Tutorial 6: General Random Variables 2

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Outline

Extend the notion of PDF to the case of multiple random variables

- Jointly continuous PDF
- Marginal PDF
- Conditional PDF
- Exercises

Continuous PDF

- Recall that X is continuous if there is a function $f(x)$ (the density) such that

$$P(X \leq t) = \int_{-\infty}^{t} f_X(x)dx$$

We generalize this to two random variables.

Jointly continuous PDF

- **Definition**

  Two random variables X and Y are jointly continuous if there is a function $f_{X,Y}(x,y)$ on R^2, called the joint probability density function, such that

$$P(X \leq s, Y \leq t) = \iint_{x \leq s, y \leq t} f_{X,Y}(x,y)dxdy$$

Jointly continuous PDF

- The integral is over $\{(x, y) : x \leq s, y \leq t\}$. We can also write the integral as

$$P(X \leq s, Y \leq t) = \int_{-\infty}^{s} \left(\int_{-\infty}^{t} f_{X,Y}(x, y) dy \right) dx$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{s} f_{X,Y}(x, y) dx \right) dy$$

Jointly continuous PDF

- In order for a function $f(x, y)$ to be a joint density it must satisfy

$$f(x, y) \geq 0$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx\, dy = 1$$

Jointly continuous PDF

- Just as with one random variable, the joint density function contains all the information about the underlying probability measure if we only look at the random variables X and Y .

- In particular, we can compute the probability of any event defined in terms of X and Y just using $f(x, y)$.

Jointly continuous PDF

Here are some events defined in terms of X and Y :

- $\{X \leq Y\}$
- $\{X^2 + Y^2 \leq 1\}$
- $\{1 \leq X \leq 4, Y \geq 0\}$

They can all be written in the form $\{(X, Y) \in B\}$ for some subset B of R^2.

Jointly continuous PDF

- **Proposition**. $For\ B\ \subset R^2$,

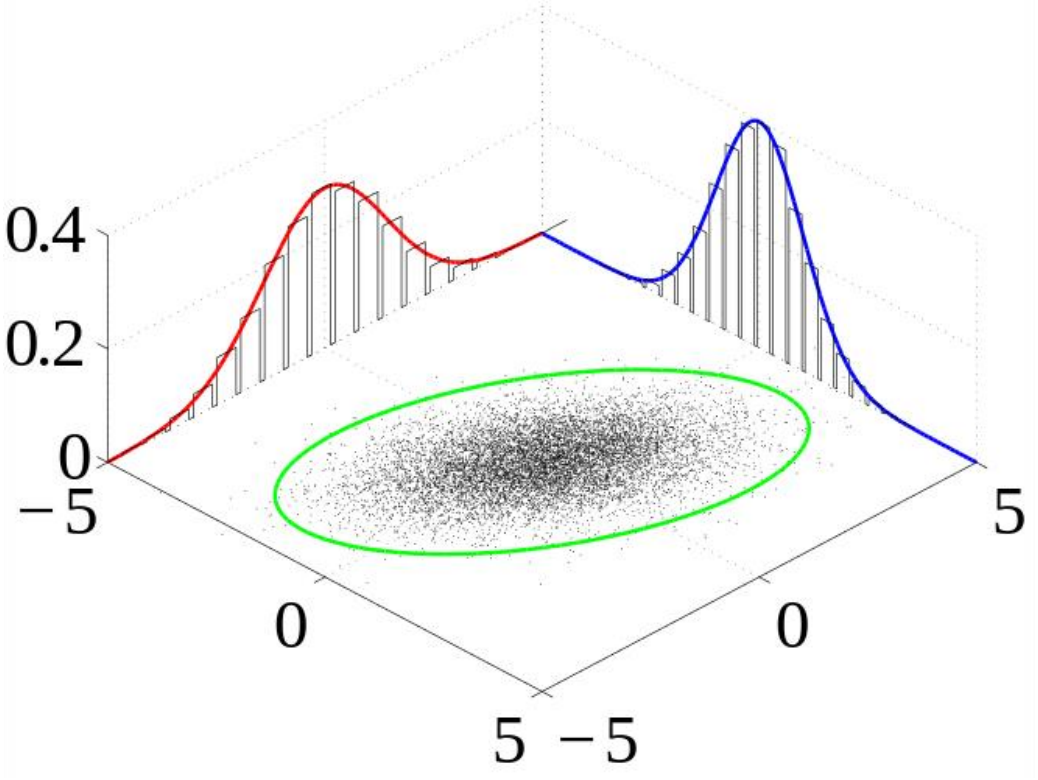
$$P\big((X,Y) \in B\big) = \iint\limits_{(x,y)\in B} f_{X,Y}(x,y)dxdy$$

The two-dimensional integral is over the subset B of R^2

Jointly continuous PDF

- Typically, when we want to actually compute this integral we have to write it as an iterated integral.

- It is a good idea to draw a picture of B to help do this.

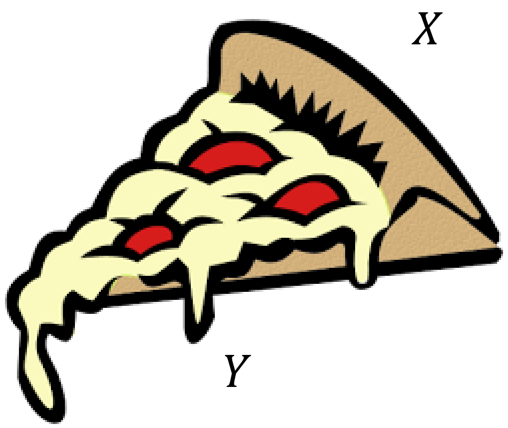


Many sample observations are shown from a joint probability distribution. The marginal densities are shown as well.

Exercise

$$f(x,y) = \begin{cases} \frac{1}{6}(x+y), & if\ 1 \leq x \leq 2\ and\ 4 \leq y \leq 5 \\ 0, & otherwise \end{cases}$$

What it the probability of $1 \leq x \leq 1.5\ and\ 4.5 \leq y \leq 5$?



Solution

$$P(1 \leq x \leq 1.5\,, 4.5 \leq y \leq 5)$$

$$= \int_{1}^{1.5} \int_{4.5}^{5} f_{X,Y}(x, y) dy\, dx$$

$$= \int_{1}^{1.5} \int_{4.5}^{5} \frac{1}{6}(x + y)\, dy\, dx$$

=0.25

Marginal PDF

- Suppose we know the joint density $f_{X,Y}(x, y)$ of X and Y . How do we find their individual densities $f_X(x)$, $f_Y(y)$. These are called marginal densities. The CDF of X is

$$F_X(x) = P(X \le x) = P(-\infty < X \le x, -\infty < Y < \infty)$$

$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f_{X,Y}(x, y) dy \right] dx$$

Marginal PDF

- Differentiate this with respect to x and we get

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

In words, we get the marginal density of X by integrating y from $-\infty$ to ∞ in the joint density.

Marginal PDF

- **Proposition.** If X and Y are jointly continuous with joint density $f_{X,Y}(x, y)$, then the marginal densities are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx$$

Conditional PDF

- **Definition.** Suppose *X* and *Y* are continuous random variables with joint probability density function *f*(*x*,*y*) and marginal probability density functions $f_X(x)$ and $f_Y(y)$, respectively. Then, the conditional PDF of Y given $X = x$ is defined as:

$$h(y|x) = f(x, y)/fX(x)$$

provided $f_X(x) > 0$.

Exercise

- Suppose the continuous random variables *X* and *Y* have the following joint probability density function:

$$f(x, y) = 3/2$$

for $x^2 \leq y \leq 1$ and $0 < x < 1$.

- What is the conditional distribution of *Y* given *X* = *x*?

Solution

- We can use the formula:

$$h(y|x) = f(x, y)/f_X(x)$$

to find the conditional PDF of *Y* given *X*.

- So, we basically have a plane, shaped like the support, floating at a constant 3/2 units above the *xy*-plane. To find $f_X(x)$ then, we have to integrate:

$$f(x,y) = 3/2$$

over the support $x^2 \leq y \leq 1$.

- That is:

$$f_X(x) = \int_{s2} f(x, y)dy$$

$$= \int_{x^2} \frac{3}{2} dy$$

$$= \frac{3}{2}(1 - x^2)$$

for $0 < x < 1$.

- Now, we can use the joint PDF $f(x, y)$ that we were given and the marginal PDF $f_X(x)$ that we just calculated to get the conditional PDF of *Y* given *X* = *x*:

$$h(y|x) = \frac{f(x, y)}{f_{X(x)}} = \frac{\frac{3}{2}}{\frac{3}{2}(1 - x^2)} = 1(1 - x^2),$$

$$0 < x < 1, x^2 \leq y \leq 1$$

- That is, given x, the continuous random variable Y is uniform on the interval $(x^2, 1)$.
- For example, if x = ¼, then the conditional PDF of Y is:

$$h(y|1/4) = 11 - \left(\frac{1}{4}\right)^2 = 1/(15/16) = 16/15$$

$$for\ 1/16 \ \leq \ y \ \leq \ 1.$$

- And, if x = ½, then the conditional PDF of Y is:

$$h(y|1/2) = 11 - \left(\frac{1}{2}\right)^2 = 1/(1-1/4) = 4/3$$

$$for\ 1/4 \ \leq \ y \ \leq \ 1.$$