Tutorial 4: Discrete Random Variables 2

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Geometric Random Variables

- The PMF of a geometric random variable is

$$p_X(k) = (1-p)^{k-1}p$$

- $E[X|X > 1] = 1 + E[X]$

Geometric Random Variables

- For $k > 1$,

$$P(X = k|X > 1) = \frac{P(X=k)}{P(X>1)} = \frac{(1-p)^{k-1}p}{\sum_{k=2}^{\infty}(1-p)^{k-1}p} = \frac{(1-p)^{k-1}p}{p(1-p)\cdot\frac{1}{1-(1-p)}}$$

$$= (1-p)^{k-2}p = p_X(k-1)$$

- $E[X|X > 1] = \sum_{k=2}^{\infty} kP(X = k|X > 1)$

$$= \sum_{k=2}^{\infty} kp_X(k-1) = \sum_{k=1}^{\infty}(k+1)p_X(k) = \sum_{k=1}^{\infty} kp_X(k) + \sum_{k=1}^{\infty} p_X(k)$$

$$= 1 + E[X]$$

Geometric Random Variables

- In general,

$$P(X = k|X > a) = \frac{P(X = k)}{P(X > a)}$$

$$= \frac{(1-p)^{k-1}p}{\sum_{k=a+1}^{\infty}(1-p)^{k-1}p} = \frac{(1-p)^{k-1}p}{p(1-p)^{a}\cdot\frac{1}{1-(1-p)}}$$

$$= (1-p)^{k-a-1}p$$

$$= p_X(k-a)$$

Geometric Random Variables

- $E[f(X)|X > a] = E[f(X + a)]$

- $E[f(X)|X > a]$

$$= \sum_{k=a+1}^{\infty} f(k)P(X = k|X > a) = \sum_{k=a+1}^{\infty} f(k)p_X(k - a)$$

$$= \sum_{k=1}^{\infty} f(k + a)p_X(k) = E[f(X + a)]$$

Example 1: Functions of random variables

- Let X be a random variable that takes value from 0 to 9 with equal probability $1/10$.
- (a) Find the PMF of the random variable $Y = X\ mod(3)$.

X	0,3,6,9	1,4,7	2,5,8
Y	0	1	2
P	$\frac{4}{10} = 2/5$	$3/10$	$3/10$

Example 1: Functions of random variables

- Let X be a random variable that takes value from 0 to 9 with equal probability $1/10$.
- (b) Find the PMF of the random variable $Y = 5\, mod(X+1)$.

X	0	1	2	3	4	5	6	7	8	9
Y	0	1	2	1	0	5	5	5	5	5

Y	0	1	2	5
P	$\frac{2}{10} = 1/5$	$\frac{2}{10} = 1/5$	$1/10$	$\frac{5}{10} = 1/2$

Example 2: Expectation, Mean, and Variance

- Let X be a random variable with PMF

$$p_X(x) = \begin{cases} \frac{x^2}{a}, & if\ x = -3,-2,-1,0,1,2,3 \\ 0, & otherwise \end{cases}$$

- (a) Find a and $E[X]$

$$a = \sum_{x=-3}^{3} x^2 = 2 \times (1 + 4 + 9) = 28$$

$$E[X] = \sum_{x=-3}^{3} x \cdot \frac{x^2}{28} = 0$$

Example 2: Expectation, Mean, and Variance

- Let X be a random variable with PMF

$$p_X(x) = \begin{cases} \dfrac{x^2}{a}, & if\ x = -3,-2,-1,0,1,2,3 \\ 0, & otherwise \end{cases}$$

- (b) What is the PMF of the random variable $Z = (X - E[X])^2$?

$$Z = X^2$$

X	0	-1,1	-2,2	-3,3
Z	0	1	4	9
P	0	$2 \times \frac{1}{28} = 1/14$	$2 \times \frac{4}{28} = \frac{2}{7}$	$2 \times \frac{9}{28} = \frac{9}{14}$

Example 2: Expectation, Mean, and Variance

- Let X be a random variable with PMF

$$p_X(x) = \begin{cases} \frac{x^2}{a}, & if\ x = -3,-2,-1,0,1,2,3 \\ 0, & otherwise \end{cases}$$

- (c) Using the result from part (b), find the variance of X.

$$var(X) = E[Z] = 1 \times \frac{1}{14} + 4 \times \frac{2}{7} + 9 \times \frac{9}{14} = \frac{99}{14}$$

Z	0	1	4	9
P	0	$2 \times 1/28 = 1/14$	$2 \times 4/28 = 2/7$	$2 \times 9/28 = 9/14$

Example 3: Expectation, Mean, and Variance

- Two coins are simultaneously tossed until one of them comes up a head and the other a tail.
- The first coin comes up a head with prob. p
- The second coin comes up a head with prob. q
- Independent
- (a) Find the PMF, the expected value, and the variance of the number of tosses.
- The number of tosses is a geometric random variable with parameter

$$p' = p(1-q) + (1-p)q$$

So the mean is $\frac{1}{p'}$ and variance is $\frac{1-p'}{p'^2}$.

Example 3: Expectation, Mean, and Variance

- Two coins are simultaneously tossed until one of them comes up a head and the other a tail.
- The first coin comes up a head with prob. p
- The second coin comes up a head with prob. q
- Independent
- (b) What is the prob. that the last toss of the first coin is a head?

$$P(first\ coin\ is\ head|last\ toss) = \frac{p(1-q)}{p(1-q)+(1-p)q}$$

Example 4: Joint PMFs of Multiple Random Variables

- A stock trader buys 100 shares of stock A and 200 shares of stock B.
- Let X and Y be the price changes of A and B, respectively.
- Assume the joint PMF of X and Y is uniform over the set of integers

$$-1 \le x \le 2, \qquad -1 \le y - x \le 1$$

- (a) Find the marginal PMFs and the means of X and Y
- $P_X(-1) = P_X(0) = P_X(1) = P_X(2) = \frac{1}{4}, E[X] = 0.5$

Y	-2	-1	0	1	2	3
P	$1/12$	$2 \times \frac{1}{12} = 1/6$	$3 \times \frac{1}{12} = 1/4$	$3 \times \frac{1}{12} = 1/4$	$2 \times \frac{1}{12} = 1/6$	$1/12$

- $E[Y] = \frac{1}{12}(-2 - 2 + 0 + 3 + 4 + 3) = \frac{6}{12} = \frac{1}{2}$

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- A stock trader buys 100 shares of stock A and 200 shares of stock B.
- Let X and Y be the price changes of A and B, respectively.
- Assume the joint PMF of X and Y is uniform over the set of integers

$$-1 \leq x \leq 2, \qquad -1 \leq y - x \leq 1$$

- (b) Find the mean of the trader's profit

$$E[100X + 200Y] = 100 \times 0.5 + 200 \times 0.5 = 150$$

Example 5: Conditioning

- Consider ten independent rolls of a 6-sided die. Let X be the number of $6s$ and let Y be the number of $1s$ obtained.
- What is the joint PMF of X and Y?
- X is a binomial random variable with $n = 10$ and $p = \frac{1}{6}$
- $P(Y = b|X = a)$ is a binomial random variable with $n = 10 - a$ and $p = \frac{1}{5}$
- So the joint PMF is

$$P(X = a, y = b) = P(X = a)P(Y = b|X = a)$$
$$= \binom{10}{a}\left(\frac{1}{6}\right)^a \left(\frac{5}{6}\right)^{10-a}\binom{10-a}{b}\left(\frac{1}{5}\right)^b\left(\frac{4}{5}\right)^{10-a-b}$$

Example 6: Independence

- Suppose that X and Y are independent, identically distributed, geometric random variables with parameter p. Show that

$$P(X=i|X+Y=n)=\frac{1}{n-1}, \qquad i=1,\ldots,n-1$$

- $P(X=i|X+Y=n)=\frac{P(X=i,X+Y=n)}{P(X+Y=n)}=\frac{P(X=i)P(Y=n-i)}{P(X+Y=n)}$
- For $i=1,\ldots,n-1$,

$$P(X=i)P(Y=n-i)=p(1-p)^{i-1}\cdot p(1-p)^{n-i-1}$$
$$=p^2(1-p)^{n-2}$$

So they are equally likely. Notice that

$$P(X+Y=n)=\sum_{i=1}^{n-1}P(X=i)P(Y=n-i)$$