Tutorial 7: General Random Variables 3

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Revise - Joint PDF

- Two continuous random variables X, Y satisfying

$$P\big((X,Y) \in B\big) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

for every subset B

Revise - Marginal Probability

- $P(X \in A) = P\big(X \in A, Y \in (-\infty, \infty)\big)$

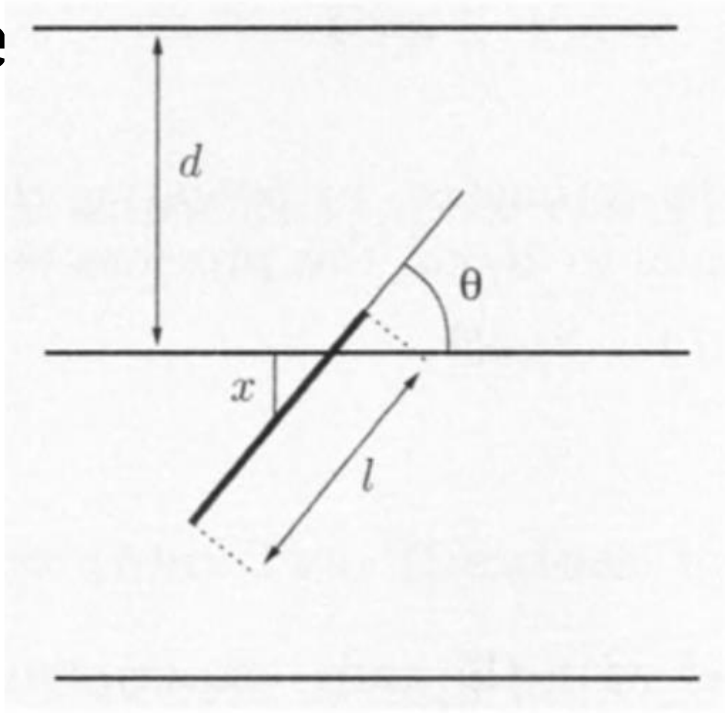
$$= \int_A \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

- The marginal PDF of X is

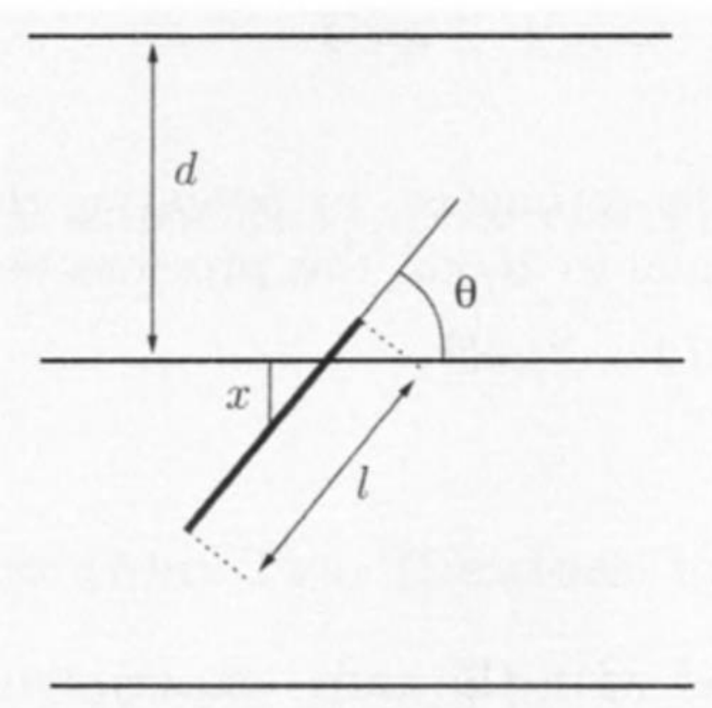
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

Example: Buffon's Needle

- A surface is ruled with parallel lines, which at distance d from each other.
- Suppose we throw a needle of length l randomly.
- What is the prob. that the needle will intersect one of the lines?



Example: Buffon's Needle



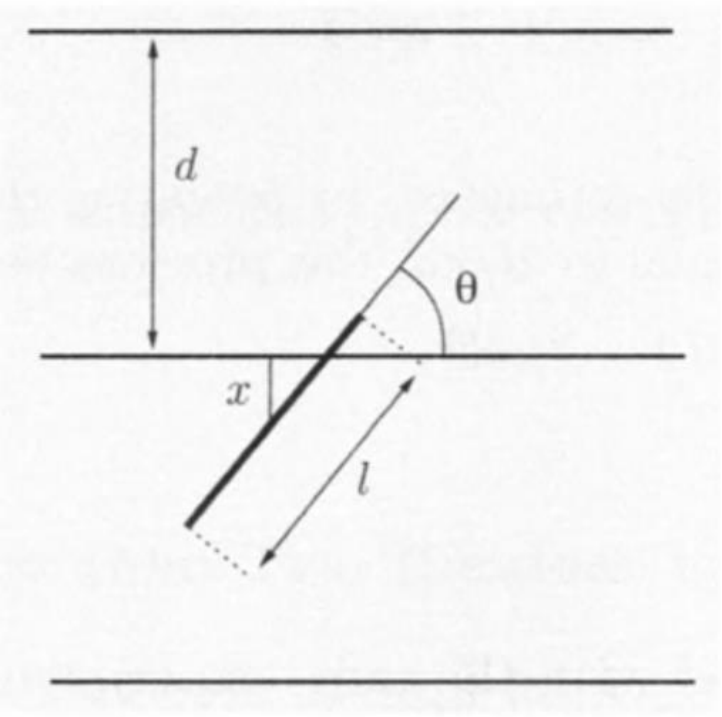
- Assume $l < d$ so that the needle cannot intersect two lines simultaneously.
- X, the distance from the middle point of the needle and the nearest of the parallel lines
- ϑ, the acute angle formed by the needle and the lines

Example: Buffon's Needle

- We model (X, ϑ) with a uniform joint PDF so that

$$f_{X,\vartheta}(x, \theta) = \begin{cases} \dfrac{4}{\pi d}, & if\ x \in \left[0, \dfrac{d}{2}\right]\ and\ \theta \in [0, \dfrac{\pi}{2}] \\ 0, & otherwise \end{cases}$$

Example: Buffon's Needle



- The needle will intersect one of the lines if and only if

$$X \leq \frac{l}{2}\sin\vartheta$$

Example: Buffon's Needle

- So the probability of intersection is

$$P\left(X \le \frac{l}{2}\sin\vartheta\right) = \iint_{X \le \frac{l}{2}\sin\theta} f_{X,\vartheta}(x,\theta)\,dx\,d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{\left(\frac{l}{2}\right)\sin\theta} dx\,d\theta = \frac{4}{\pi d}\int_0^{\pi/2}\left(\frac{l}{2}\right)\sin\theta\,d\theta$$

$$= \frac{2l}{\pi d}(-\cos\theta)\Big|_0^{\frac{\pi}{2}} = \frac{2l}{\pi d}$$

Example: Multivariate Gaussian

- Let $X \in R^n$ be a n-dimensional normal random variable. The PDF of X is then

$$f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$$

Example: Multivariate Gaussian

- The mean and variance

$$E(X) = \mu$$
$$V(X) = \Sigma$$

(Recall)

$$f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

Example: Multivariate Gaussian

- To draw: suppose we have $Z_i \sim \mathbb{N}(0,1)$ i.i.d and

$$Z \quad = (Z_1, \ldots, Z_n)$$

- We have

$$\mu + AZ \sim \mathbb{N}(\mu, \Sigma)$$

where $AA^T = \Sigma$.

Example: Multivariate Gaussian

- Element-wise variance

$$\Sigma_{ij} = Cov(X_i, X_j)$$

- Degenerate case

$$\Sigma = diag(v_1, v_2, \dots, v_n)$$

Example: Multivariate Gaussian

- Fact: in degenerate case, elements of X are independent
- Proof: As the coefficient of $x_i x_j$ is zero in $(x-\mu)^T \Sigma^{-1} (x-\mu)$, the PDF can be decomposed into the product of x_i part and x_j part.

Example: Multivariate Gaussian

- For conditional distribution: Gaussian variable condition on Gaussian variable is still Gaussian. We omit the proof.