# Quantum Computing (Fall 2013) Instructor: Shengyu Zhang.

# Lecture 9 Quantum Information 1: operations and distance

In this lecture, we give some background knowledge which is necessary for the future lectures on quantum information theory.

## Quantum operations

We’ve encountered some quantum operations, including unitary operation and orthogonal measurements. One can of course have other operations, such as adding a quantum system and discarding part of a system. In general, one can use arbitrary sequence of the above operations to an existing system A, such as attaching a system B, applying a unitary U on AB, removing A, and measure B. This is physically clear, but mathematically not friendly to process. Is there a characterization for all these implementable quantum operations? It turns out to be yes, though the answer is slightly complicated. We will mention two most commonly used representations, one is operator-sum, and another is CPTP maps.

Recall that a general (possibly mixed) quantum state in space of dimension is a square matrix with and .

Suppose that a joint system (A,B) is in a quantum state . Then removing system B means to trace out B and get .

Suppose that system A is in state and system B is in state which is not entangled with system A. Adding B to A results in the state .

### operator-sum

For the first one, sometimes also called *Kraus representation*, let’s first examine a closely related operation as an easy start. We’ve learned orthogonal measurements, but we can use adding/removing systems to get more measurements. For example, first add a system, then make an orthogonal measurement, and finally discard some subsystem. What’s the resulting measurement? It turns out that all such measurements can be described by the so-called *POVM measurements*. It is just a collection of operators s.t. . When we use this measurement on a quantum state , we observe outcome with probability and the post-measurement state becomes . The post-measurement state is random; it is with probability , thus overall it is a mixed state . Note that the orthogonal measurement () is a special case because we can just take , and note that and .

The general quantum operation is quite similar. It turns out that for any quantum operation, one can associate a collection of operators in s.t. , and when operated on a state , the system is changed to . Note that the operator preserves the trace:

**Exercise**. What’s the operator-sum representation for the POVM measurement ? (hint: ).

### CPTP maps

The second representation is by completely positive and trace preserving (CPTP) map. Suppose the starting state is and an operation is , then the ending state is . Mathematically, maps linear operators (like ) to linear operators (like ). is physically implementable if and only if is a CPTP map.

Now we explain the term precisely. The “TP” part is easy to describe and understand: it requires the map to preserve trace. Thus, it maps a quantum state , which has trace 1, to another quantum state which should also have trace 1.

The “CP” part is a bit more complicated. Recall that a square matrix corresponds to a quantum state iff and . While the trace part is handled by the above “TP” property, the psd part is handled by the “CP” property. It is tempting to just require to preserve positivity: . Such kind of is called *positive*. Unfortunately it’s not enough, because there are examples that when attaching a new system B, is not positive any more. Physically, if we attach a new system B but operate only on the original system A, the whole system should remain a valid quantum system, namely should hold. For this to hold, being positive is not enough. So we put this further requirement in, and call a map *completely positive* (CP) if is positive for any B.

Now we know that

a map is quantum admissible, i.e. implementable,   
iff with s.t.

iff is CPTP.

## Distance measures of classical distributions

Suppose that we have two probability distributions and over a sample space . Two standard distance measures are

* **trace distance**:
  + . iff
  + It’s a metric. Triangle inequality holds.
* **fidelity**:
  + . iff
  + It’s not a metric. Triangle inequality doesn’t hold.

## Distance measures of quantum states

We now extend the trace distance and fidelity to the quantum case. Suppose that and are two quantum mixed states in the same space. Recall that for a real function and for a Hermitian where ’s are the eigenvalues, one can define .

* **trace distance**: .
  + If and commute, i.e. , then , where is the vector of eigenvalues of , and similarly for .
  + Unitary operations don’t change trace distance: .
  + , where the maximum is over all projectors .
  + , where the maximum is over all POVM , and are distributions of outcomes obtained by applying on and , respectively.   
    This implies that the trace distance between and is the largest difference one can tell by a measurement.
  + is a metric. Triangle inequality holds.
  + Quantum operations don’t increase trace distance. (One cannot make two states more distinguishable by operating on them.) .
  + Strong convexity: .
* **fidelity**:
  + It’s symmetric: .
  + If and commute, then .
  + iff .
  + iff and have orthogonal supports.
  + Special case of pure state(s): , .
  + .
  + , for any fixed purification of .
  + , where the maximum is over all POVM , and are distributions of outcomes obtained by applying on and , respectively.
  + .
  + .

A basic relation between the two distance measures is as follows.

## Note

There are a couple of excellent references for quantum information. Part III of [NC00] is still very good. [Wil13] is a new book with emphasis on quantum Shannon theory. [Wat11] contains more other stuff and it is somewhat closer to computer science perspectives. [KSV02] (Chapter 11 and 12) was one of the earliest introductions to channel distance by diamond norm.

## Reference

[KSV02] A. Yu. Kitaev, A. H. Shen , M. N. Vyalyi, **Classical and Quantum Computation**, *American Mathematical Society*, 2002.

[NC00] Michael Nielsen and Isaac Chuang, **Quantum Computation and Quantum Information**, *Cambridge University Press*, 2000.

[Wat11] John Watrous, [Lecture notes for ***Theory of Quantum Information***](https://cs.uwaterloo.ca/~watrous/CS766/LectureNotes/all.pdf)*,* Fall 2011.

[Wil13] Mark M. Wilde, **Quantum Information Theory**, *Cambridge University Press*, 2013