Tutorial 1: Sample Space and Probability 1

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Probability models

A probability model is an assignment of probabilities to every element of the sample space.

Probabilities are nonnegative and add up to one.

Examples

$$S = \{\ \mathrm{HH},\ \mathrm{HT},\ \mathrm{TH},\ \mathrm{TT}\ \}$$

$$\tfrac{1}{4} \quad \tfrac{1}{4} \quad \tfrac{1}{4} \quad \tfrac{1}{4}$$

models a pair of coins with equally likely outcomes

Elements of a Probabilistic Model

- Event: a subset of sample space.
  - $A \subseteq \Omega$ is a set of possible outcomes
  - *Example*. $A = \{HH, TT\}$, the event that the two coins give the same side.
- The probability law assigns our knowledge or belief to an event A a number $P(A) \geq 0$.
  - It specifies the likelihood of any outcome.

Probability Axioms

1. (*Non-negativity*) $P(A) \geq 0$, for every event A.

2. (*Additivity*) For any two disjoint events A and B, $P(A \cup B) = P(A) + P(B)$
   In general, if $A_1, A_2, \ldots$ are disjoint events, then
   $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

3. (*Normalization*) $P(\Omega) = 1$.

Question

If 4 boys and 4 girls randomly sit in a row, what is the probability of the following seating arrangement

(a) the boys and the girls are each to sit together

The total number of all arrangements is

$$8! = 40320$$

The number of target arrangements is

$$4! \times 4! \times 2 = 1152$$

The probability

$$\frac{\#\{target\ arrangements\}}{\#\{all\ arrangements\}} = \frac{1152}{40320} = \frac{1}{35}$$

Question

If 4 boys and 4 girls randomly sit in a row, what is the probability of the following seating arrangement

(b) only the boys must sit together

The total number of all arrangements is

$$8! = 40320$$

The number of target arrangements is

$$4! \times 5! = 2880$$

The probability

$$\frac{\#\{target\ arrangements\}}{\#\{all\ arrangements\}} = \frac{2880}{40320} = \frac{1}{14}$$

Question

If 4 boys and 4 girls randomly sit in a row, what is the probability of the following seating arrangement

(c) no two people of the same sex are allowed to sit together

The total number of all arrangements is

$$8! = 40320$$

The number of target arrangements is

$$4! \times 4! \times 2 = 1152$$

The probability

$$\frac{\#\{target\ arrangements\}}{\#\{all\ arrangements\}} = \frac{1152}{40320} = \frac{1}{35}$$

Question

Eight balls are randomly withdrawn from an urn that contains 8 red, 8 blue, and 8 green balls. Find the probability that

(a) 4 red, 2 blue, and 2 green balls are withdrawn

The total number of all combinations is

$$\binom{24}{8}=735471$$

The number of target combinations is

$$\binom{8}{4}\times\binom{8}{2}\times\binom{8}{2}=54880$$

The probability

$$\frac{\#\{target\ combinations\}}{\#\{all\ combinations\}}=\frac{54880}{735471}$$

Question

Eight balls are randomly withdrawn from an urn that contains 8 red, 8 blue, and 8 green balls. Find the probability that

(b) at least 3 red balls are withdrawn

The total number of all combinations is

$$\binom{24}{8}=735471$$

The number of target combinations is

$$\binom{24}{8}-\binom{8}{2}\times\binom{16}{6}-\binom{8}{1}\times\binom{16}{7}-\binom{16}{8}=406857$$

The probability

$$\frac{\#\{target\ combinations\}}{\#\{all\ combinations\}}=\frac{12329}{22287}$$

Question

Eight balls are randomly withdrawn from an urn that contains 8 red, 8 blue, and 8 green balls. Find the probability that

(c) all withdrawn balls are of the same color

The total number of all combinations is

$$\binom{24}{8}=735471$$

The number of target combinations is

$$3$$

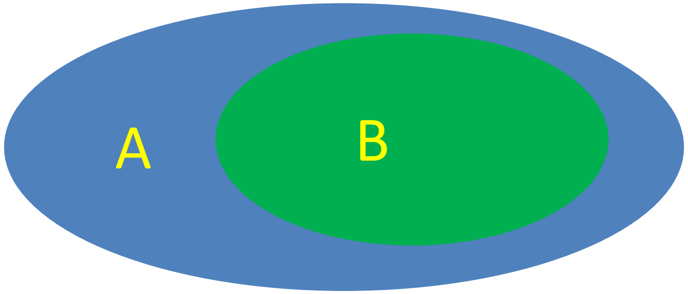
The probability

$$\frac{\#\{target\ combinations\}}{\#\{all\ combinations\}} = \frac{1}{245247}$$

Question

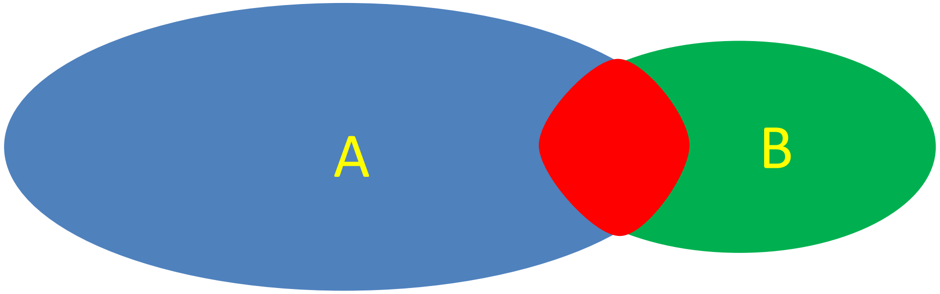
Suppose P(A) = $\frac{2}{3}$ and P(B) = $\frac{2}{5}$. Show that $\frac{1}{15} \le P(A \cap B) \le \frac{2}{5}$

Best case:



$P(A \cap B) \le \frac{2}{5}$

Worst case:



$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1$

$P(A \cap B) \ge \frac{1}{15}$

Question

Assume that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(a) a royal flush? (A hand of 10, J, Q, K, A of the same suit.)

There are 4 suits in total. So the probability is

$$P = \frac{4}{\binom{52}{5}}$$

Question

Assume that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(b) a straight flush? (A hand of five adjacent values of the same suit, but not a royal flush. Note that A, 2, 3, 4, 5 counts as a straight flush but K, A, 2, 3, 4 doesn't.)

For each suit, the number of straight flush is

$$9$$

There are 4 suits, so the total number of straight flush

$$9 \cdot 4 = 36$$

Question

Assume that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(b) a straight flush? (A hand of five adjacent values of the same suit, but not a royal flush. Note that A, 2, 3, 4, 5 counts as a straight flush but K, A, 2, 3, 4 doesn't.)

The probability is

$$P = \frac{36}{\binom{52}{5}}$$

Question

Assume that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(c) a flush? (A hand of the same suit, but it is not a straight flush or royal flush.)

In one suit, the number of special flush is 10.

The number of flush in one suit

$$\binom{13}{5} - 10 = 1227$$

Question

Assume that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(c) a flush? (A hand of the same suit, but it is not a straight flush or royal flush.)

The probability is

$$P = \frac{4 \cdot 1227}{\binom{52}{5}}$$

Question

Assume that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(d) a full house? (A hand with three cards of the same value, plus two cards with the same value as each other.)

$X =$ the value of three cards $Y =$ the value of two cards

All possible pairs of {X,Y}

$$\binom{13}{2} \cdot 2 = 156$$

Question

Assume that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(d) a full house? (A hand with three cards of the same value, plus two cards with the same value as each other.)

Choose three cards of the same value

$$\binom{4}{3}$$

Choose two cards of the same value

$$\binom{4}{2}$$

Question

Assume that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(d) a full house? (A hand with three cards of the same value, plus two cards with the same value as each other.)

The probability is

$$P = \frac{156 \cdot \binom{4}{3} \cdot \binom{4}{2}}{\binom{52}{5}}$$

Conditional Probability

- *Definition*. Conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

where we assume that $P(B) > 0$.

  - If $P(B) = 0$: then $P(A|B)$ is undefined.

- *Fact*. $P(A|B)$ form a legitimate probability law satisfying the three axioms.

Question

Let A and B be events with $P(A) > 0$ and $P(B) > 0$. We say that an event B suggests an event A if $P(A|B) > P(A)$, and does not suggest event A if $P(A|B) < P(A)$.

(a) Show that B suggests A if and only if A suggests B.

$$P(A|B) = P(A \cap B)/P(B)$$

B suggests A if and only if

$$P(A \cap B) > P(A)P(B)$$

Equivalent to A suggesting B

Question

Let A and B be events with $P(A) > 0$ and $P(B) > 0$. We say that an event B suggests an event A if $P(A|B) > P(A)$, and does not suggest event A if $P(A|B) < P(A)$.

(b) Show that B suggests A if and only if B^c does not suggest A. Assume that $P(B^c) > 0$.

B^c does not suggest A if and only if

$$P(A \cap B^c) < P(A)P(B^c)$$

We have

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

And

$$P(B^c) = 1 - P(B)$$

Question

Let A and B be events with $P(A) > 0$ and $P(B) > 0$.
We say that an event B suggests an event A if $P(A|B) > P(A)$, and does not suggest event A if $P(A|B) < P(A)$.
(b) Show that B suggests A if and only if B^c does not suggest A. Assume that $P(B^c) > 0$.

Substituting in the previous inequality

$$P(A) - P(A \cap B) < P(A)(1 - P(B))$$

Or

$$P(A \cap B) > P(A)P(B)$$

Equivalent to A suggesting B

Question

We know that a treasure is located in one of two places, with probabilities β and $1 - \beta$, respectively, where $0 < \beta < 1$. We search the first place and if the treasure is there, we find it with probability $p > 0$.

Show that the event of not finding the treasure in the first place suggests that the treasure is in the second place

Let A and B be the events

$$A = \{\text{the treasure is in the second place}\}$$

$$B = \{\text{not find the treasure in the first place}\}$$

Question

We know that a treasure is located in one of two places, with probabilities β and $1-\beta$, respectively, where $0<\beta<1$. We search the first place and if the treasure is there, we find it with probability $p>0$.

Show that the event of not finding the treasure in the first place suggests that the treasure is in the second place

By the total probability theorem

$$P(B) = P(A^c)P(B|A^c) + P(A)P(B|A)$$
$$= \beta(1-p) + (1-\beta)$$

Question

We know that a treasure is located in one of two places, with probabilities β and $1-\beta$, respectively, where $0<\beta<1$. We search the first place and if the treasure is there, we find it with probability $p>0$.
Show that the event of not finding the treasure in the first place suggests that the treasure is in the second place

By the total probability theorem

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{1-\beta}{\beta(1-p)+(1-\beta)}$$

$$=\frac{1-\beta}{1-\beta p}>1-\beta=P(A)$$

Thanks

Question are welcome.