Tutorial 2: Sample Space and Probability 2

Baoxiang WANG bxwang@cse

Spring 2017

Probability models

A probability model is an assignment of probabilities to every element of the sample space.

Probabilities are nonnegative and add up to one.

Examples

$$S = \{\ \mathrm{HH},\ \mathrm{HT},\ \mathrm{TH},\ \mathrm{TT}\ \}$$

$$\tfrac{1}{4} \quad \tfrac{1}{4} \quad \tfrac{1}{4} \quad \tfrac{1}{4}$$

models a pair of coins with equally likely outcomes

Elements of a Probabilistic Model

- Event: a subset of sample space.
  - $A \subseteq \Omega$ is a set of possible outcomes
  - *Example*. $A = \{HH, TT\}$, the event that the two coins give the same side.
- The probability law assigns our knowledge or belief to an event A a number $P(A) \geq 0$.
  - It specifies the likelihood of any outcome.

Probability Axioms

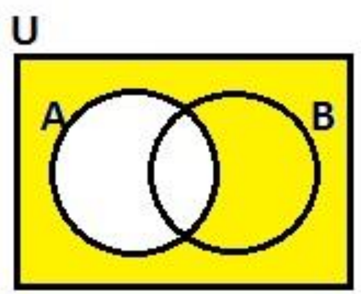
1. (*Non-negativity*) $P(A) \geq 0$, for every event A.

2. (*Additivity*) For any two disjoint events A and B, $P(A \cup B) = P(A) + P(B)$
   In general, if $A_1, A_2, \ldots$ are disjoint events, then
   $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

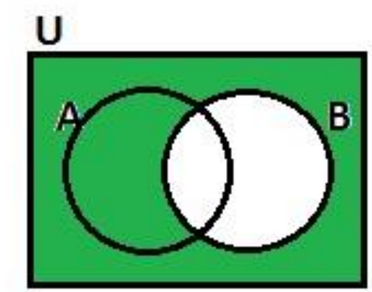
3. (*Normalization*) $P(\Omega) = 1$.

De Morgan’s laws

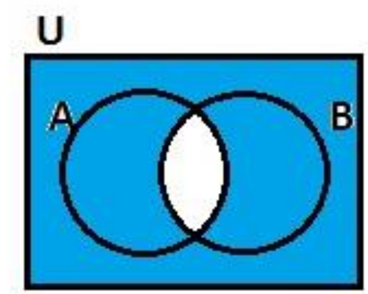
$(A \cup B)^{\mathrm{c}} = \mathrm{A}^{c} \cap B^{c}$ $\quad$ $(A \cap B)^{\mathrm{c}} = \mathrm{A}^{c} \cup B^{c}$



A'



B'



A'uB'

$$(A_1 \cup \cdots \cup A_n)^{\mathrm{c}} = A_1^c \cap \cdots \cap A_n^c$$

$$(A_1 \cap \cdots \cap A_n)^{\mathrm{c}} = A_1^c \cup \cdots \cup A_n^c$$

Question

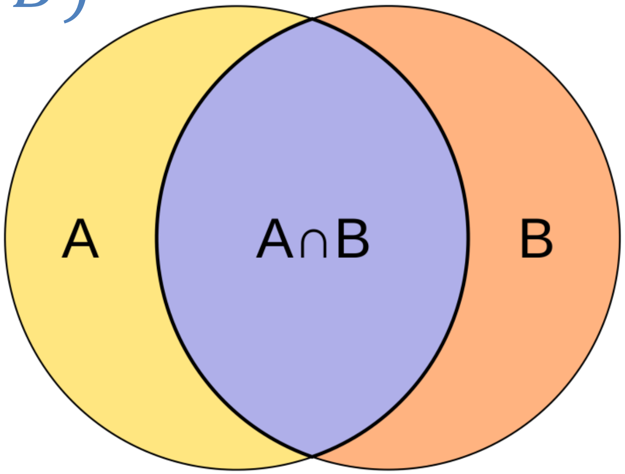
Bonferroni's inequality.

Prove that for any two events A and B, we have

$$P(A \cap B) \geq P(A) + P(B) - 1$$

$P(A) + P(B) - P(A \cap B) = P(A \cup B)$

$P(A \cup B) \leq 1$



Question

Romeo and Juliet have a date.

Each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being equally likely.

The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived.

Question: What is the probability that they will meet?

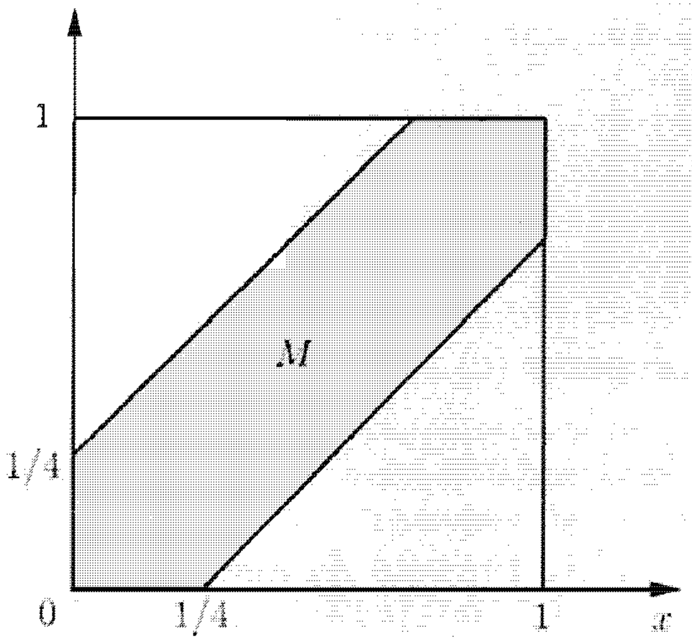
Question

Sample space: the unit square $[0,1] \times [0,1]$,

Its elements are the possible pairs of delays.

"equally likely" pairs of delays: let $P(A)$ for event $A \subseteq \Omega$ be equal to A's "area".

This satisfies the axioms.



$$M = \left\{(x, y): |x - y| \leq \frac{1}{4}, 0 \leq x \leq 1, 0 \leq y \leq 1\right\}$$

Question

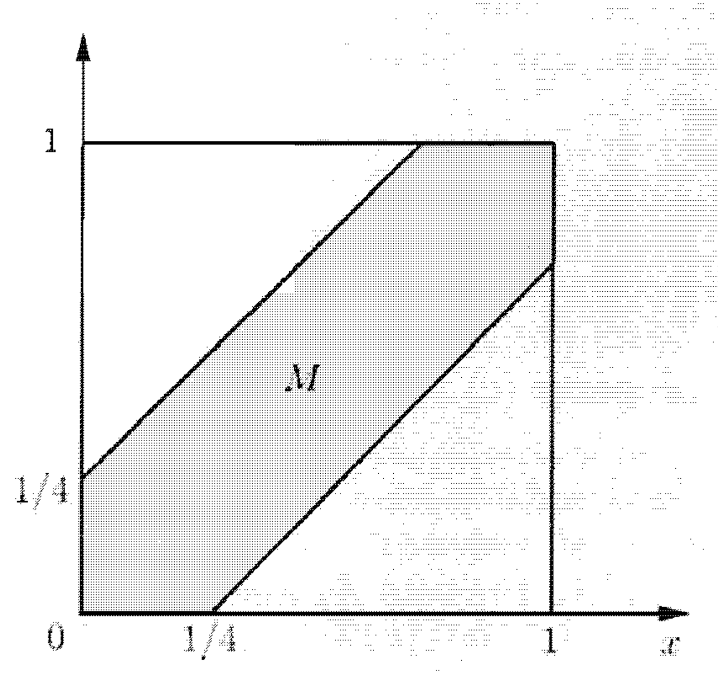
The event that Romeo and Juliet will meet is the shaded region.

Its probability is calculated to be 7/16.

$= 1 -$ the area of the two unshaded triangles

$= 1 - 2 \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right)/2$

$= 7/16.$



$$M = \left\{(x, y): |x - y| \leq \frac{1}{4}, 0 \leq x \leq 1, 0 \leq y \leq 1\right\}$$

Question

A parking lot contains 100 cars, k of which happen to be lemons. We select m of these cars at random and take them for a test drive. Find the probability that n of the cars tested turn out to be lemons.

The sample space

$$\Omega = \{\text{random choose } m \text{ cars}\}$$

The size of sample space

$$|\Omega| = \binom{100}{m}$$

Question

A parking lot contains 100 cars, k of which happen to be lemons. We select m of these cars at random and take them for a test drive. Find the probability that n of the cars tested turn out to be lemons.

The event n cars are lemons ($n \leq k$ and $n \leq m$)

$$\binom{k}{n} \cdot \binom{100-k}{m-n}$$

The probability

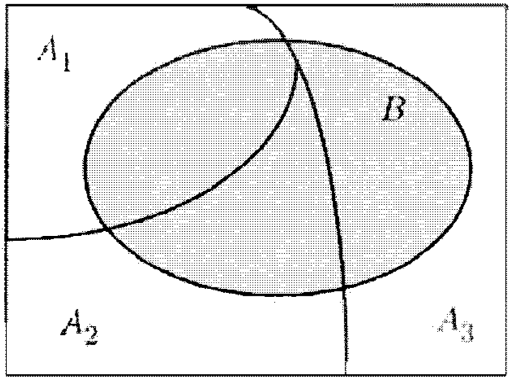
$$P = \binom{k}{n} \cdot \binom{100-k}{m-n} / \binom{100}{m}$$

Question: Conditioning

- Consider ten independent rolls of a 6-sided die. Let X be the number of $6s$ and let Y be the number of $1s$ obtained.
- What is the joint PMF of X and Y?
- X is a binomial random variable with $n = 10$ and $p = \frac{1}{6}$
- $P(Y = b|X = a)$ is a binomial random variable with $n = 10 - a$ and $p = \frac{1}{5}$
- So the joint PMF is

$$P(X = a, y = b) = P(X = a)P(Y = b|X = a)$$
$$= \binom{10}{a}\left(\frac{1}{6}\right)^a (\frac{5}{6})^{10-a}\binom{10-a}{b}(\frac{1}{5})^b(\frac{4}{5})^{10-a-b}$$

Total Probability Theorem



- Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space. Assume $P(A_i) > 0$ for all i. Then, for any event B, we have

$$\begin{aligned} P(B) &= P(A_1 \cap B) + \cdots + P(A_n \cap B) \\ &= P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n) \end{aligned}$$

- Indeed, B is the the disjoint union of $(A_1 \cap B)$, …, $(A_n \cap B)$.
- The second equality is given by

$$P(A_i \cap B) = P(A_i)P(B|A_i).$$

Question

We know that a treasure is located in one of two places, with probabilities β and $1-\beta$, respectively, where $0 < \beta < 1$. We search the first place and if the treasure is there, we find it with probability $p > 0$.

Show that the event of not finding the treasure in the first place suggests that the treasure is in the second place

Let A and B be the events

$A = \{\text{the treasure is in the second place}\}$

$B = \{\text{not find the treasure in the first place}\}$

Question

We know that a treasure is located in one of two places, with probabilities β and $1-\beta$, respectively, where $0<\beta<1$. We search the first place and if the treasure is there, we find it with probability $p>0$.

Show that the event of not finding the treasure in the first place suggests that the treasure is in the second place

By the total probability theorem

$$P(B) = P(A^c)P(B|A^c) + P(A)P(B|A)$$
$$= \beta(1-p) + (1-\beta)$$

Question

We know that a treasure is located in one of two places, with probabilities β and $1-\beta$, respectively, where $0<\beta<1$. We search the first place and if the treasure is there, we find it with probability $p>0$.

Show that the event of not finding the treasure in the first place suggests that the treasure is in the second place

By the total probability theorem

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{1-\beta}{\beta(1-p)+(1-\beta)}$$

$$=\frac{1-\beta}{1-\beta p}>1-\beta=P(A)$$

Independence

- Independent Events

- A, B are independent $\Longleftrightarrow$ B provides no information of A

$$P(A|B) = P(A)$$

- Equivalently

$$P(A \cap B) = P(A)P(B)$$

Question

A hunter has two hunting dogs. On a two-path road, each dog will choose the correct path with probability p independently. Let each dog choose a path, and if they agree, take that one, and if they disagree, randomly pick a path. Is this strategy better than just letting one of the two dogs decide on a path?

Both dogs agree on the correct path

$$p \cdot p = p^2$$

The dogs disagree

$$p(1-p) \cdot 2$$

Question

A hunter has two hunting dogs. On a two-path road, each dog will choose the correct path with probability p independently. Let each dog choose a path, and if they agree, take that one, and if they disagree, randomly pick a path. Is this strategy better than just letting one of the two dogs decide on a path?

Hunter chooses the correct path when dogs disagree

$$p(1-p)$$

The probability that hunter chooses the correct path

$$p^2 + p(1-p) = p$$

Question

There are n different power plants. And each one can produce enough electricity to supply the entire city. Now suppose that the ith power plants fails independently with probability p_i.

(a) What is the probability that the city will experience a black-out?

Let A denote the event that the city experiences a black-out.

Since the power plants fail independently

$$P(A) = \prod_{i=1}^{n} p_i$$

Question

There are n different power plants. And each one can produce enough electricity to supply the entire city. Now suppose that the ith power plants fails independently with probability p_i.

(b) Now suppose two power plants are necessary to keep the city from a black-out. Find the probability that the city will experience a black-out.

The city will black out if either all n or any $n - 1$ power plants fail

These two events are mutually exclusive

Question

There are n different power plants. And each one can produce enough electricity to supply the entire city. Now suppose that the ith power plants fails independently with probability p_i.

(b) Now suppose two power plants are necessary to keep the city from a black-out. Find the probability that the city will experience a black-out.

$$P(A) = \prod_{i=1}^{n} p_i + \sum_{i=1}^{n} \left((1 - p_i) \prod_{j \neq i} p_j \right)$$

Thanks

Question are welcome.