Tutorial 5: General Random Variables 1

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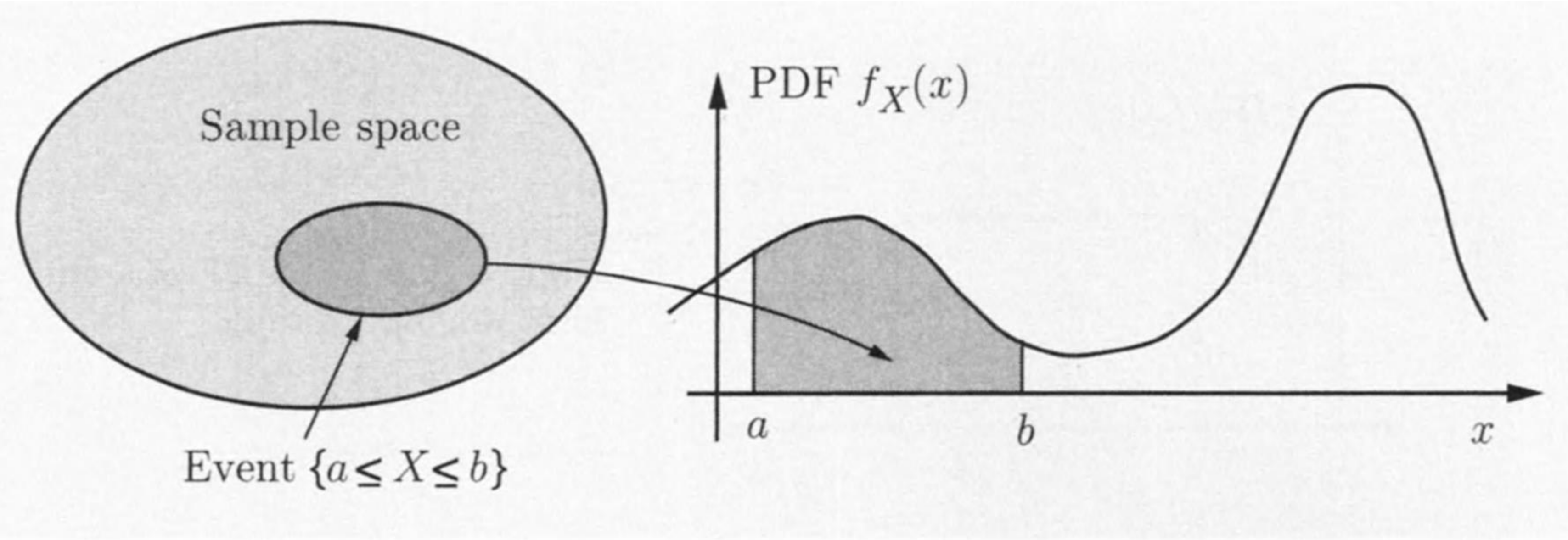
Outline

- Continuous random variables, PDFs, CDFs
- Uniform distributions
- Exponential distributions
- Normal distributions
- Laplace distributions

Continuous r.v. and PDFs

- A random variable X is called continuous if there is a function $f_X \geq 0$, called the probability density function of X, or PDF, s.t.
$$P(X \in B) = \int_B f_X(x)dx$$
for every subset $B \subseteq \mathbb{R}$.
- In particular, when $B = [a, b]$,
$$P(a \leq X \leq b) = \int_a^b f_X(x)dx$$
is the area under the graph of PDF.

Continuous r.v. and PDFs



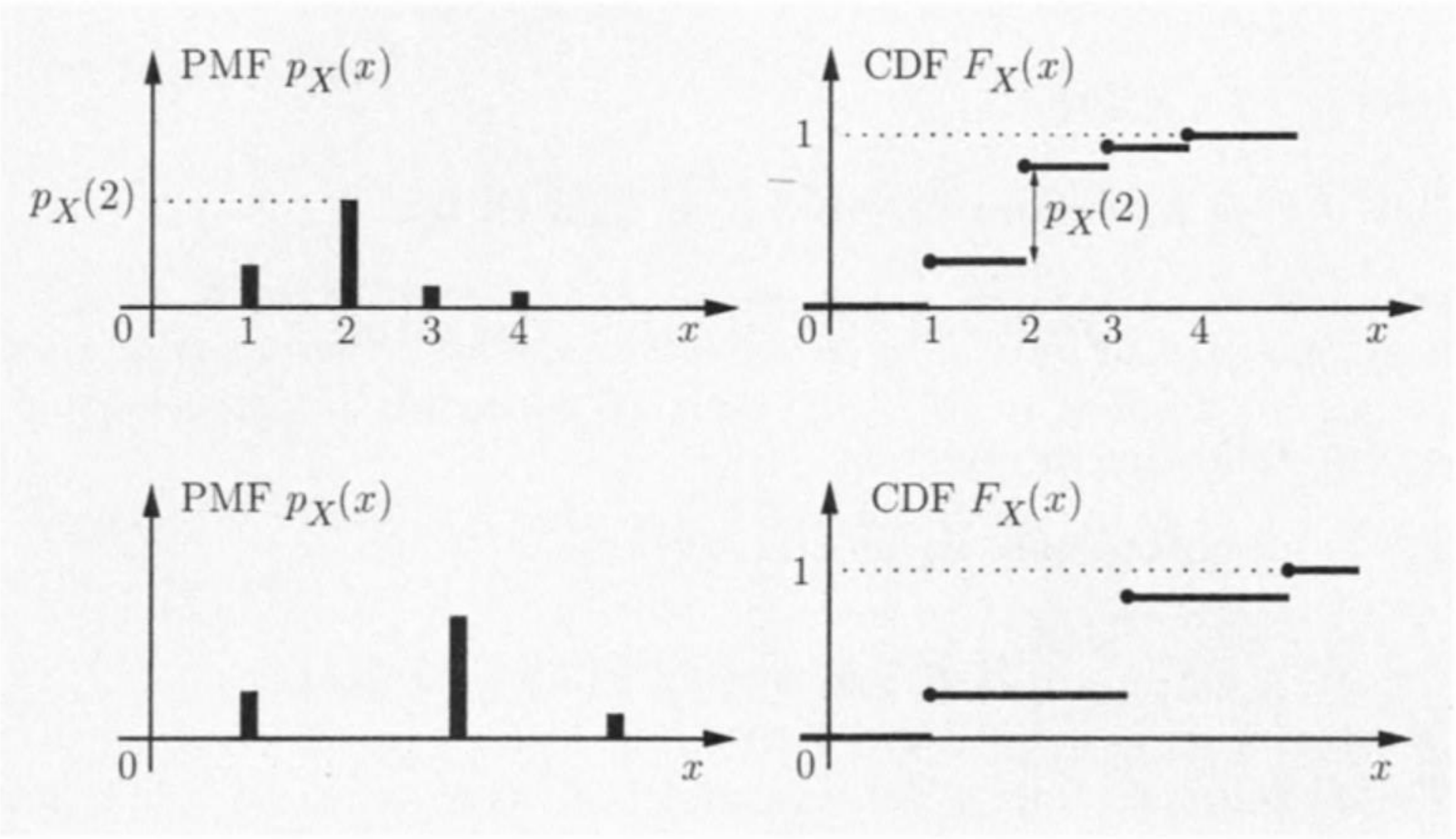
Cumulative Distribution Function

- The cumulative distribution function, or CDF, of a random variable X is

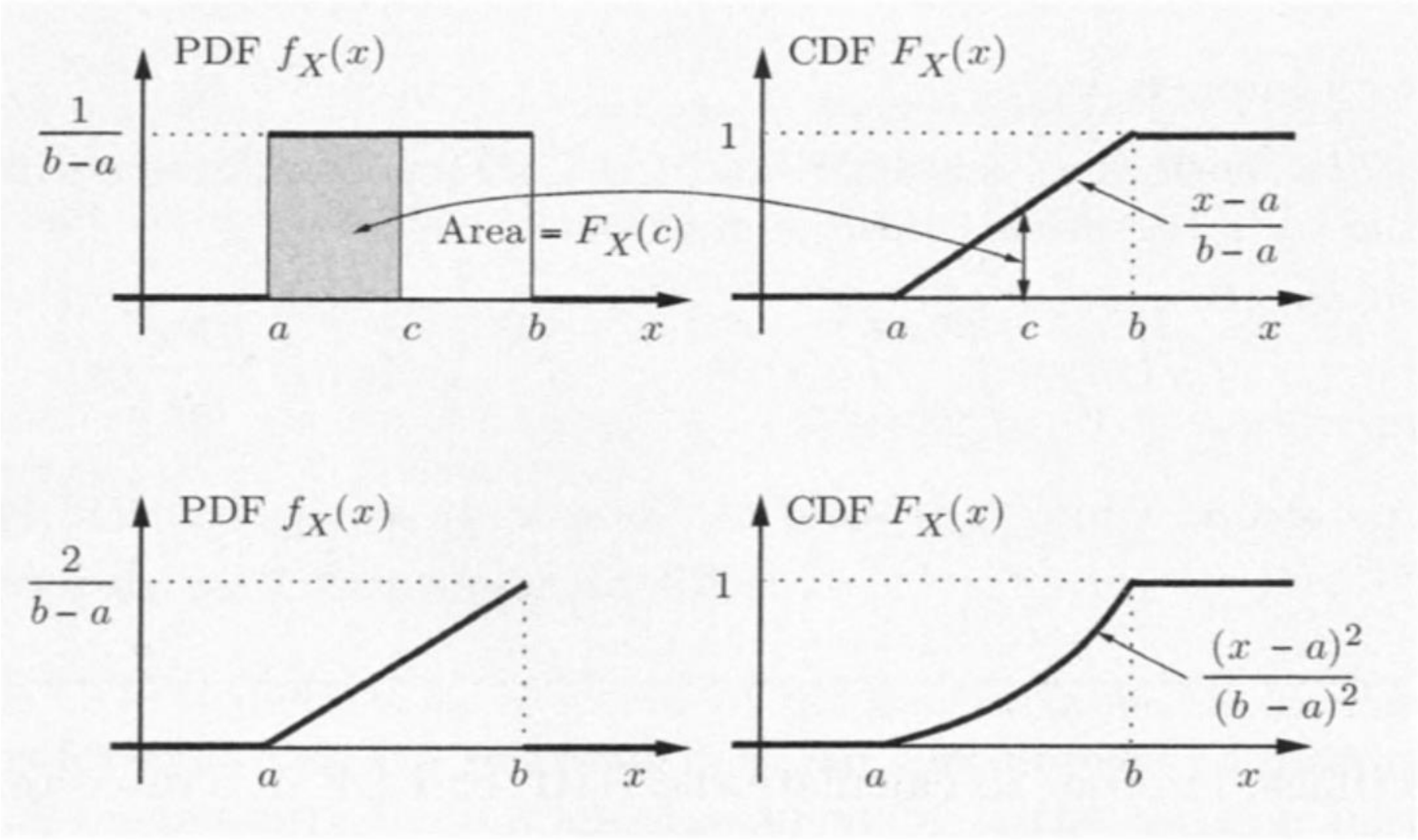
$$F_X(x) = P(X \leq x)$$
$$= \begin{cases} \sum_{k \leq x} p_X(x), & \text{discrete} \\ \int_{-\infty}^{x} f_X(x)dx, & \text{continuous} \end{cases}$$

- The CDF $F_X(x)$ "accumulates" probability "up to" the value x.

CDF for discrete case



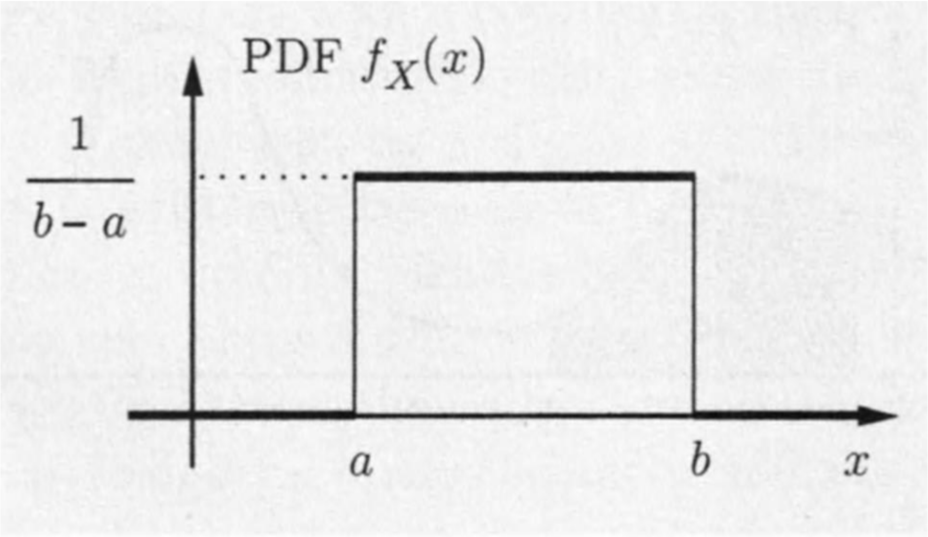
CDF for continuous case



Uniform distributions

- Its PDF has the form

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$



Example 1

- Let X be uniformly distributed in the unit interval $[0,1]$. Consider the random variable $Y = g(X)$, where

$$g(x) = \begin{cases} 1, & if\ x \leq 2/3, \\ 2, & if\ x > 2/3. \end{cases}$$

- Find the expected value of Y.

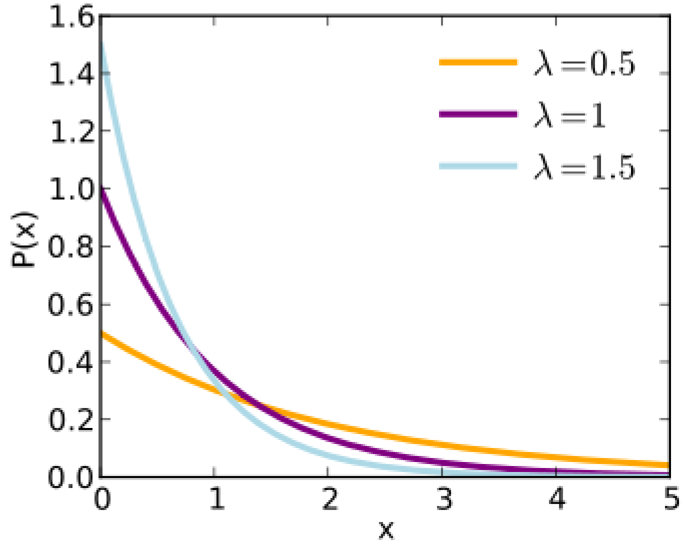
X	$[\mathbf{0}, \mathbf{2}/\mathbf{3}]$	$(\frac{\mathbf{2}}{\mathbf{3}}, \mathbf{1}]$
Y	1	2
P	$2/3$	$1/3$

$$E[Y] = 1 \times \frac{2}{3} + 2 \times \frac{1}{3} = \frac{4}{3}$$

Exponential distributions

- An exponential random variable has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



- Mean: $\frac{1}{\lambda}$, mode:0, variance: $\frac{1}{\lambda^2}$
- Its CDF is

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = \left(-e^{-\lambda t}\right)\Big|_0^x$$

$$= 1 - e^{-\lambda x}$$

Example 2

- Alice goes to a bank to make a withdrawal, and is equally likely to find 0 or 1 customers ahead of her.
- The service time of the customer ahead, if present, is exponentially distributed with parameter λ.
- Find the CDF of Alice's waiting time.

When $x \geq 0$

$$F_X(x) = P(X \leq x) = \frac{1}{2} + \frac{1}{2} F_{exp}(x) = \frac{1}{2} + \frac{1}{2}\left(1 - e^{-\lambda x}\right) = 1 - \frac{1}{2} e^{-\lambda x}$$

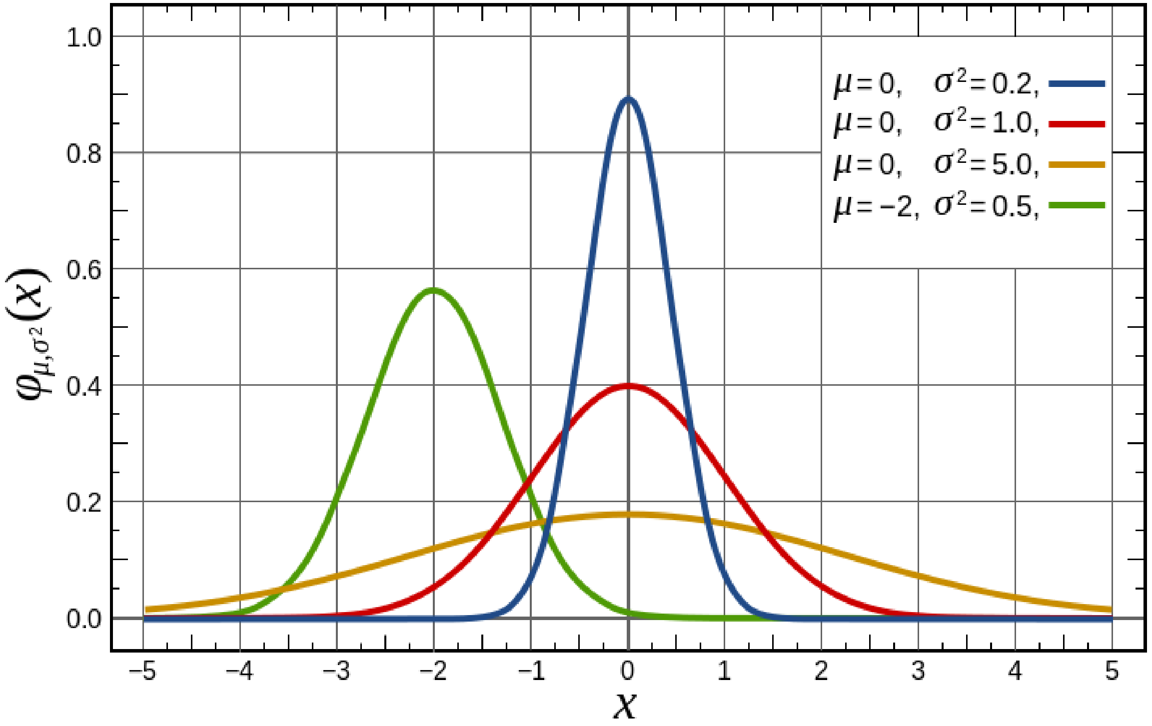
Normal distributions

- A continuous random variable X is normal, or Gaussian, if it has a PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for some $\sigma > 0$.

- Mean: μ, Mode: μ, variance: σ^2



Example 3: Normal random variables

- Let X and Y be normal random variables with means 0 and 1, respectively, and variance 1 and 4, respectively.
- (a) Find $P(X \leq 1.5)$ and $P(X \leq -1)$.

$$P(X \leq 1.5) = \Phi(1.5) = 0.9932$$
$$P(X \leq -1) = \Phi(-1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

Example 3: Normal random variables

- Let X and Y be normal random variables with means 0 and 1, respectively, and variance 1 and 4, respectively.
- (b) Find the PDF of $(Y-1)/2$

Since Y is a normal random variable with mean 1 and variance 4, so $(Y-1)/2$ is a standard normal random variable.

- (c) Find $P(-1 \leq Y \leq 1)$

$$P(-1 \leq Y \leq 1) = P\left(-1 \leq \frac{Y-1}{2} \leq 0\right) = \Phi(0) - \Phi(-1)$$

$$= \Phi(0) - \left(1 - \Phi(1)\right) = \Phi(1) - 0.5 = 0.8413 - 0.5 = 0.3413$$

Example 4:

- Let X be a normal random variable with zero mean and standard deviation σ.
- Use the normal tables to compute the probabilities of the events $\{X \geq k\sigma\}$ and $\{|X| \leq k\sigma\}$ for $k = 1,2,3$.

- X/σ is a standard normal random variable
- $P(X \geq k\sigma) = P\left(\frac{X}{\sigma} \geq k\right) = 1 - P\left(\frac{X}{\sigma} \leq k\right) = 1 - \Phi(k)$
- $P(|X| \leq k\sigma) = P\left(-k \leq \frac{X}{\sigma} \leq k\right) = P\left(\frac{X}{\sigma} \leq k\right) - P\left(\frac{X}{\sigma} \leq -k\right) = \Phi(k) - \Phi(-k) = 2\Phi(k) - 1$

Example 5: Laplace random variable

- Let X has the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

where λ is a positive scalar.

- (a) Verify that f_X satisfies the normalization condition

$$\int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda|x|}\,dx = \int_{-\infty}^{0} \frac{\lambda}{2} e^{\lambda x}\,dx + \int_{0}^{\infty} \frac{\lambda}{2} e^{-\lambda x}\,dx = 2\int_{0}^{\infty} \frac{\lambda}{2} e^{-\lambda x}\,dx$$

$$= \left(-e^{-\lambda x}\right)\Big|_0^\infty = 1$$

Example 5: Laplace random variable

- Let X has the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

  where λ is a positive scalar.

- (b) Find the mean and variance of X

$$E[X] = \int_{-\infty}^{\infty} x \frac{\lambda}{2} e^{-\lambda|x|} dx = 0$$

Example 5: Laplace random variable

- Let X has the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

where λ is a positive scalar.

- (b) Find the mean and variance of X

$$var[X] = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda|x|} dx = 2\int_0^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda x} dx = \int_0^{\infty} x^2 d(-e^{-\lambda x})$$

$$= x^2(-e^{-\lambda x})\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} 2x dx) = \frac{2}{\lambda}\int_0^{\infty} x d(-e^{-\lambda x})$$

$$= \frac{2}{\lambda} x(-e^{-\lambda x})\Big|_0^{\infty} + \frac{2}{\lambda}\int_0^{\infty} e^{-\lambda x} dx = \frac{2}{\lambda^2}(-e^{-\lambda x})\Big|_0^{\infty} = \frac{2}{\lambda^2}$$

Example 5: Laplace random variable

- Let X has the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

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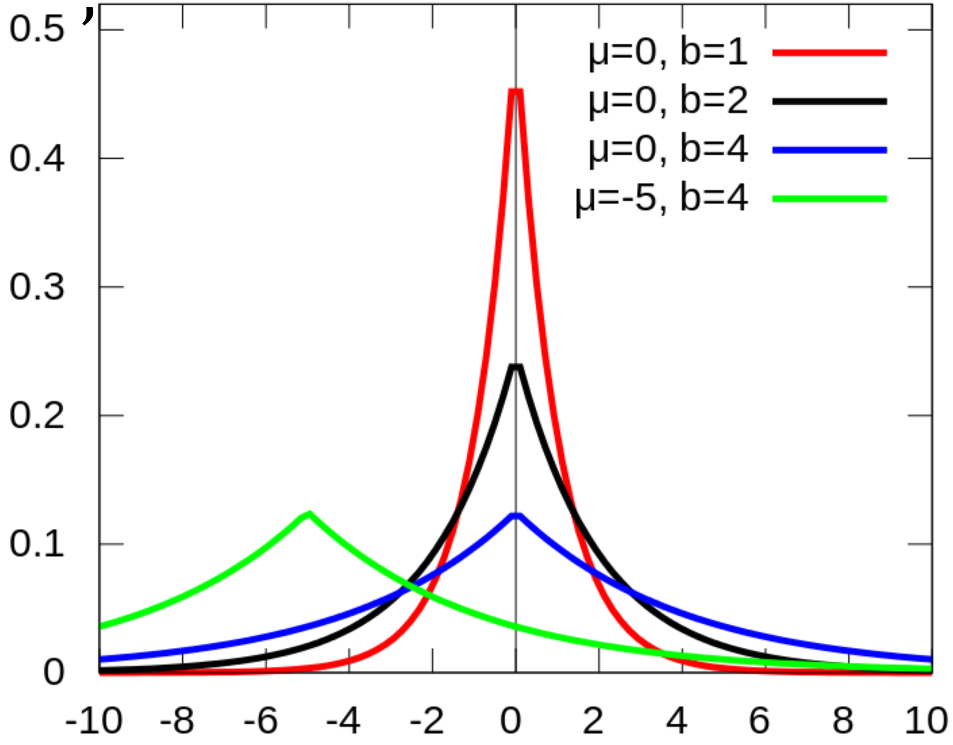

- $b = \frac{1}{\lambda}$
- This is for the general form

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x-\mu|}$$