Tutorial 8: Further Topics on Random Variables 1

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Derived distributions

- Given $Y = g(X)$ of a continuous random variable X and PDF of X, how to calculate the PDF of Y?
  - Two step approach

Calculation of PDF of $Y = g(X)$

1. Calculate the CDF F_Y of Y using the formula

$$F_Y(y) = P(g(X) \leq y) = \int_{\{x | g(x) \leq y\}} f_X(x) dx$$

2. Differentiate F_Y to obtain the PDF f_Y of Y:

$$f_Y(y) = \frac{dF_Y}{dy}(y).$$

Calculation of PDF of $Y = g(X)$

- Linear case:
  - The PDF of $Y = aX + b$ in terms of the PDF X.

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- Monotonic Case:
  - Suppose that g is monotonic and that for some function h and all x in the range of X we have

$$y = g(x) \text{ if and only if } x = h(y).$$

  - Assume that h is differentiable.

$$f_Y(y) = f_X(h(y)) \left|\frac{dh}{dy}(y)\right|$$

Example 1

- Let X be a random variable with PDF f_X. Find the PDF of the random variable $Y = |X|$,
  - (a) when $f_X(x) = \begin{cases} \frac{1}{3}, if \ -2 < x \leq 1, \\ 0, \quad otherwise; \end{cases}$
  - (b) when $f_X(x) = \begin{cases} 2e^{-2x}, if \ x > 0, \\ 0, \quad otherwise; \end{cases}$
  - (c) for general $f_X(x)$.

Example 1

- Since $Y = |X|$, you can visualize the PDF for any given y as

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y), & if\ y \geq 0 \\ 0, & if\ y < 0 \end{cases}$$

- Also note that since $Y = |X|$, $Y \geq 0$.

Example 1

- (a) Since $f_X(x) = \begin{cases} \frac{1}{3}, if \;\; -2 < x \leq 1, \\ 0, \qquad\quad otherwise; \end{cases}$

So, $f_X(x)$ for $-1 \;\leq\; x \;\leq\; 0$ gets added to $f_X(x)$ for $0 \;\leq\; x \;\leq\; 1$:

$$f_Y(y) = \begin{cases} \dfrac{2}{3}, if\ 0 \leq y < 1, \\ \dfrac{1}{3}, if\ 1 \leq y < 2, \\ 0, \qquad\quad otherwise; \end{cases}$$

Example 1

- (b) Here we are told $X > 0$. So there are no negative values of X that need to be considered. Thus

$$\mathrm{f}_Y(\mathrm{y}) = f_X(x) = \begin{cases} 2e^{-2x}, if\ x > 0, \\ 0, \quad otherwise. \end{cases}$$

- (c) As explained in the beginning,

$$f_Y(y) = f_X(y) + f_X(-y).$$

Example 2: archer shooting

- Two archers shoot at a target.
- The distance of each shot from the center of the target is uniformly distributed from 0 to 1, independent of the other shot.
- Question: What is the PDF of the distance of the winning shot from the center?

Example 2: archer shooting

- Let X and Y be the distances from the center of the first and second shots, respectively.
- Z: the distance of the winning shot:
$$Z = \min\{X, Y\}.$$
- Since X and Y are uniformly distributed over $[0,1]$,
- we have, for all $z \in [0, 1]$,
$$P(X \geq z) = P(Y \geq z) = 1 - z.$$

Example 2: archer shooting

- Thus, using the independence of X and Y, we have for all $z \in [0,1]$,

$$\begin{aligned} F_Z(z) &= 1 - P(\min\{X, Y\} \geq z) \\ &= 1 - P(X \geq z)P(Y \geq z) \\ &= 1 - (1 - z)^2. \end{aligned}$$

- Differentiating, we obtain

$$f_Z(z) = \begin{cases} 2(1 - z), & \text{if } 0 \leq z \leq 1. \\ 0, & \text{otherwise.} \end{cases}$$

Covariance

- The covariance of two random variables X and Y are defined as

$$\mathrm{cov}(X,Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])].$$

- Alternatively

$$\mathrm{cov}(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y].$$

Correlation Coefficient

- For any random variable X, Y with nonzero variances, the correlation coefficient $\rho(X, Y)$ of them is defined as

$$\rho(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}.$$

- It may be viewed as a normalized version of the covariance $\operatorname{cov}(X, Y)$.
  - Recall $\operatorname{cov}(X, X) = \operatorname{var}(X)$.
  - It's easily verified that

$$-1 \leq \rho(X, Y) \leq 1$$

Example 3

- A restaurant serves three fixed-price dinners costing \$12, \$15, and \$20. For a randomly selected couple dinning at this restaurant, let $X = the\ cost\ of\ the\ man's\ dinner$ and $Y = the\ cost\ of\ the\ woman's\ dinner$. If the joint PMF of X and Y is assumed to be, what is $\text{cov}(X, Y)$?

$p(x, y)$		y = 12	15	20
	12	.05	.05	.10
x	15	.05	.10	.35
	20	0	.20	.10

Example 3

- $\text{Cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] = 276.7 - 15.9 \cdot 17.45 = -0.755$