Full characterization of quantum correlated equilibria

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Abstract. Quantum game theory aims to study interactions of people (or other agents) using quantum devices with possibly conflicting interests. Recently Zhang studied some quantitative questions in general quantum strategic games of growing sizes [Zha12]. However, a fundamental question not addressed there is the characterization of quantum correlated equilibria (QCE). In this paper, we answer this question by giving a sufficient and necessary condition for an arbitrary state ρ being a QCE. In addition, when the condition fails to hold for some player i, we give an explicit positive-operator valued measurement (POVM) for that player to achieve a strictly positive gain of payoff. Finally, we give some upper bounds for the maximum gain by playing quantum strategies over classical ones, and the bounds are tight for some games.

1 Introduction

Game theory studies the interaction of different players with possibly conflicting goals [OR94,FT91,VNRT07]. Equilibrium is a central solution concept which characterizes the situation in which no player likes to deviate from the current strategy provided that all other players do not change theirs. In strategic games, or games in strategic forms, each player i has a set S_i of strategies, and when playing the game, all k players choose their strategies at the same time. They then get payoffs according to their payoff functions which, in general, depend on all the players' strategies. If each player i chooses her strategy s_i from a distribution p_i (on her own strategy space), then the joint distribution $p = p_1 \times \cdots \times p_k$ is a (mixed) Nash equilibrium if no player i can increase her average payoff by changing her distribution p_i to any other p'_i. A fundamental theorem by Nash says that any game with a finite set of strategies has at least one Nash equilibrium [Nas51].

Aumann [Aum74] gave an important generalization of Nash equilibrium, called correlated equilibrium (CE), where a Referee selects a joint strategy $s = (s_1, \ldots, s_k)$ from some distribution p and suggests s_i to the i-th player. The joint distribution p is a correlated equilibrium (CE) if no player i, when sees only her part s_i, can improve her expected payoff by deviating from this suggested strategy.

The notion of correlated equilibria captures the optimal solution in natural games such as the Traffic Light and the Battle of the Sexes ([VNRT07], Chapter 1). Let us review the first one for illustration. Suppose that two cars, one heading east and the other heading north, drive to an intersection at the same time. Both cars have choices of crossing and stopping. If both choose to cross, then an accident would happen, in which case both players suffer a lot. If exactly one car chooses to cross, then it does not need to wait and thus gets payoff 1, and the other car stops and waits, having payoff 0. If both cars stop then both have payoff 0. The payoff is summarized by the following payoff bimatrix, where in each entry, the first number is the payoff for Player 1 and the second is for Player 2.

	Cross	Stop
Cross	(-100,-100)	(1,0)
Stop	(0,1)	(0,0)

There are two pure Nash equilibria in this game, namely (Cross,Stop) and (Stop,Cross). But neither of them is fair, since it clearly prefers one car to the other. Therefore, different cars have different preferences over these two Nash equilibria. In the language of games, it is the issue of which equilibrium the players should agree on. There is actually a third Nash equilibrium, a mixed one: Each car crosses with probability 1/101. This solves the fairness issue, but loses the efficiency: The total expected payoff is very small (0) because most likely both cars would stop. Even worse, there is a positive probability of

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car crash. The issue in the real world is easily solved by introducing a traffic light, from which each car gets a signal. Each signal can be viewed as a random variable uniformly distributed on {red, green}. But the two random signals/variables are designed to be perfectly correlated that if one is red, then the other is green. This is actually a correlated equilibrium, i.e. a distribution over {Cross,Stop}×{Cross,Stop} with half probability on (Cross,Stop) and half on (Stop,Cross). It is easy to verify that it simultaneously achieves high payoff, fairness, and zero-probability of car accident.

The set of CE also has good mathematical properties. For example, the set of CE is convex, with Nash equilibria being some of the vertices of the polytope. Computationally, we can find the best CE (of any game with constant number of players), measured by a weighted summation of individual payoffs, simply by solving a linear program. This is in contrast with the fact that finding one Nash equilibrium is PPAD-hard [DGP09,CDT09]. Other nice properties include that a natural learning dynamics lead to an approximate variant of CE ([VNRT07], Chapter 4), and all CE in a graphical game with n players and $\log(n)$ maximum degree can be found in polynomial time ([VNRT07], Chapter 7).

In the quantum world, quantum game theory focuses on the study of the interaction of people using quantum computers with conflicting interests. Indeed, quantization of classical strategic games has drawn much attention in the past decade. Despite the rapid accumulation of literature on quantum games [EWL99,BH01b,LJ03,FA03,FA05,DLX$^+$02a,DLX$^+$02b,PSWZ07], the whole picture of the area is not as clean as one desires, partially due to controversy in models [BH01a,vEP02,CT06] and partially due to lack of studies for general games. Recently, Zhang initialized systematic studies in a simpler yet rich, arguably more natural quantization of classical games in strategic form [Zha12][3]; also see that paper for a review of the existing literature under the name of "quantum games". Other than the model, what mainly distinguishes that work from previous ones is the generality of the classical game it studies: Unlike previous work focusing on specific games of small sizes or refereed extensive games, the paper studies general strategic games of growing sizes. In addition, rather than aiming at qualitative questions such as whether playing quantum strategies has any advantage as in previous work, [Zha12] studies quantitative questions such as how much quantum advantage a general game can have. The work triggered some follow-up studies [KZ12,ZWC$^+$12,JSWZ13] on efficiency of quantum games in various settings and measures.

Solution concepts such as Nash equilibrium and correlated equilibrium are naturally extended to the quantum model. While quantum Nash equilibria can be easily characterized as all product states whose probability distribution by the measurement in the computational basis is a classical Nash equilibrium, quantum correlated equilibria seem to be much more elusive. Given the importance of correlated equilibria in game theory and computer science, it is desirable to understand quantum correlated equilibria well, and indeed a correspondence between classical and quantum correlation equilibria was established in [Zha12], which also answers the questions such as the hardness of finding a quantum Nash or correlated equilibrium. However, one fundamental question that [Zha12] did not address is the following: For an arbitrary (classical) strategic game, can we characterize all the quantum correlated equilibria (QCE) in the quantum game?

In this paper, we answer this question by giving the following sufficient and necessary condition for any given game and any state ρ. Recall that in a classical game with k players, each player t has a set S_t of strategies. This space of strategies extends to $H_t = \mathbb{C}^{S_t}$ in the quantized game, and each player can apply an arbitrary completely positive and trace preserving (CPTP) map in her space. We use notation S_{-t} to denote the set $\prod_{i\neq t} S_i$.

Theorem 1 *A quantum state ρ in space $H = \bigotimes_{i=1}^k H_i$ is a QCE if and only if for each player t, when we write $\rho = [\rho^{i_1 i_2}_{j_1 j_2}]_{i_1 j_1, i_2 j_2}$, where $i_1, i_2 \in [m] \stackrel{\mathsf{def}}{=} \{1, 2, ..., m\}$ and $j_1, j_2 \in [n]$ with $m = |S_t|$ and $n = |S_{-t}|$, we have*

$$B_i \stackrel{\mathsf{def}}{=} \Big[\sum_{j=1}^{n} \rho^{i_1 i_2}_{jj}(a_{ij} - a_{i_1 j})\Big]_{i_1 i_2} \preceq 0, \quad \forall i \in [m]. \tag{1}$$

Here a_{ij} is the payoff of Player t when she takes strategy i and the others take the joint strategy indexed by j. (Note that the matrices B_i's are different for different players.)

[3] The author later learned that a similar model was also considered in [Mey04] by Meyer, though only *qualitative* studies are conducted and only on *pure* states are studied there, whereas [Zha12] emphasizes quantitative studies on general mixed states.

Let us compare this with the definition of QCE. For a given strategic quantum game, a quantum state ρ is a QCE if and only if no player could obtain a positive gain by any local quantum operation Φ. Compared to this definition, the above characterization eliminates the quantifier $\forall\Phi$, and makes checking the QCE property as easy as checking positivity of some matrices.

The proof of the theorem is based on semi-definite programming (SDP) duality, which makes the proof desirably simple. However, using the duality also fails to provide a good intuition for the above characterization. To give more intuitions for the condition, and also to provide a constructive way for a player to gain if the above condition is not satisfied, we give an explicit operation for the player to get a strictly positive payoff. This can serve as an operational explanation for the necessity of the condition. This part is more technical, and consists of two steps. First, we show that a weaker condition

$$\sum_j \rho_{jj}^{i_1 i_2} a_{i_1 j} = \sum_j \rho_{jj}^{i_1 i_2} a_{i_2 j}, \quad \forall i_1, i_2 \in [m], \tag{2}$$

is needed for ρ to be a QCE, and if this condition is violated, then we give an explicit *unitary* operation (followed by the measurement in the computational basis) by which the player has a positive gain. In the next step, we show that if Eq. (2) holds, but the condition in the characterization does not hold, then we can use further spectral properties provided by Eq. (2) to construct an explicit positive-operator valued measurement(POVM) by which the player has a positive gain.

Finally, if a quantum state ρ is not a QCE, we also give two upper bounds on the maximum gain a player could achieve. There results are based on analyzing the structure of B_i's. We also provide an example to show that there upper bounds could be (almost) tight.

The paper is organized as follows. Some preliminary notions are introduced in Section 2. In Section 3 we give the sufficient and necessary condition, and in Section 4 the necessity part is reproved constructively, which can be regarded as the operational explanation of this condition. In Section 5, we obtain some upper bounds of the gain when ρ is not a QCE. Some open problems are listed in Section 6.

2 Definitions and notation

Matrix theory A matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if $A^\dagger = A$, or equivalently, A has a spectral decomposition and all eigenvalues are real numbers. A matrix $A \in \mathbb{C}^{n\times n}$ is positive (semi-definite), written as $A \succeq 0$, if $\langle\psi|A|\psi\rangle \geq 0$ for all column vectors $|\psi\rangle$. Equivalently, $A \succeq 0$ if and only if A has a spectral decomposition and all eigenvalues are nonnegative numbers. Thus all positive matrices are Hermitians. Define $A \preceq 0$ if $-A \succeq 0$.

Quantum computing A pure quantum state in a vector space H is a unit vector in ℓ_2 norm, usually denoted by the ket notation $|\cdot\rangle$. A mixed quantum state is represented by a density matrix, *i.e.* a positive matrix ρ whose trace is 1. An linear operator Φ acting on a vector space of matrices is completely positive if $\Phi \otimes I_n$ is positive for all n. A quantum admissible operation is a completely positive linear operator that preserves the trace, usually shortened as a CPTP (completely positive and trace preserving) map. A particular class of quantum operations are measurements. A general measurement, called POVM measurement, is a collection of positive operators $\{E_i\}$ satisfying that $E_i \succeq 0$ and $\sum_i E_i = I$. When we use this measurement on a mixed state ρ, a label i will be observed with probability $p_i = \mathtt{tr}(E_i\rho)$; the two above conditions just imply that the probability $p_i \geq 0$ and $\sum_i p_i = 1$.

2.1 Classical strategic games

Suppose that in a classical game there are k players. Each player i has a set S_i of strategies. To play the game, each player i selects a strategy s_i from S_i. We use $s = (s_1, \ldots, s_k)$ to denote the *joint strategy* selected by the players and $S = S_1 \times \ldots \times S_k$ to denote the set of all possible joint strategies. Each player i has a utility function $u_i : S \to \mathbb{R}$, specifying the *payoff* or *utility* $u_i(s)$ to player i on the joint strategy s. We use subscript $-i$ to denote the set $[k] - \{i\}$, so s_{-i} is $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k)$.

Definition 1 *A* pure Nash equilibrium *is a joint strategy* $s = (s_1, \ldots, s_k) \in S$ *satisfying*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \qquad \forall i \in [k], \forall s_i' \in S_i.$$

A (mixed) Nash equilibrium (NE) *is a product probability distribution* $p = p_1 \times \ldots \times p_k$*, where each* p_i *is a probability distributions over* S_i*, satisfying*

$$\sum_{s_{-i}} p_{-i}(s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} p_{-i}(s_{-i})u_i(s'_i, s_{-i}), \quad \forall i \in [k],\ \forall s_i, s'_i \in S_i \text{ with } p_i(s_i) > 0.$$

There are various extensions of (mixed) Nash equilibria. Aumann [Aum74] introduced a relaxation called *correlated equilibrium*. This notion assumes an external party, called Referee, to draw a joint strategy $s = (s_1, ..., s_k)$ from some probability distribution p over S, possibly correlated in an arbitrary way, and to suggest s_i to Player i. Note that Player i only sees s_i, thus the rest strategy s_{-i} is a random variable over S_{-i} distributed according to the conditional distribution $p|_{s_i}$, the distribution p conditioned on the i-th part being s_i. Now p is a correlated equilibrium if any Player i, upon receiving a suggested strategy s_i, has no incentive to change her strategy to a different $s'_i \in S_i$, assuming that all other players stick to their received suggestion s_{-i}.

Definition 2 *A* correlated equilibrium (CE) *is a probability distribution* p *over* S *satisfying*

$$\sum_{s_{-i}} p(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} p(s_i, s_{-i})u_i(s'_i, s_{-i}), \qquad \forall i \in [k],\ \forall s_i, s'_i \in S_i.$$

Notice that a correlated equilibrium p is an Nash equilibrium if p is a product distribution.

2.2 Quantum strategic games

In this paper we consider quantum games which allow the players to use strategies quantum mechanically. We assume the basic background of quantum computing; see [NC00] and [Wat08] for comprehensive introductions. The set of admissible super operators, or equivalently the set of CPTP maps, of density matrices in Hilbert spaces H_A to H_B, is denoted by $\mathsf{CPTP}(H_A, H_B)$. We write $\mathsf{CPTP}(H)$ for $\mathsf{CPTP}(H, H)$.

For a strategic game being played quantumly, each player i has a Hilbert space $H_i = span\{s_i : s_i \in S_i\}$, and a joint strategy can be any quantum state ρ in $H = \otimes_i H_i$. The players are supposed to measure the state ρ in the computational basis, giving a distribution over the set S of classical joint strategies, and yielding a payoff for each player. Therefore the (expected) payoff for player i on joint strategy ρ is

$$u_i(\rho) = \sum_s \langle s|\rho|s\rangle u_i(s). \tag{3}$$

Please refer to [Zha12] for more explanations of the model.

Corresponding to changing strategies in a classical game, now each player i can apply an arbitrary CPTP operation on H_i. So the natural requirement for a state being a quantum Nash equilibrium is that each player cannot gain by applying any admissible operation on her strategy space. The concepts of quantum Nash equilibrium, and quantum correlated equilibrium, and quantum approximate equilibrium are defined in the following, where we overload the notation by writing Φ_i for $\Phi_i \otimes I_{-i}$ if no confusion is caused.

Definition 3 *A* quantum Nash equilibrium (QNE) *is a quantum strategy* $\rho = \rho_1 \otimes \cdots \otimes \rho_k$ *for some mixed states* ρ_i*'s on* H_i*'s satisfying*

$$u_i(\rho) \geq u_i(\Phi_i(\rho)), \qquad \forall i \in [k],\ \forall \Phi_i \in \mathsf{CPTP}(H_i).$$

Note that in the above definition, only produce states are allowed. One can also consider general quantum states, leading to the following notion.

Definition 4 *A* quantum correlated equilibrium (QCE) *is a quantum strategy* ρ *in* H *satisfying*

$$u_i(\rho) \geq u_i(\Phi_i(\rho)), \qquad \forall i \in [k],\ \forall \Phi_i \in \mathsf{CPTP}(H_i).$$

An ϵ-approximate quantum correlated equilibrium (ϵ-QCE) *is a quantum strategy* ρ *in* H *satisfying that*

$$u_i(\Phi_i(\rho)) \leq u_i(\rho) + \epsilon, \qquad \forall i \in [k], \forall \Phi_i \in \mathsf{CPTP}(H_i).$$

When we later characterize quantum correlated equilibrium, we will need that no player can increase her payoff, so a condition is required for each player. For easy presentation, we fix an arbitrary player, say, Player t, and consider the possible increase of her payoff by local operations. Write the state as

$$\rho = \left[\rho^{i_1 i_2}_{j_1 j_2}\right]_{i_1 j_1, i_2 j_2},$$

where $i_1, i_2 \in H_t$ and $j_1, j_2 \in H_{-t}$. Suppose that the dimensions of H_t and H_{-t} are m and n, respectively.

3 Characterization of quantum correlated equilibrium

We will first give an explicit expression of Player t's gain of payoff by applying a POVM measurement $\{E_i\}$ (compared to the measurement in the computational basis). Recall that S_t is the set of strategies of Player t, and $S_{-t} = S_1 \times \cdots \times S_{t-1} \times S_{t-1} \times \cdots \times S_k$ is the set of joint strategies of other players. Under this notation, denote by $\{a_{ij}\}$ the payoff of Player t when she takes strategy i and the others take the joint strategy indexed by j.

Lemma 2 *Suppose that Player t uses a POVM measurement $E = \{E_i : i \in S_t\}$ and other players use $M = \{|j\rangle\langle j| : j \in S_{-t}\}$ to measure their parts in the computational basis. Then the gain of Player t's payoff by applying E than measuring in the computational basis $\{|i\rangle\langle i| : i \in S_t\}$ is*

$$\textit{Gain} \stackrel{\mathsf{def}}{=} u_1\big((E \otimes M)\rho\big) - u_1(\rho) = \sum_i tr(E_i B_i), \qquad \textit{where } B_i = \Big[\sum_{j \in S_{-t}} \rho^{i_1 i_2}_{jj}(a_{ij} - a_{i_1 j})\Big]_{i_1 i_2}.$$

Proof. The probability of choosing strategies (i, j) is $tr((E_i \otimes |j\rangle\langle j|)\rho) = \sum_{i_1, i_2} E_i(i_1, i_2)^* \rho^{i_1 i_2}_{jj}$. Note that

$$\sum_{ij} \sum_{i_1 i_2} E_i(i_1, i_2)^* \rho^{i_1 i_2}_{jj} a_{i_1 j} = \sum_{i_1 i_2 j} \big(\sum_i E_i(i_1, i_2)^*\big) \rho^{i_1 i_2}_{jj} a_{i_1 j} = \sum_{i_1 j} \rho^{i_1 i_1}_{jj} a_{i_1 j} = \sum_{ij} \rho^{ii}_{jj} a_{ij} \tag{4}$$

where the second equality is because $\{E_i\}$, as a POVM measurement, satisfies $\sum_i E_i(i_1, i_2) = \delta_{i_1, i_2}$. Therefore,

$$\begin{aligned}
\textit{Gain} &= \sum_{ij} \Big(\sum_{i_1, i_2} E_i(i_1, i_2)^* \rho^{i_1 i_2}_{jj} - \rho^{ii}_{jj}\Big) a_{ij} && (5)\\
&= \sum_{ij} \sum_{i_1, i_2} E_i(i_1, i_2)^* \rho^{i_1 i_2}_{jj} a_{ij} - \sum_{ij} \rho^{ii}_{jj} a_{ij} && (6)\\
&= \sum_{ij} \sum_{i_1, i_2} E_i(i_1, i_2)^* \rho^{i_1 i_2}_{jj} a_{ij} - \sum_{ij} \sum_{i_1, i_2} E_i(i_1, i_2)^* \rho^{i_1 i_2}_{jj} a_{i_1 j} && (7)\\
&= \sum_{i} \sum_{i_1, i_2} E_i(i_1, i_2)^* \sum_j \rho^{i_1 i_2}_{jj} (a_{ij} - a_{i_1 j}) && (8)\\
&= \sum_i tr(E_i^\dagger B_i) = \sum_i tr(E_i B_i), && (9)
\end{aligned}$$

where we have utilized Eq.(4) and the definition of B_i.

The above lemma immediately gives a sufficient condition for a state ρ being a QCE.

Theorem 3 *If for each player, the corresponding $B_i \preceq 0$ holds for all $i \in [m]$, then ρ is a QCE.*

Proof. By the above lemma, the gain $\sum_i tr(E_i B_i) \leq 0$ because each $B_i \preceq 0$ and each $E_i \succeq 0$. Since this holds for all possible POVM measurement $\{E_i\}$, ρ is a QCE by definition.

Next we will use SDP duality to show that the condition is also necessary.

Theorem 4 *Suppose that ρ is a QCE. Then for each player t, if we write $\rho = [\rho^{i_1 i_2}_{j_1 j_2}]_{i_1 j_1, i_2 j_2}$, where $i_1, i_2 \in [m]$ and $j_1, j_2 \in [n]$ with $m = dim(H_t)$ and $n = dim(H_{-t})$, then we have*

$$B_i \stackrel{\mathsf{def}}{=} \Big[\sum_{j=1}^n \rho^{i_1 i_2}_{jj} (a_{ij} - a_{i_1 j})\Big]_{i_1 i_2} \preceq 0, \quad \forall i \in [m]. \tag{10}$$

Proof. Since ρ is a QCE, Player t cannot increase her payoff by applying any POVM measurement. Therefore, the value of the following maximization problem

$$\begin{aligned}\max \quad & \sum_i tr(E_i B_i) \\ s.t. \quad & E_i \succeq 0, \quad \forall i \in [m], \\ & \sum_{i=1}^m E_i = I_m\end{aligned}$$

is equal to 0. This maximization problem is a Semidefinite program (SDP), and it has the following dual SDP problem. (See [Hel02,VB96] for more details of SDP.)

Dual:

$$\begin{aligned}\min \quad & tr(Y) \\ s.t. \quad & Y \succeq B_i, \ \forall i \in [m]\end{aligned}$$

According to the strong duality theorem, these two SDP problems have the same value. Suppose Y is an optimal solution of the dual, then we have $tr(Y) = 0$. Note that each $Y(i,i) \geq B_i(i,i) = 0$, so $tr(Y) = 0$ implies that all $Y(i,i) = 0$, and thus $(Y - B_i)(i,i) = 0$. But note that as a feasible solution, Y satisfies $Y - B_i \succeq 0$, so actually the entries of the entire i-th row of $Y - B_i$ are all 0's. Since the i-th row of B_i is also 0 by its definition, the i-th row of Y is that of $Y - B_i$ plus that of B_i, which is equal to 0. Applying this argument to all i, we reach the conclusion that the entire $Y = 0$, giving the claimed relation $B_i \preceq 0$.

Since a negative matrix B_i is a Hermitian, and $\rho_{jj}^{i_1 i_2} = (\rho_{jj}^{i_2 i_1})^*$ (as $\rho \succeq 0$ is a Hermitian), we have an immediate corollary as follows.

Corollary 5 *If ρ is a QCE, then $\forall i_1, i_2 \in [m]$,*

$$\sum_j \rho_{jj}^{i_1 i_2} a_{i_1 j} = \sum_j \rho_{jj}^{i_1 i_2} a_{i_2 j}. \tag{11}$$

Both necessary conditions in Theorem 4 and the above corollary are not constructive in the sense that if ρ is not a QCE, they do not provide an explicit POVM measurement to realize a strictly positive gain of payoff. We will resolve this issue in the next section.

4 A constructive proof of the characterization

In the last section, we give two necessary conditions Eq. (10) and Eq. (11), the first of which is also sufficient (while the second is not by itself). In this section, we will give explicit local operations to increase the payoff if these conditions are not satisfied. We will first study in Section 4.1 the violation of Eq. (11), in which case a local *unitary* operation is explicitly given to achieve a positive gain. Based on this result, we will then consider in Section 4.2 the general scenario of Eq. (10) being violated, in which case we will exhibit an explicit POVM measurement with a positive gain for the player.

4.1 Eq. (11) violated: gain by an explicit local unitary

Lemma 6 *If $\sum_j \rho_{jj}^{i_1 i_2} a_{i_1 j} \neq \sum_j \rho_{jj}^{i_1 i_2} a_{i_2 j}$ for some $i_1, i_2 \in [m]$, then there exists an explicit unitary only on $span\{|i_1\rangle, |i_2\rangle\}$ to make an increase of payoff for the player.*

Proof. Consider the unitary operator U defined by

$$\begin{aligned} U|i_1\rangle &= u_{11}|i_1\rangle + u_{12}|i_2\rangle, \\ U|i_2\rangle &= u_{21}|i_1\rangle + u_{22}|i_2\rangle. \end{aligned}$$

The new probability distribution of strategy after the operation of $U^\dagger$ on $span\{|i_1\rangle, |i_2\rangle\}$ and identity on other i's is

$$p_{ij} = Tr((U|i\rangle\langle i|U^\dagger \otimes |j\rangle\langle j|)\rho). \tag{12}$$

where we overload the notation by writing U for $U \otimes I_{[m]-\{i_1,i_2\}}$. Note that when $i \in [m]-\{i_1,i_2\}$, $p_{ij} = Tr(|i\rangle\langle i| \otimes |j\rangle\langle j|)\rho) = \rho_{jj}^{ii}$. Thus, the gain of Player 1 by the operation Ψ is

$$\begin{aligned}
Gain &= \sum_{ij} \Big(p_{ij} - \rho_{jj}^{ii}\Big) a_{ij} \\
&= \sum_{j} \Big(p_{i_1 j} - \rho_{jj}^{i_1 i_1}\Big) a_{i_1 j} + \sum_{j} \Big(p_{i_2 j} - \rho_{jj}^{i_2 i_2}\Big) a_{i_2 j} \\
&= \sum_{j} \Big(\sum_{a,b=1}^{2} \rho_{jj}^{i_a i_b} u_{1,i_a}^* u_{1,i_b} - \rho_{jj}^{i_1 i_1}\Big) a_{i_1 j} + \sum_{j} \Big(\sum_{a,b=1}^{2} \rho_{jj}^{i_a i_b} u_{2,i_a}^* u_{2,i_b} - \rho_{jj}^{i_2 i_2}\Big) a_{i_2 j} \\
&= \sum_{j} \Big(|u_{11}|^2 \rho_{jj}^{i_1 i_1} + u_{11}^* u_{12} \rho_{jj}^{i_1 i_2} + u_{11} u_{12}^* \rho_{jj}^{i_2 i_1} + |u_{12}|^2 \rho_{jj}^{i_2 i_2} - \rho_{jj}^{i_1 i_1}\Big) a_{i_1 j} \\
&\quad + \sum_{j} \Big((|u_{21}|^2 \rho_{jj}^{i_1 i_1} + u_{21}^* u_{22} \rho_{jj}^{i_1 i_2} + u_{21} u_{22}^* \rho_{jj}^{i_2 i_1} + |u_{22}|^2 \rho_{jj}^{i_2 i_2} - \rho_{jj}^{i_2 i_2}\Big) a_{i_2 j}.
\end{aligned}$$

Since U is a unitary operation, we have

$$u_{11}^* u_{12} + u_{21}^* u_{22} = 0, \quad u_{11} u_{12}^* + u_{21} u_{22}^* = 0, \tag{13}$$

and

$$|u_{11}|^2 + |u_{21}|^2 = 1, \quad |u_{12}|^2 + |u_{22}|^2 = 1. \tag{14}$$

Thus, we obtain

$$\begin{aligned}
Gain = u_{11}^* u_{12} \sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_1 i_2} + u_{11} u_{12}^* \sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_2 i_1} \\
-|u_{12}|^2 \sum_j \Big(a_{i_2 j} - a_{i_1 j}\Big) \rho_{jj}^{i_2 i_2} - |u_{21}|^2 \sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_1 i_1},
\end{aligned}$$

Since $\sum_j (a_{i_1 j} - a_{i_2 j}) \rho_{jj}^{i_1 i_2} \neq 0$, we have $\sum_j (a_{i_1 j} - a_{i_2 j}) \rho_{jj}^{i_2 i_1} = \sum_j (a_{i_1 j} - a_{i_2 j}) (\rho_{jj}^{i_1 i_2})^* \neq 0$ as well. Define a positive real number c by

$$c = \frac{\max\Big\{\sum_j \Big(a_{i_2 j} - a_{i_1 j}\Big) \rho_{jj}^{i_2 i_2}, \sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_1 i_1}\Big\}}{\Big|\sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_1 i_2}\Big|}.$$

which is just to make

$$\sum_j \Big(a_{i_2 j} - a_{i_1 j}\Big) \rho_{jj}^{i_2 i_2} < c \Big|\sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_1 i_2}\Big|,$$

and

$$\sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_1 i_1} < c \Big|\sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_1 i_2}\Big|.$$

Now one could choose $u_{11} = \sqrt{1-x}$, and $u_{12} = e^{ir}\sqrt{x}$, where x is a positive real number, and r is a proper real number such that $u_{11}^* u_{12} \sum_j (a_{i_1 j} - a_{i_2 j}) \rho_{jj}^{i_1 i_2}$ is also a positive real number. It can be checked that if $0 < x < \frac{1}{c^2+1}$, we have

$$\frac{u_{11}}{|u_{12}|} = \sqrt{\frac{1-x}{x}} > c,$$

Finally, note that $|u_{12}|^2 = |u_{21}|^2$. We then have

$$\begin{aligned}
Gain &= 2|u_{11}| \cdot |u_{12}| \Big|\sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_1 i_2}\Big| \\
&\quad -|u_{12}|^2 \sum_j \Big(a_{i_2 j} - a_{i_1 j}\Big) \rho_{jj}^{i_2 i_2} - |u_{21}|^2 \sum_j \Big(a_{i_1 j} - a_{i_2 j}\Big) \rho_{jj}^{i_1 i_1} \\
&> 0.
\end{aligned}$$

4.2 Eq. (10) violated: gain by an explicit POVM measurement

In the rest of this section, we assume that condition Eq. (11) holds, because otherwise there exists an explicit local unitary operation to increase the payoff.

First note that under this assumption, all matrices $B_i = \left[\sum_j \rho_{jj}^{i_1 i_2}(a_{ij} - a_{i_1 j})\right]_{i_1 i_2}$ are Hermitians. Indeed, we have

$$
\begin{aligned}
B_i(i_2, i_1)^* &= \sum_j (\rho_{jj}^{i_2 i_1})^* (a_{ij} - a_{i_2 j}) && \text{(because } a_{ij}, a_{i_2 j} \in \mathbb{R}\text{)} \qquad (15)\\
&= \sum_j \rho_{jj}^{i_1 i_2}(a_{ij} - a_{i_2 j}) && \text{(because } \rho \text{ is Hermitian)} \qquad (16)\\
&= \sum_j \rho_{jj}^{i_1 i_2}(a_{ij} - a_{i_1 j}) && \text{(by Eq. (11))} \qquad (17)
\end{aligned}
$$

Therefore all B_i's have spectral decompositions.

Now suppose that $B_i \preceq 0$ is not true for some $i \in [m]$. Without loss of generality, assume that $i = 1$, namely B_1 has a positive eigenvalue. We denote it by λ, and suppose the corresponding eigenvector (with unit ℓ_2 norm) is $|\psi\rangle$. Note that the first row of B_1 contains all 0's, and since we assumed Eq. (11), so is the first column. This allows us to write $|\psi\rangle$ as $|\psi\rangle = (0, v_2, v_3, ..., v_m)^T$. Since at least one $v_i \neq 0$, for the notational convenience, we assume $|v_2| \neq 0$ without loss of generality.

In the following, we will construct a local POVM measurement $\{E_i\}$ by which Player 1 can strictly increase her payoff, which will complete the proof. Set $\{E_i\}$ to be

$$
E_1 = \varepsilon|\psi\rangle\langle\psi| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon|v_2|^2 & \varepsilon v_2 v_3^* & \cdots & \varepsilon v_2 v_m^* \\ 0 & \varepsilon v_2^* v_3 & \varepsilon|v_3|^2 & \cdots & \varepsilon v_3 v_m^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \varepsilon v_2^* v_m & \varepsilon v_3^* v_m & \cdots & \varepsilon|v_m|^2 \end{pmatrix}, \qquad (18)
$$

$$
E_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & d_{2,2} & -\varepsilon v_2 v_3^* & \cdots & -\varepsilon v_2 v_m^* \\ 0 & -\varepsilon v_2^* v_3 & d_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\varepsilon v_2^* v_m & 0 & \cdots & d_{2,m} \end{pmatrix}, \qquad (19)
$$

$$
E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & d_{3,3} & -\varepsilon v_3 v_4^* & \cdots & -\varepsilon v_3 v_m^* \\ 0 & 0 & -\varepsilon v_3^* v_4 & d_{3,4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -\varepsilon v_3^* v_m & 0 & \cdots & d_{3,m} \end{pmatrix}, \qquad (20)
$$

$$
\vdots
$$

$$
E_{m-1} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & d_{m-1,m-1} & -\varepsilon v_{m-1} v_m^* \\ 0 & \cdots & 0 & -\varepsilon v_{m-1}^* v_m & d_{m-1,m} \end{pmatrix}, \qquad (21)
$$

$$
E_m = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & d_{m,m} \end{pmatrix}. \qquad (22)
$$

Here, $\varepsilon < 1$ is a small positive number that will be determined later. For any fixed ε, we choose $d_{i,j}$'s as follows. Firstly, note that we have the relationship

$$E_1 + E_2 + ... + E_m = I, \tag{23}$$

by which one can obtain that $d_{2,2} = 1 - \varepsilon|v_2|^2$. Let $d_{2,k} = (\varepsilon|v_2 v_k|)^2/d_{2,2}$, thus $d_{2,2}d_{2,k} = (\varepsilon|v_2 v_k|)^2$. After fixing E_2, $d_{3,3}$ can also be obtained by using $\sum_i E_i = I$. In general, we have

$$d_{i,i} = 1 - \epsilon|v_i|^2 - d_{2,i} - \cdots - d_{i-1,i}, \quad \forall i \geq 3, \tag{24}$$

$$d_{i,k} = \epsilon^2|v_i v_k|^2/d_{i,i}, \quad \forall i \geq 2,\ k \geq i+1 \tag{25}$$

By an induction on i, it is not difficult to see that for any fixed B_i's, for $\epsilon \to 0$, it holds that

$$d_{i,i} = 1 - \epsilon|v_i|^2 - O(\epsilon^2) = 1 - O(\epsilon) > 0, \quad \forall i \geq 2 \tag{26}$$

and

$$d_{i,k} = \epsilon^2|v_i v_k|^2/d_{i,i} = O(\epsilon^2), \quad \forall i \geq 2, \forall k \geq i+1. \tag{27}$$

It can be checked that $E_1 = \varepsilon|\psi\rangle\langle\psi| + |1\rangle\langle 1| \succeq 0$, and every other E_i is also positive because it has nonnegative diagonal entries and is actually diagonally dominant Hermitian for sufficiently small ϵ. Besides, since the way we defined $\{E_i\}$ satisfies $\sum_i E_i = I$, $\{E_i\}$ is a legal POVM measurement.

Next we calculate the gain of the Player by using $\{E_i\}$ as in Lemma 2. Since $\langle\psi|B_1|\psi\rangle = \lambda$, we have

$$Tr(E_1 B_1) = \langle 1|B_1|1\rangle + \epsilon\langle\psi|B_1|\psi\rangle = \epsilon\langle\psi|B_1|\psi\rangle = \epsilon\lambda. \tag{28}$$

For $i = 2, ..., m$, note that the only nonzero off-diagonal entries of E_i are on the i-th row and column, but those entries in B_i are zero; this is the reason why we chose the POVM measurement $\{E_i\}$. So only the diagonal entries of E_i and B_i contribute to $Tr(E_i^T B_i)$, and the contribution is $\sum_{k=i}^m d_{i,k}(\sum_j \rho_{jj}^{kk}(a_{ij} - a_{kj}))$. Therefore,

$$\begin{aligned} Gain = \sum_i Tr(E_i B_i) &= \varepsilon\lambda + \sum_{i=2}^m \sum_{k=i}^m d_{i,k}\Big(\sum_j \rho_{jj}^{kk}(a_{ij} - a_{kj})\Big) && (29) \\ &= \varepsilon\lambda + \sum_{i=2}^m \sum_{k=i+1}^m d_{i,k}\Big(\sum_j \rho_{jj}^{kk}(a_{ij} - a_{kj})\Big) \qquad \text{(because } B_i(i,i) = 0\text{)} && (30) \\ &= \varepsilon\lambda \pm O(\epsilon^2) \qquad \text{(because of Eq. (27))} && (31) \end{aligned}$$

Here note that m and B_i are all fixed and only ϵ approaches to 0. So for sufficiently small ϵ, the gain is strictly positive.

5 Upper bounds for the gain

In the above sections, we have shown how to determine whether a given quantum state is a QCE or not. In this section, we consider those quantum states that are not QCE. According to the definition of QCE, one can find a proper POVM measurement $\{E_i\}$ such that some player can get a strictly positive payoff gain by this operation. A natural question is, how much is the maximal gain? In the following theorems we provide two simple upper bounds as first-step attempts.

Theorem 7 *Suppose that the maximum eigenvalue of* $B_i = \left[\sum_j \rho_{jj}^{i_1 i_2}(a_{ij} - a_{i_1 j})\right]_{i_1 i_2}$ *is* λ_i. *Let* $\lambda = \max_i \lambda_i$, *then we have*

$$Gain \leq m\lambda. \tag{32}$$

Proof. Since $B_i \preceq \lambda_i I$ and each E_i is positive, it holds that

$$Tr(E_i B_i) \leq Tr(E_i(\lambda_i I)) = \lambda_i Tr(E_i).$$

Thus,

$$\begin{aligned}
Gain &= \sum_i Tr(E_i B_i) && (33)\\
&\le \sum_i \lambda_i Tr(E_i) && (34)\\
&\le \lambda \sum_i Tr(E_i) \qquad (Tr(E_i) \ge 0) && (35)\\
&= m\lambda \qquad (\sum_i Tr(E_i) = Tr(\sum_i E_i) = m). && (36)
\end{aligned}$$

Another bound is the following.

Theorem 8 *Suppose that the eigenvalues of* $B_i = \left[\sum_j \rho_{jj}^{i_1 i_2}(a_{ij} - a_{i_1 j})\right]_{i_1 i_2}$ *are* $\lambda_{i1}, ..., \lambda_{im}$. *Then*

$$Gain \le \sum_{ij:\lambda_{ij}>0} \lambda_{ij}. \tag{37}$$

Proof. Suppose the spectral decomposition of B_i is $B_i = \sum_{j\in[m]} \lambda_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|$. Then

$$\begin{aligned}
Gain &= \sum_i Tr(E_i B_i) && (38)\\
&= \sum_i Tr\big(E_i \sum_{j\in[m]} \lambda_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|\big) && (39)\\
&\le \sum_{ij:\lambda_{ij}>0} \lambda_{ij} \langle\psi_{ij}|E_i|\psi_{ij}\rangle && (40)\\
&\le \sum_{ij:\lambda_{ij}>0} \lambda_{ij} \qquad (\langle\psi_{ij}|E_i|\psi_{ij}\rangle \le \langle\psi_{ij}|I|\psi_{ij}\rangle = 1) && (41)
\end{aligned}$$

The above bounds can be pretty tight. By the dual SDP, it is not hard to see that the gain is the following value:

$$\begin{array}{ll} \min & tr(Y) \\ s.t. & Y \succeq B_i, \ \forall i \in [m] \end{array}$$

Consider the following example:

$$u_1 = \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{m\times n}, \quad \rho = \begin{bmatrix} \frac{1}{mn} & & \\ & \ddots & \\ & & \frac{1}{mn} \end{bmatrix}_{mn\times mn}$$

It is not hard to verify that

$$B_1 = diag(0, 1/m, ..., 1/m), \quad B_2 = ... = B_m = diag(-1/m, 0, ..., 0).$$

Therefore, the gain is $tr(B_1) = (m-1)/m$, which matches the second bound, and is also close to the first bound 1.

6 Concluding remarks: complexity of verification and some open problems

The paper gives a complete characterization of the fundamental concept of quantum correlated equilibria (QCE). For states that are not QCE, we give an explicit oeration for a player perform to obtain a strictly positive gain in payoff. From Theorem 1, it is not hard to get a polynomial-time algorithm for testing QCE. That is, for a given game (where the k payoff functions $a_i : S_1 \times ... \times S_k \to \mathbb{R}$ are given as input) and a quantum state ρ (where each entry of ρ as a density matrix is given), one only needs to check whether, for each player t, all the matrices B_i are negative semi-definite. Each B_i is of dimension $|S_t| \times |S_t|$, and each entry is a summation of $|S_{-t}|$ numbers. Since computing the eigenvalues takes only polynomial time, overall the complexity of deciding whether ρ is a QCE is polynomial in the input size.

Some open problems are left for future explorations.

1. In Section 4, we showed that if the condition is not satisfied for Player i, then the player can use a POVM measurement to obtain a strictly positive gain. A natural question is whether the POVM measurement can be replaced by a unitary operation? In general, is the maximum gain always achieved by a unitary operation?
2. Can the condition be simplified if ρ is a pure state?
3. Can we improve the bounds in Section 5?
4. Can we have a nice characterization of ϵ-approximate QCE? Results in Section 5 provide sufficient conditions for ϵ-QCE, where the ϵ is the one of the given upper bounds. But it is desirable to obtain more nontrivial results along this line.

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