Tutorial 10: Limit Theorems

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Markov and Chebyshev Inequalities

- These inequalities use the mean and possibly the variance of a random variable to draw conclusions on the probabilities of certain events.
- primarily useful in situations
  - exact values or bounds for the mean and variance of a random variable X are easily computable.
  - but the distribution of X is either unavailable or hard to calculate.

Markov inequality

- If a random variable X can only take nonnegative values, then

$$P(X \geq a) \leq \frac{E[X]}{a}, \quad \text{for all } a > 0$$

Loosely speaking, it asserts that

- if a nonnegative random variable has a small mean, then the probability that it takes a large value must also be small.

Example 1

- Let X be uniformly distributed in the interval $[0,4]$ and note that $E[X] = 2$.
- Then, the Markov inequality asserts that

$$P(X \geq 2) \leq \frac{2}{2} = 1.$$

$$P(X \geq 3) \leq \frac{2}{3} = 0.67.$$

$$P(X \geq 4) \leq \frac{2}{4} = 0.5.$$

Example 1

- By comparing with the exact probabilities

$$P(X \geq 2) = 0.5.$$
$$P(X \geq 3) = 0.25.$$
$$P(X \geq 4) = 0.$$

- We see that the bounds provided by the Markov inequality can be quite loose.

Chebyshev inequality

- If X is a random variable with mean μ and variance σ^2, then

$$P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}, \qquad \text{for all } c > 0$$

Example 2: Uninformative case

- Let X be uniformly distributed in $[0,4]$.
- Let us use the Chebyshev inequality to bound the probability that $|X - 2| \geq 1$.
- We have $\sigma^2 = 16/12 = 4/3$, and

$$P(|X - 2| \geq 1) \leq 4/3$$

Which is uninformative.

Example 2: Uninformative case

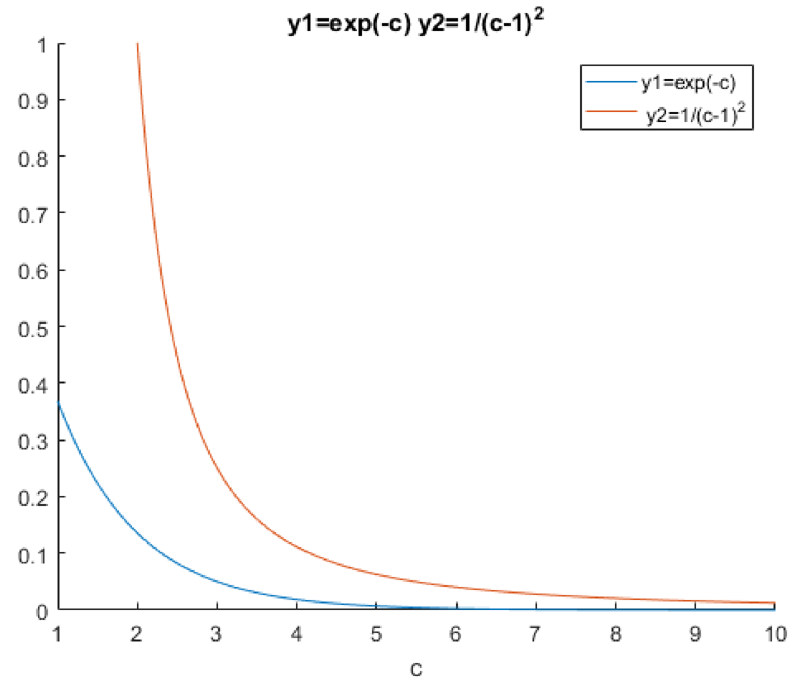
- let X be exponentially distributed with parameter $\lambda = 1$, so that $E[X] = var(X) = 1$.
- For $c > 1$, using the Chebyshev inequality, we obtain $P(x \geq c) = P(X - 1 \geq c - 1)$

$$\leq P(|X - 1| \geq c - 1)$$

$$\leq \frac{1}{(c-1)^2}$$

Example 2: Uninformative case

- $\frac{1}{(c-1)^2}$ is again conservative compared to the exact answer $P(X \geq c) = e^{-c}$



Weak law of large numbers

- The law asserts that the sample mean of a large number of independent identically distributed random variables is very close to the true mean, with high probability.

Weak law of large numbers

- Consider a sequence $X_1, X_2, \ldots$ of independent identically distributed random variables with mean μ and variance σ^2,
- and define the sample mean by

$$M_n = \frac{X_1 + \cdots + X_n}{n}$$

Weak law of large numbers

- We have

$$\mathrm{E}[M_n] = \frac{E[X_1]+\cdots+E[X_n]}{n} = \frac{n\mu}{n} = \mu$$

and, using independence,

$$\mathrm{var}(M_n) = \frac{var(X_1 + \cdots + X_n)}{n^2}$$

$$= \frac{var(X_1) + \cdots + var(X_n)}{n^2}$$

$$\frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Weak law of large numbers

- Let $X_1, X_2, \ldots$ be independent identically distributed random variables with mean μ.

- For every $\epsilon > 0$, we have

$$P(|M_n - \mu| \geq \epsilon)$$
$$= P\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \epsilon\right)$$
$$\to 0, \qquad as\, n \to \infty$$

Convergence in Probability

- Let $Y_1, Y_2, \ldots$ be a sequence of random variables, and let a be a real number.

- We say that the sequence Y_n converges to a in probability, if for every $\epsilon > 0$, we have

$$\lim_{n \to \infty} \mathrm{P}(|Y_n - a| \geq \epsilon) = 0.$$

Example 3

- Consider a sequence of independent random variables X_n that are uniformly distributed in the interval $[0,1]$,
- and let

$$Y_n = \min\{X_1, \ldots X_n\}.$$

Example 3

- The sequence of values of Y_n cannot increase as n increases,
- and it will occasionally decrease
- whenever a value of X_n that is smaller than the preceding values is obtained.

Example 3

- Thus, we expect that Y_n converges to zero. Indeed, for $\epsilon > 0$,we have using the independence of the X_n,

$$\begin{aligned} P(|Y_n - 0| \geq \epsilon) &= P(X_1 \geq \epsilon, \dots X_n \geq \epsilon) \\ &= P(X_1 \geq \epsilon), \dots P(X_n \geq \epsilon) \\ &= (1 - \epsilon)^n \end{aligned}$$

Example 3

- In particular,

$$\lim_{n\to\infty} P(|Y_n - 0| \geq \epsilon) = \lim_{n\to\infty} (1-\epsilon)^n = 0$$

Since this is true for every $\epsilon > 0$, we conclude that Y_n converges to zero, in probability.

Example 4

- In order to estimate f. the true fraction of smokers in a large population, Alvin selects n people at random. His estimator M_n is obtained by dividing S_n, the number of smokers in his sample, by n, i.e. $M_n = \frac{S_n}{n}$. Alvin Chooses the sample size n to be the smallest possible number for which the Chebyshev inequality yields a guarantee that,

$$P(|M_n - f| \geq \epsilon) \leq \delta,$$

- where ϵ and δ are some pre-specified tolerances. Determine how the value of n recommended by the Chevyshev inequality changes in the following cases.
- A) The value of epsilon is reduced to half its original value.
- B) The probability delta is reduced to half its original value.

Example 4

- Hint:

$$\mathrm{P}(|M_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}, \text{ for any } \epsilon > 0$$

- A)$n' = 4n$
- B)$n' = 2n$