Tutorial 3: Discrete Random Variables 1

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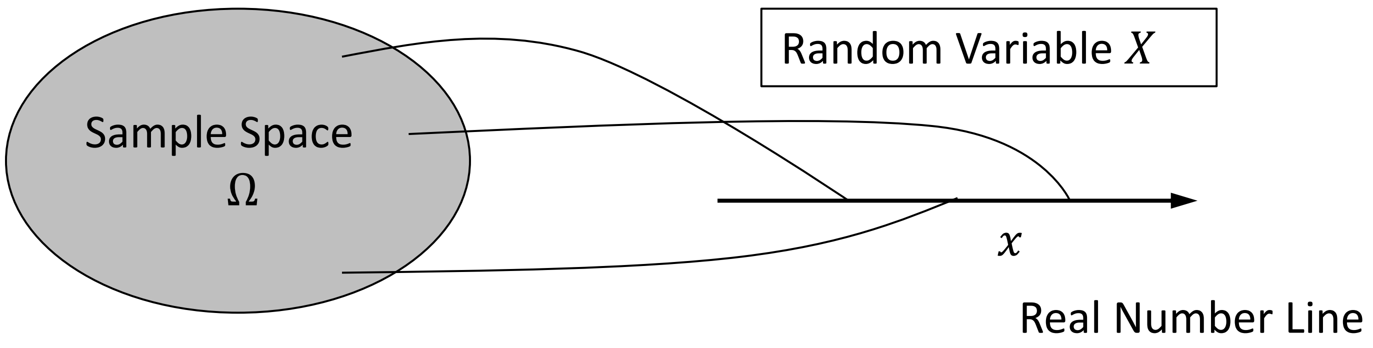
Spring 2017

Discrete Random Variable



- A random variable is discrete if its range is finite or countably infinite.
- Each discrete random variable has an associated **probability mass function** (**PMF**).

toss a coin	Head	Tail
Probability	$\frac{1}{2}$	$\frac{1}{2}$

Example 1: Discrete random variables

Experiment	Random Variable	Possible Values
Making 100 sales call	# sales	0,1,…,100
Inspect 70 radios	# defective	0,1,…,70
Answer 20 questions	# correct	0,1,…,20
Count cars at toll between 11:00-1:00	# cars arriving	0,1,2,……

Example 2: Soccer Game

- 2 games this weekend
- 0.4 probability----not losing the first game
- 0.7 probability----not losing the second game
- equally likely to win or tie
- independent
- 2 points for a win, 1 for a tie and 0 for a loss.

- Find the PMF of the number of points that the team earns in this weekend.

Example 2: Soccer Game

	X_1	2	1	0
X_2	P	0.2	0.2	0.6
2	0.35	0.07	0.07	0.21
1	0.35	0.07	0.07	0.21
0	0.3	0.06	0.06	0.18

$X = X_1 + X_2$	**4**	**3**	**2**	**1**	**0**
P	0.07	0.14	0.34	0.27	0.18

Compare: Continuous r.v. and PDFs

- A random variable X is called continuous if there is a function $f_X \geq 0$, called the probability density function of X, or PDF, s.t.

$$P(X \in B) = \int_B f_X(x)dx$$

for every subset $B \subseteq \mathbb{R}$.

- In particular, when $B = [a, b]$,

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx$$

is the area under the graph of PDF.

Continuous r.v. and PDFs

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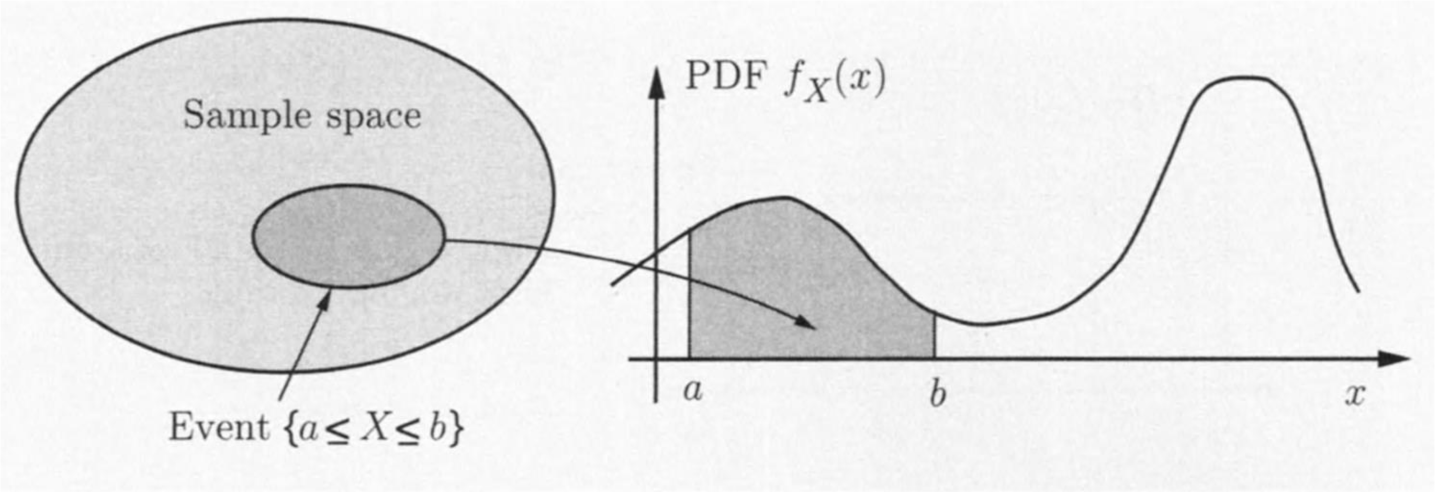
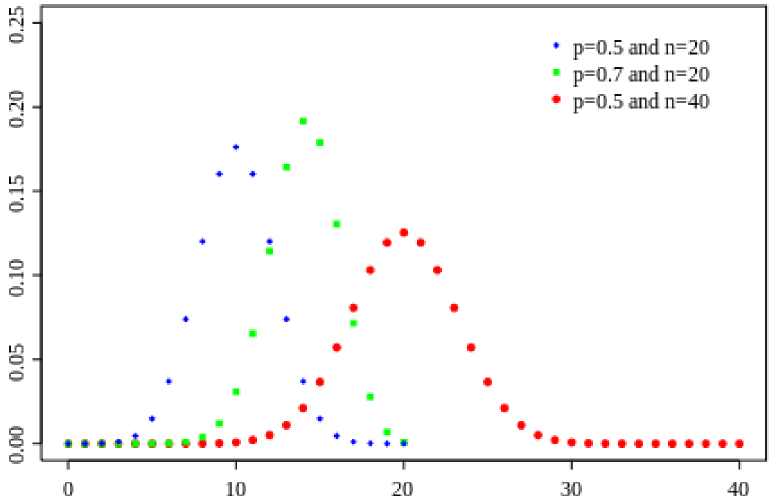


Typical Discrete Random Variables

- We review typical variables, and understand them using examples
  - Binomial Random Variable
  - Poisson Random Variable
  - Poisson Limit Theorem

The Binomial Random Variable

- We refer to X as a binomial random variable with parameters n and p .
- For $k = 0,1,\dots,n$
- $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$



- Mean: np, Mode:$[(n+1)p]$, variance:$np(1-p)$
- Notice that $\binom{n}{k} = \binom{n}{n-k}$, so $Bin(k|n,p) = Bin(n-k|n,1-p)$.

Example 3: Thinking Challenge

- You are taking a multiple choice test with 20 questions. Each question has 4 choices. The total score is 100 and each question has full score 5.
- (a) Clueless on question 1, you decide to guess. What is the chance you will get it right?

$$\frac{1}{4}$$

- (b) If you guessed all 20 questions, what is the probability that you get a score of exact 60?

$$\binom{20}{12}\left(\frac{1}{4}\right)^{12}\left(\frac{3}{4}\right)^{8}$$

Example 3: Thinking Challenge

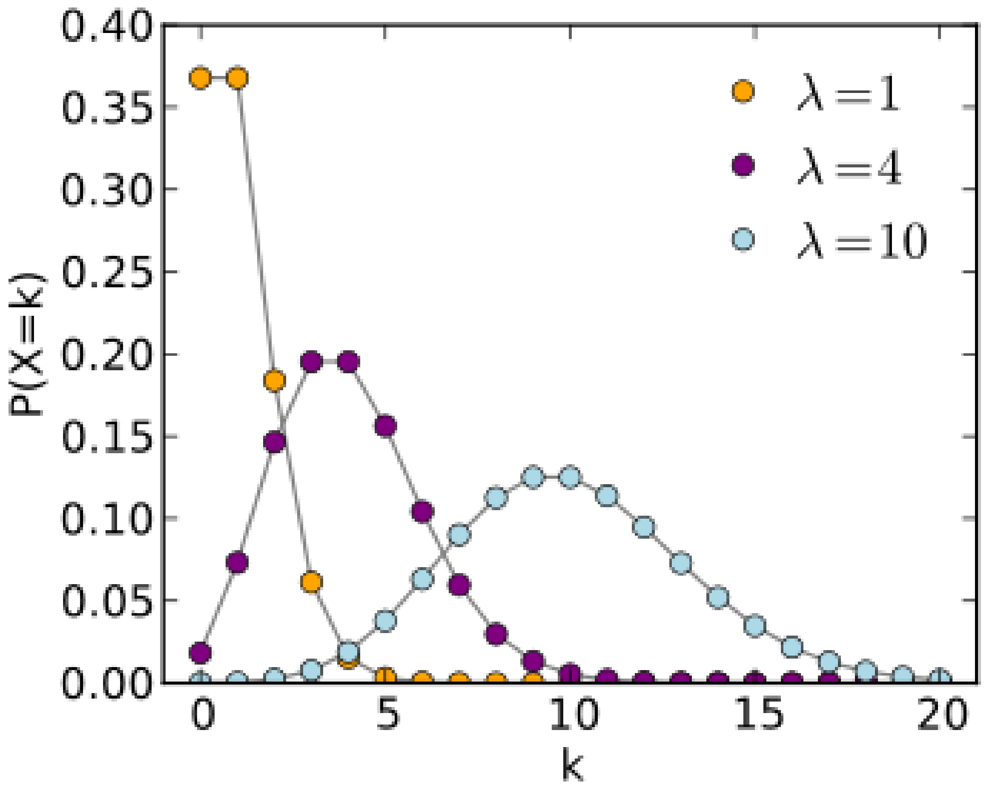
- You are taking a multiple choice test with 20 questions. Each question has 4 choices. The total score is 100 and each question has full score 5.
- (c) If you guessed all 20 questions, what is the probability that you get a score no less than 80?

$$\sum_{k=16}^{20} \binom{20}{k} \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{20-k}$$

- What is the result? Estimate?

The Poisson Random Variable

- A Poisson random variable X takes nonnegative integer values.
- The PMF
- $p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots,$



- Mean: λ, Mode: $[\lambda]$, variance: λ

Poisson Limit Theorem

- If $n \to \infty, p \to 0$ such that $np \to \lambda$, then

$$Bin(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k} \to \ e^{-\lambda} \frac{\lambda^k}{k!} = Poi(k|\lambda)$$

- $\sum_{k=16}^{20} \binom{20}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{20-k} = \sum_{k=0}^{4} \binom{20}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{20-k}$
- $\approx \sum_{k=0}^{4} Poi\left(k \middle| 20 \times \frac{3}{4}\right) = \sum_{k=0}^{4} e^{-15} \frac{15^k}{k!} \approx 8.5664 \times 10^{-4}$
- very small!!

Example 4: Birthday

- You go to a party with other 500 guests.
- What is the probability that exactly one other guest has the same birthday as you?
- (exclude birthdays on Feb 29.)

$$\binom{500}{1}\left(\frac{1}{365}\right)^{1}\left(\frac{364}{365}\right)^{499}$$

- By Poisson approximation, $\approx Poi(1|\frac{500}{365}) = e^{-\frac{500}{365}}\frac{500}{365} \approx 0.3481.$

Example 5: Phone calls

- The number of calls coming per minute is a Poisson random variable with mean 3.
- (a) Find the probability that no calls come in a given 1 minute period.

$$P = Poi(0|3) = e^{-3}$$

- (b) Find the probability that at least two calls will arrive in a given two minute period. (Assume independency.)
- $P(X_1 + X_2 \geq 2)$
  $= 1 - P(X_1 + X_2 < 2)$
  $= 1 - P(X_1 = X_2 = 0) - P(X_1 = 0, X_2 = 1)$
  $- P(X_1 = 1, X_2 = 0) = 1 - e^{-3}e^{-3} - e^{-3}e^{-3}3 - e^{-3}3e^{-3}$
  $= 1 - 7e^{-6}$

Example 6: Chess match

- Alice and Bob play a chess match
- the first player to win a game wins the match
- 0.4, the probability of Alice won
- 0.3, the probability of Bob won
- 0.3, the probability of a draw
- independent
- (a) What is the probability that Alice wins the match?

$$P = 0.4 + 0.3 \times 0.4 + 0.3^2 \times 0.4 + \cdots = 0.4 \times \frac{1}{1 - 0.3} = \frac{4}{7}$$

Example 6: Chess match

- Alice and Bob play a chess match
- the first player to win a game wins the match
- 0.4, the probability of Alice won
- 0.3, the probability of Bob won
- 0.3, the probability of a draw
- independent
- (b) What is the PMF of the duration of the match?
  $P(X = 1) = 0.7, \mathrm{P}(\mathrm{X} = 2) = 0.3 \times 0.7, P(X = 3) = 0.3^2 \times 0.7$
- So X follows geometric distribution with $p = 0.7$ on positive integers $\{1,2,3, \dots\}$.

Example 7: Functions of random variables

- Let X be a random variable that takes value from 0 to 9 with equal probability $1/10$.
- (a) Find the PMF of the random variable $Y = X\ mod(3)$.

X	0,3,6,9	1,4,7	2,5,8
Y	0	1	2
P	$\frac{4}{10} = 2/5$	$3/10$	$3/10$

Example 7: Functions of random variables

- Let X be a random variable that takes value from 0 to 9 with equal probability $1/10$.
- (b) Find the PMF of the random variable $Y = 5\ mod(X + 1)$.

X	0	1	2	3	4	5	6	7	8	9
Y	0	1	2	1	0	5	5	5	5	5

Y	0	1	2	5
P	$\frac{2}{10} = 1/5$	$\frac{2}{10} = 1/5$	$1/10$	$\frac{5}{10} = 1/2$