Upper bound for the success probability of unambiguous discrimination among quantum states

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One strategy to the discrimination problem is to identify the state with certainty, leaving a possibility of undecidability. This paper gives an upper bound for the maximal success probability of unambiguous discrimination among *n* states. This bound coincides with the known IDP limit when two states are considered.

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Quantum state discrimination is a classically interesting and important problem [1]. A quantum system is prepared in a number of known, finite set of states, and we hope to determine what quantum state the system was actually in with the minimum probability of error. Ivanovic [2], Dieks [3], and Peres [4] consider the problem under the special requirement that one must identify the state with certainty, leaving a possibility of undecidability. They find a higher probability of discrimination than merely using von Neumann measurement on a single-qubit state by adding an auxiliary quantum system, and the best result along their approach in the case of discrimination between two quantum states is $1-|\langle p|q\rangle|$, where p and q are the two states to be distinguished. Jaeger and Shimony extend the problem in Ref. [5] to the case of unequal *priori* probabilities, and get the result as $1-2\sqrt{rs}|\langle p|q\rangle|$, where r and s are the *priori* probabilities of the two states. This result is also discussed by Ban in Ref. [6] in the context of quantum communications. However, all these papers are along the Ivanovic's approach and demonstrate only as far as their approach is considered that the probability of correct classification they get is the optimal one. This paper extends their work by showing that the bound they obtain is also the best one in a more general context: if one wants to unambiguously distinguish two quantum states only by arbitrary generalized measurements (POVMs), then the maximal probability of successful classification is $1-2\sqrt{rs}|\langle p|q\rangle|$.

Another naturally intriguing extension is to prepare more than two states to be discriminated. Peres and Terno [7] give a solution to the problem of optimal distinction of three states having arbitrary *priori* probabilities and arbitrary detection values. More generally, for the distinction of n quantum states, an important fact shown by Chefles in Ref. [8] is that only linear independent states can be unambiguously discriminated. Another result, which solves a kind of special case known as equally probable symmetrical states, was given in Ref. [9]. But the optimal unambiguous distinction of arbitrary n quantum states is still unknown. This note gives an upper bound on this optimal value, and shows that the upper bound coincides with the known result $1-2\sqrt{rs}|\langle p|q\rangle|$ in the two-state case.

In what follows, we assume a quantum system is prepared in one of the n states $|\psi_1\rangle, \ldots, |\psi_n\rangle$ in a k-dimension Hilbert space with probabilities $p_1, \ldots, p_n$, respectively, where k is an arbitrary positive integer. We hope to identify the state of the system by one or more measurements. A measurement is described by a set of linear operators $\{M_m\}$ such that $\Sigma_m M_m^\dagger M_m = I$. If the state of the quantum system is $|\psi\rangle$ before the measurement then the probability that result m occurs is $\langle\psi|M_m^\dagger M_m|\psi\rangle$, and the post-measurement state is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}.$$

A measurement can also be described by a POVM measurement, which is a set of positive operators $\{E_m\}$ such that $\Sigma_m E_m = I$. Similarly, If the state of the quantum system is $|\psi\rangle$ before the measurement then the probability that result m occurs is $\langle\psi|E_m|\psi\rangle$, and the post-measurement state is

$$\frac{\sqrt{E_m}|\psi\rangle}{\sqrt{\langle\psi|E_m|\psi\rangle}}.$$

To present our results formally, we need an auxiliary definition. The definition gives the probability of unambiguous identification among $|\psi_1\rangle, \ldots, |\psi_n\rangle$ by measurement $\{M_m\}$.

Definition 1. Suppose a quantum system is prepared in one of the n states $|\psi_1\rangle, \ldots, |\psi_n\rangle$ in a k-dimension Hilbert space with probabilities of $p_1, \ldots, p_n$, respectively. The probability of unambiguous identification by measurement $\{M_m\}$ is defined as follows:

$$D(p_1, \ldots, p_n, |\psi_1\rangle, \ldots, |\psi_n\rangle, \{M_m\}) = \sum_{i=1}^{n} \sum_{\substack{M_m|\psi_i\rangle \neq 0 \\ M_m|\psi_j\rangle = 0, \forall j \neq i}} p_i \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle.$$

Intuitively, the ith summand of the right-hand side of the defining equation is the probability with which one can assert with certainty that the system is prepared in state $|\psi_i\rangle$. Thus the total summation of the right-hand side is the success probability of unambiguous discrimination among states $|\psi_1\rangle, \ldots, |\psi_n\rangle$.

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For simplicity, we often write $D(\{M_m\})$ instead of $D(p_1,\dots,p_n,|\psi_1\rangle,\dots,|\psi_n\rangle,\{M_m\})$ if no confusion is caused.

One of our main results is the following theorem, which gives an upper bound for $D(\{M_m\})$.

Theorem 1. For any measurement $\{M_m\}$, we have

$$D_m(\{M_m\})\leqslant 1-\frac{1}{n-1}\sum_{i\neq j}\sqrt{p_ip_j}|\langle\psi_i|\psi_j\rangle|.$$

Proof. It follows from Definition 1 that

$$\begin{aligned}D_m(\{M_m\})&=\sum_{i=1}^{n}\sum_{\substack{M_m|\psi_i\rangle\neq 0\\ M_m|\psi_j\rangle=0,\forall j\neq i}} p_i\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle\\
&=\sum_{i=1}^{n}p_i\langle\psi_i|\sum_{\substack{M_m|\psi_i\rangle\neq 0\\ M_m|\psi_j\rangle=0,\forall j\neq i}} M_m^\dagger M_m|\psi_i\rangle\\
&=\sum_{i=1}^{n}p_i\Bigg(\langle\psi_i|\sum_{\substack{M_m|\psi_i\rangle\neq 0\\ M_m|\psi_j\rangle=0,\forall j\neq i}} M_m^\dagger M_m|\psi_i\rangle\\
&\quad+\langle\psi_i|\sum_{i'\neq i}\sum_{\substack{M_m|\psi_{i'}\rangle\neq 0\\ M_m|\psi_j\rangle=0,\forall j\neq i'}} M_m^\dagger M_m|\psi_i\rangle\Bigg)\\
&=\sum_{i=1}^{n}p_i\langle\psi_i|E^{(n-1)}|\psi_i\rangle\\
&=\sum_{i=1}^{n}p_i\langle\psi_i|I-E^{(n)}-E|\psi_i\rangle,\end{aligned}$$

where we write

$$E^{(k)}=\sum\{M_m^\dagger M_m:\text{the number of }|\psi_j\rangle\text{'s with }M_m|\psi_j\rangle=0\text{ is exactly }k\},$$

and $E=\Sigma_{k=0}^{n-2}E^{(k)}$.

Intuitively, $E^{(k)}$ denotes the summation of $M_m^\dagger M_m$ that there are exactly k $|\psi_j\rangle$'s such that $M_m|\psi_j\rangle=0$.

Then

$$\begin{aligned}D_m(\{M_m\})&=\sum_{i=1}^{n}p_i\langle\psi_i|I-E^{(n)}-E|\psi_i\rangle\\
&=1-\sum_{i=1}^{n}p_i\langle\psi_i|E|\psi_i\rangle\\
&=1-\frac{1}{n-1}\sum_{i\neq j}\frac{1}{2}(p_i\langle\psi_i|E|\psi_i\rangle+p_j\langle\psi_j|E|\psi_j\rangle).\end{aligned}$$

Note that all $E^{(k)}$'s and E are positive. Hence

$$\begin{aligned}p_i\langle\psi_i|E|\psi_i\rangle+p_j\langle\psi_j|E|\psi_j\rangle&=p_i|\sqrt{E}|\psi_i\rangle|^2+p_j|\sqrt{E}|\psi_j\rangle|^2\\
&\geqslant 2\sqrt{p_ip_j}|\sqrt{E}|\psi_i\rangle||\sqrt{E}|\psi_j\rangle|\\
&\geqslant 2\sqrt{p_ip_j}|\langle\psi_i|\sqrt{E}\sqrt{E}|\psi_j\rangle|\\
&=2\sqrt{p_ip_j}|\langle\psi_i|E|\psi_j\rangle|.\end{aligned}$$

The latter equality is derived from Cauchy-Schwartz inequality. Now

$$\begin{aligned}D_m(\{M_m\})&=1-\frac{1}{n-1}\sum_{i\neq j}\frac{1}{2}(p_i\langle\psi_i|E|\psi_i\rangle+p_j\langle\psi_j|E|\psi_j\rangle)\\
&\leqslant 1-\frac{1}{n-1}\sum_{i\neq j}\sqrt{p_ip_j}|\langle\psi_i|E|\psi_j\rangle|\\
&=1-\frac{1}{n-1}\sum_{i\neq j}\sqrt{p_ip_j}|\langle\psi_i|I-E^{(n)}\\
&\quad-E^{(n-1)}|\psi_j\rangle|\\
&=1-\frac{1}{n-1}\sum_{i\neq j}\sqrt{p_ip_j}|\langle\psi_i|I|\psi_j\rangle-\langle\psi_i|E^{(n)}|\psi_j\rangle\\
&\quad-\langle\psi_i|E^{(n-1)}|\psi_j\rangle|\\
&=1-\frac{1}{n-1}\sum_{i\neq j}\sqrt{p_ip_j}|\langle\psi_i|\psi_j\rangle|.\end{aligned}$$

Note that in the above proof, we make no assumption of the dimensionality of the prepared states. So the result can be applied to systems of any number of qubits.

In the case of discrimination of two states, i.e., $n=2$, the above bound is exactly the extended IDP limit in Ref. [5]. This shows that the IDP limit reserves in the looser requirement. In other words, by general measurements one cannot distinguish two states better than extended IDP limit. Note that the result holds for any dimension of the prepared states, therefore, the method of adding quantum systems, as assumed in [2–5], cannot get higher value than the above upper bound.

Another concern about the question is to consider the discrimination by more than once. Because any sequence of measurements can be formulated as a single generalized measurement, to which Definition 1 and Theorem 1 apply, one cannot achieve a better result by performing measurements more than once.

In conclusion, we examine the general distinction of n quantum states, and give an upper bound for success probability of unambiguous discrimination. The bound coincides with the known extended IDP limit when two states are concerned. Many times of measurements cannot ameliorate the result.

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