failure probability of the unambiguous discrimination

$$
\begin{aligned}
P_0 &= \sum_{i=1}^{n} p_i |\langle \psi_i | M_0^\dagger M_0 | \psi_i \rangle| \\
&\geqslant \sqrt{\frac{n}{n-1} \sum_{i\neq j} p_i p_j |M_0|\psi_i\rangle|^2 |M_0|\psi_j\rangle|^2} \\
&= \sqrt{\frac{n}{n-1} \sum_{i\neq j} p_i p_j |\langle \psi_i | I - M_1^\dagger M_1 - \cdots - M_n^\dagger M_n | \psi_j\rangle|^2} \\
&= \sqrt{\frac{n}{n-1} \sum_{i\neq j} p_i p_j |\langle \psi_i | \psi_j\rangle|^2}
\end{aligned} \tag{23}
$$

as desired.

Furthermore, Eq. (23) can be rewritten as follows:

$$
P_0 \geqslant n \sqrt{\frac{1}{n(n-1)} \sum_{i\neq j} p_i p_j |\langle \psi_i | \psi_j \rangle|^2}, \tag{24}
$$

the second multiplier on the right side of which is just the arithmetic mean of $\{(\sqrt{p_i p_j}|\langle \psi_i|\psi_j\rangle|)^2 : i\neq j\}$. But we know by the power-mean inequality that if we define $P^{(k)}$ as

$$
\sqrt[k]{\frac{1}{n(n-1)} \sum_{i\neq j} (\sqrt{p_i p_j}|\langle \psi_i|\psi_j\rangle|)^k}, \tag{25}
$$

then $P^{(k+1)} \geqslant P^{(k)}$ for $k=1,2,\ldots$. It is easy to find that the bound obtained in Ref. [7] is $nP^{(1)}$, while the one presented in this paper is just $nP^{(2)}$.

Since $P^{(1)} \leqslant P^{(2)} \leqslant P^{(3)} \leqslant \cdots$, can we further increase the lower bound to $nP^{(3)}$ or even greater? Unfortunately, this turns out to be impossible. In fact, one can find a counterexample as follows. Suppose $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$ is an orthonormal basis of the Hilbert space, $|\psi_i\rangle = |i\rangle$ for $i=1,2,\ldots,n-1$ and $|\psi_n\rangle = \sqrt{1-\epsilon}|n-1\rangle + \sqrt{\epsilon}|n\rangle$ for a small enough positive number ϵ. The prior probability for each $|\psi_i\rangle$ is $p_i = 1/n$. Then we can easily find that $|\langle \psi_i | M_0^\dagger M_0 | \psi_i\rangle| = 0$ for $i=1,2,\ldots,n-2$ and $|\langle \psi_i | M_0^\dagger M_0 | \psi_i \rangle| \simeq 1$ for $i=n-1,n$. Then $P_0 = 2/n$ and

$$
P^{(k)} \simeq \frac{1}{n} \sqrt[k]{\frac{2}{n(n-1)}}. \tag{26}
$$

It is easy to see that when $k \geqslant 3$,

$$
P_0 = \frac{2}{n} < nP^{(k)} = \sqrt[k]{\frac{2}{n(n-1)}}, \tag{27}
$$

so, as claimed, this indicates that we could not improve the result to $nP^{(3)}$. In fact, from Eq. (27) we know that for an arbitrary positive integer n, if

$$
k > r_n = \log_{2/n} \frac{2}{n(n-1)} = 1 + \log_{2/n} \frac{1}{n-1}, \tag{28}
$$

then there is an example in which $P_0 < nP^{(k)}$. Therefore, the lower bound on P_0 could not be greater than $nP^{(r_n)}$. But what about $nP^{(k)}$, where $2 < k < r_n$? The question remains for further study.

IV. CONCLUSION

In this paper, we derive a lower bound on the inconclusive probability of unambiguous discrimination among n states by using the constraint of no signaling. It improves the bound presented in Ref. [7], and when the optimal discrimination can be reached, these two bounds coincide with each other. An alternative method of constructing an appropriate measurement to prove the lower bound is also presented.

By carefully observing the format of the probability bound given in this paper and the one in Ref. [7], we find they are all special cases of a quantity which is introduced as $P^{(k)}$ in Eq. (25). Since the probability bounds in Ref. [7] and the present paper are, respectively, $P^{(1)}$ and $P^{(2)}$, we naturally want to know whether $P^{(k)}$ of a larger parameter k can serve as a better bound of the inconclusive probability in the unambiguous discrimination among n independent quantum states. It turns out that we are able to give a negative answer to this question whenever $k \geqslant 3$, but the case for $2 < k < r_n$ is still open.

ACKNOWLEDGMENT

This work was partially supported by the National Key Project for Basic Research (Grant No. 1998030509).

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