CMSC5706 Topics in Theoretical Computer Science

Week 7: Stable Matching

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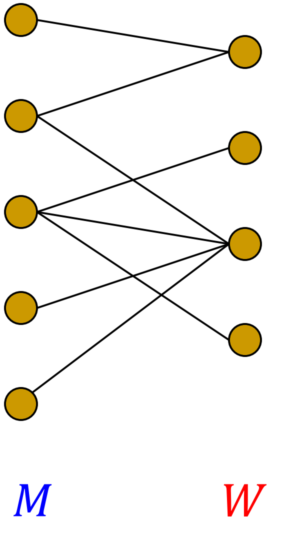
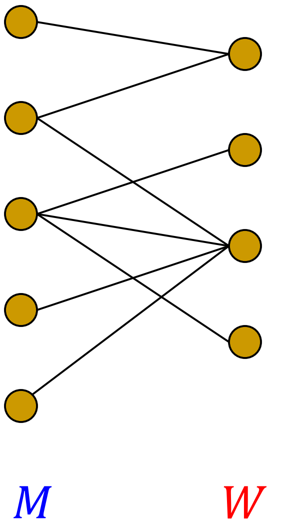
Bipartite graph

- (Undirected) Bipartite graph:
- $G = (V, E)$ for which V can be partitioned into two parts
  - $V = M \cup W$ with $M \cap W = \emptyset$,
- And all edges $e = (m, w)$ have $m \in M$ and $w \in W$.



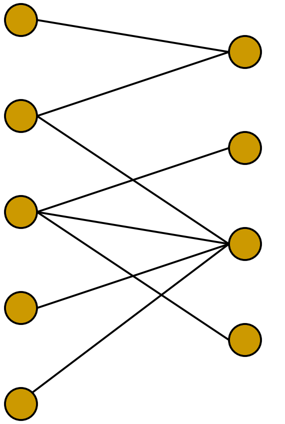
Matching, maximum matching

- Matching: a collection of vertex-disjoint edges
  - a subset $E' \subseteq E$ s.t. no two edges $e, e' \in E'$ are incident.
- $|E'|$: size of matching.
- Maximum matching: a matching with the maximum size.
- This lecture: matching in a bipartite graph



Perfect matching

- There may be some vertices not incident to any edge.
- Perfect matching: a matching with no such isolated vertex.
  - needs at least: $|M| = |W|$
- We'll assume $|M| = |W|$ in the rest of the lecture.



Men's Preference

- Suppose a man sees these women.




- He has a preference among them.
  - What's your preference list?
- Different men may have different lists.

Women's preference

- Women also have their preference lists.

- Assume no tie.
  - The general case can be handled similarly.

Setting

- n men, n women
- Each man has a preference list of all women
- Each woman has a preference list of all men
- We want to match them.

$w_1 > w_2 > w_3 > w_4$	m_1	w_1	$m_3 > m_1 > m_2 > m_4$
$w_1 > w_2 > w_3 > w_4$	m_2	w_2	$m_3 > m_4 > m_1 > m_2$
$w_2 > w_1 > w_3 > w_4$	m_3	w_3	$m_1 > m_4 > m_2 > m_3$
$w_3 > w_2 > w_4 > w_1$	m_4	w_4	$m_4 > m_1 > m_3 > m_2$

Setting

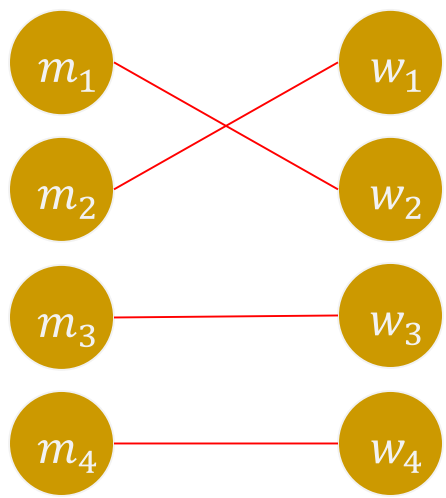
- Consider this matching.
- And this pair (m_1, w_1).
  - m_1 is matched to w_2, but he likes w_1 more.
  - w_1 is matched to m_2, but she likes w_1 more.
- What if m_1 and w_1 meet one day?

$w_1 > w_2 > w_3 > w_4$

$w_1 > w_2 > w_3 > w_4$

$w_2 > w_1 > w_3 > w_4$

$w_3 > w_2 > w_4 > w_1$



$m_3 > m_1 > m_2 > m_4$

$m_3 > m_4 > m_1 > m_2$

$m_1 > m_4 > m_2 > m_3$

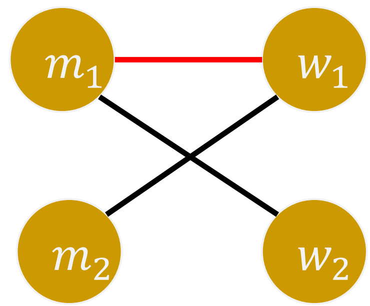
$m_4 > m_1 > m_3 > m_2$

A stability property

- Suppose there are two couples with these preferences.

$w_1 > w_2$

$w_1 > w_2$



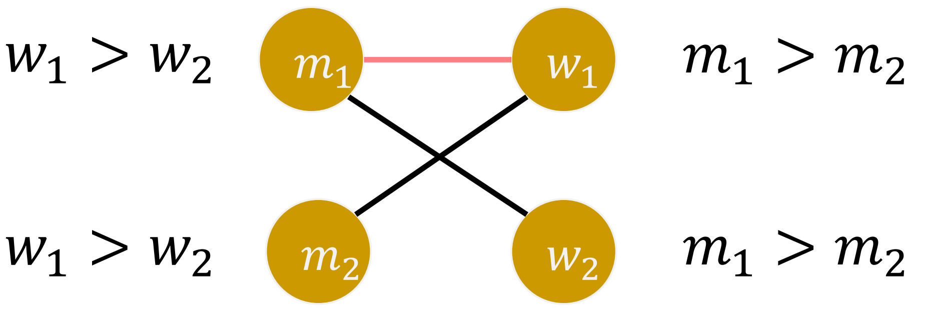
$m_1 > m_2$

$m_1 > m_2$

- The marriage is unstable, because m_1 and w_1 like each other more than their currently assigned ones!

Stability

- Such a pair is called a blocking pair.



- *Question: Can we have a matching without any blocking pair?*
  - Such a matching is then called a stable matching.

Real applications

- If you think marriage is a bit artificial since there is no centralized arranger, here is a real application.
- Medical students work as interns at hospitals.
  - In the US more than 20,000 medical students and 4,000 hospitals are matched through a clearinghouse, called NRMP (National Resident Matching Program).

Real applications

- Students and hospitals submit preference rankings to the clearinghouse, who uses a specified rule to decide who works where.

- *Question: What is a good way to match students and hospitals?*

More than one question

- *Question: Does a stable matching always exist?*
- *Question: If yes, how to find one?*
- *Question: What mathematical / economic properties it has?*

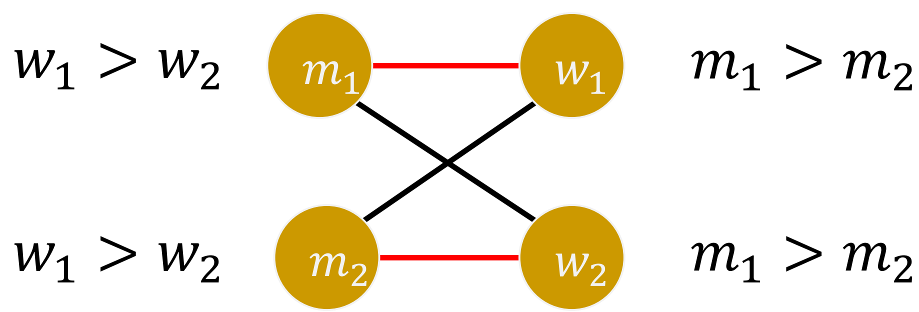
Good news: Stable matchings always exist.

- Theorem (Gale-Shapley) For any given preference lists, there always exists a stable matching.

- They actually gave an algorithm, which bears some resemblance to real marriages.

Consider a simple dynamics

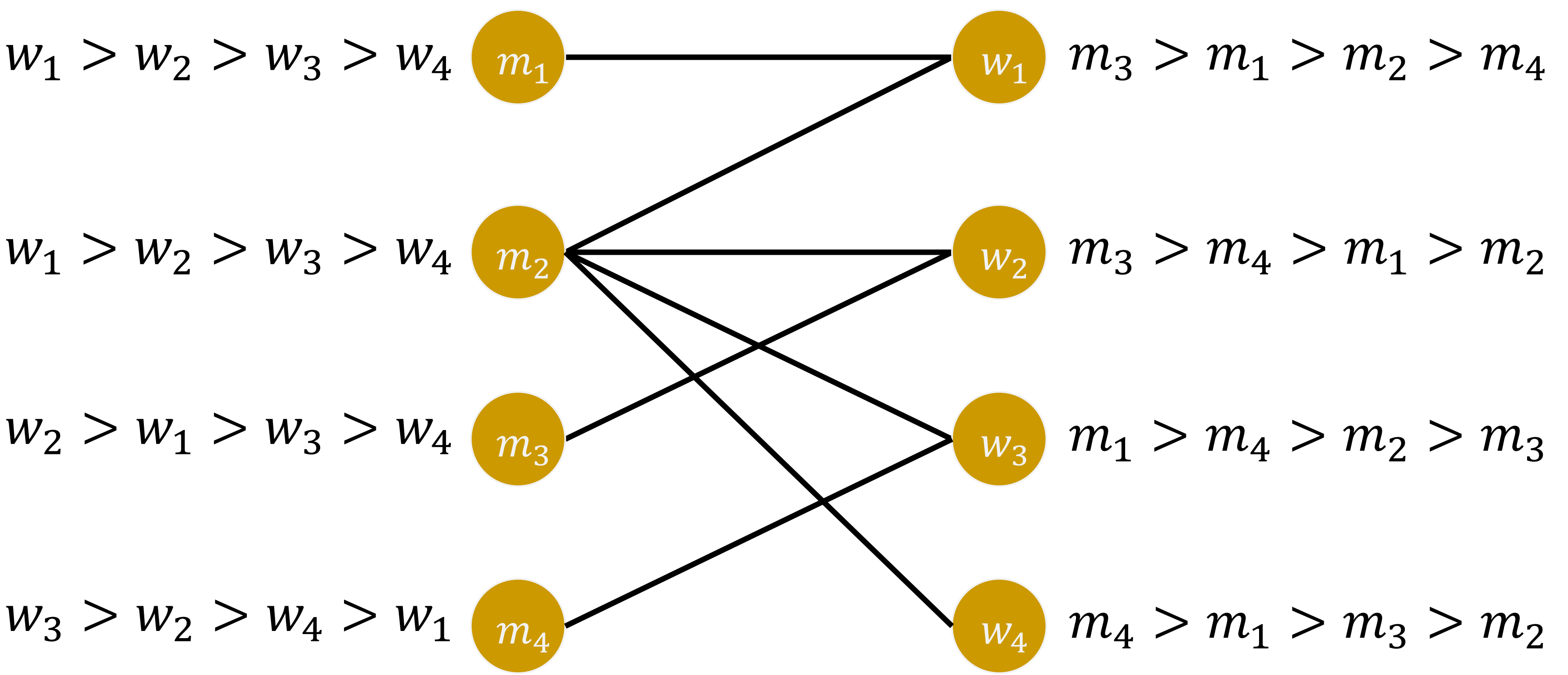
- ∀ matching f, ∀ blocking pair (m, w),
  - Remove the old pairing $\big(m, f(m)\big)$ and $\big(w, f(w)\big)$
    - $f(m)$: the woman matched to m in f.   ($f(w)$: similar.)
  - Match m and w
  - Match $f(m)$ and $f(w)$
- Question: Would repeating this finally lead to a stable matching?



Example

- Can you find an counterexample?
- Next we’ll give an algorithm that actually works.
- Let’s first run the algorithm on an example.

Algorithm by an example



Gale-Shapley (Deferred-Acceptance) Algorithm

- Initially all men and women are free
- **while** there is a man m who is free and hasn't proposed to every woman
  - choose such a man m arbitrarily
  - let w be the highest ranked woman in m's preference list to whom m hasn't proposed yet
  - *// next: m proposes to w*
  - **if** w is free, **then** (m, w) become engaged
  - **else**, suppose w is currently engaged to m'
    - **if** w prefers m' to m, **then** m remains free
    - **if** w prefers m to m', **then** (m, w) becomes engaged and m' becomes free
- Return the set of engaged pairs as a matching

Analysis of the algorithm

- We will show the following:

1. The algorithm always terminates…
2. … in $O(n^2)$ steps, // n men and n women.
3. and generates a stable matching.

Some observations

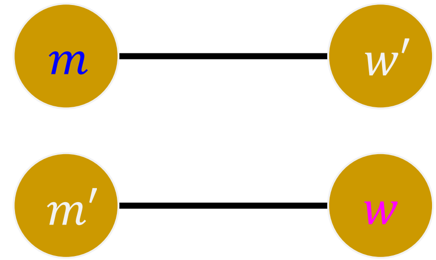
- In each iteration, one man m proposes to a new woman w.
- For any man: The women he proposes to get worse and worse
  - according to his preference list
- Because he proposes to a new woman only when the previous one dumps him
  - forcing him to try next (worse!) ones.

Time bound

- Each man proposes at most n steps.
  - since his proposed women are worse and worse
- There are n men.
- Therefore: at most n^2 proposals.
- Since each iteration has exactly one proposal, there are at most n^2 iterations.
- Theorem. Gale-Shapley algorithm terminates after at most n^2 iterations.

Correctness

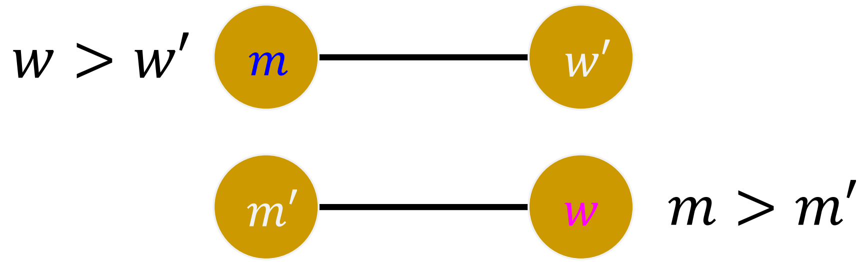
$w > w'$



$m > m'$

- Suppose the algorithm returns a matching f with a blocking pair (m, w),
  - i.e. m prefers w to w' and w prefers m to m', where w' and m' are their current partner.
- Note: m's last proposal was to w'; see the algorithm.
- m has proposed to w before to w'.
  - Since m proposes from best to worst.
- But at the end of the day, w chose m'
- So m' also proposed to w at some point.

Correctness



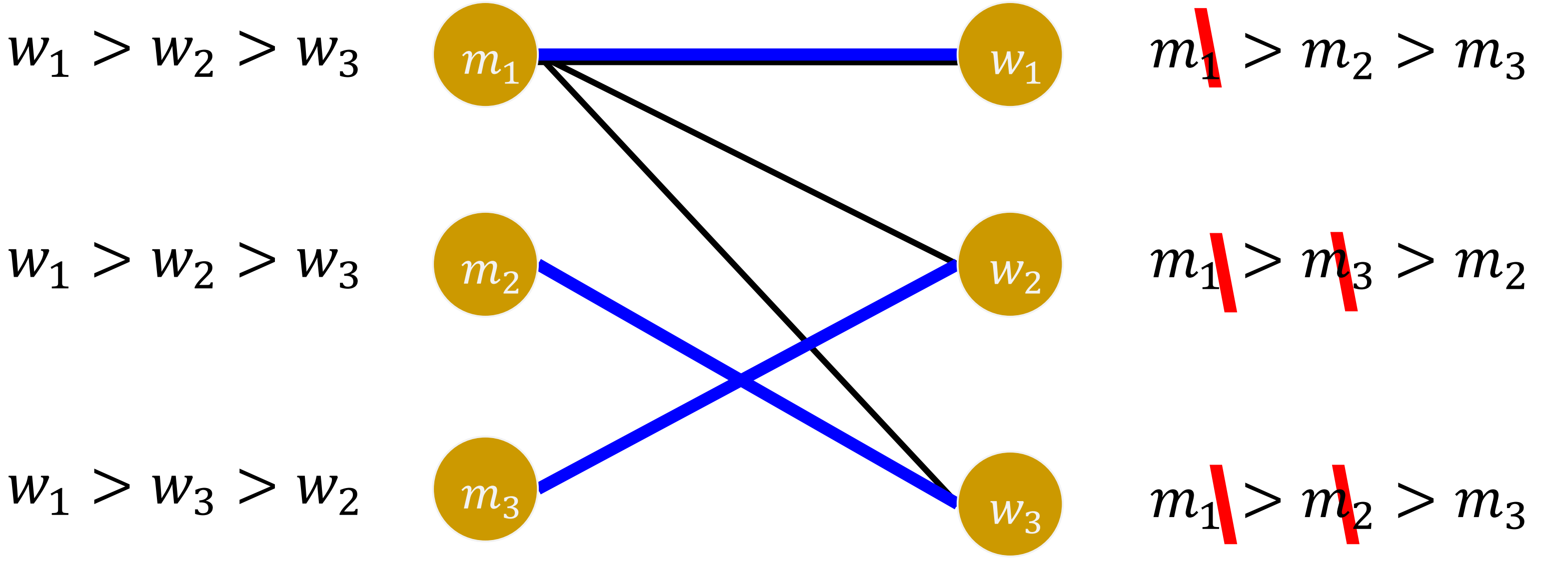
- Suppose the algorithm returns a matching f with a blocking pair (m, w),
  - i.e. m prefers w to w' and w prefers m to m', where w' and m' are their current partner.
- So both m and m' proposed to w.
- And w finally married m' instead of m.
- No matter who, m or m', proposed first, w prefers m' to m.
- A contradiction to our assumption.

Some observations

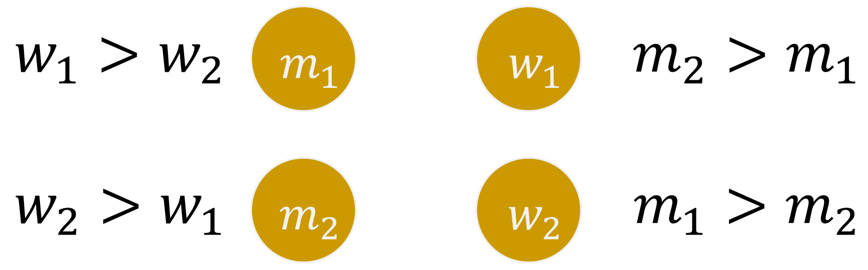
- For any man: His fiancé gets worse and worse (according to his preference list)
  - because he changes fiancé only when the previous one dumps him, forcing him to try next (worse!) ones.
- For any woman: Her fiancé gets better and better (according to her preference list)
  - because she changes fiancé only when a better man proposes to her.

Women propose?

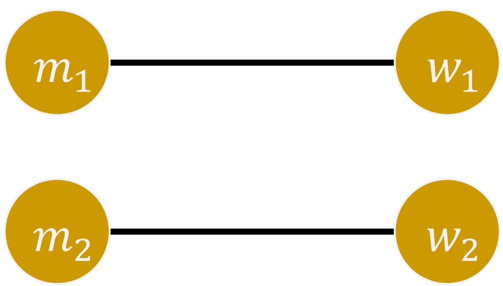
- What if women propose?



Which stable matching is better?

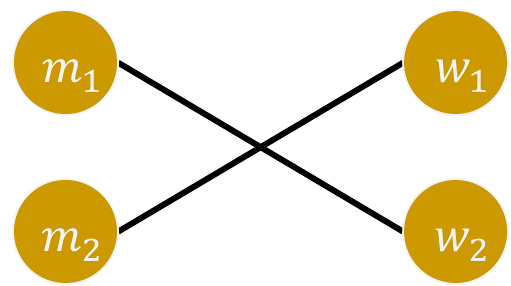


GS algorithm: men propose



- As a man, which matching you prefer?
  - What if you are m_1? What if you are m_2?

GS algorithm: women propose



- As a woman, which matching you prefer?
  - What if you are w_1? What if you are w_2?

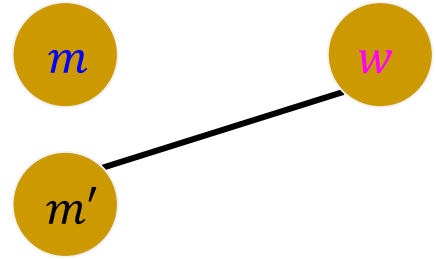
Stable Matching by G-S, men propose

- For any man m, his set of valid partners is
  $vp(m) = \{w\colon\ f(m) = w \text{ for some stable matching } f\}$
- $best(m)$: the best $w \in vp(m)$.
  - "best": according to m's preference.
- Theorem. Gale-Shapley algorithm matches all men m to $best(m)$.
- Implications:
  - different orders of free men picked do not matter
  - for any men $m_1 \neq m_2$, $best(m_1) \neq best(m_2)$

Proof

- For contradiction, assume that some m^* is matched to worse than $w^* = best(m^*)$.
- Since m^*proposes in the decreasing order, m^* must be rejected by w^* in the course of the GS algorithm.
- Note that $w^* \in vp(m^*)$. So there exists a man rejected by his valid partner.

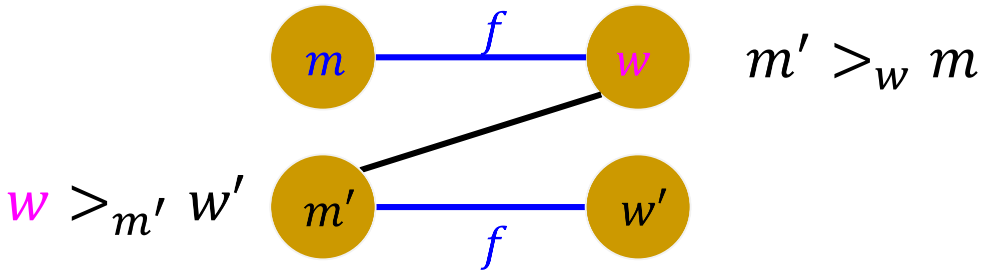
Proof



$m' >_w m$

- Consider the first such moment t that some m is rejected by some $w \in vp(m)$.
- Since m proposes in the decreasing order, $w = best(m)$.
- What triggers the rejection?
  - Either m proposed but was turned down (w prefers her current partner),
  - or w broke her engagement to m in favor of a better proposal.
- In either case, at moment t, w is engaged to a man m' whom she prefers to m, i.e., $m' >_w m$.

Proof



- By def of $best(m)$, ∃ a stable matching f assigning m to w.
- Assume that m' is matched to $w' \neq w$ in f.
- At moment t, m is *first* man rejected by someone in $vp(m)$.
- So no one in $vp(m')$, including w', rejected m' by now.
  - $w' \in vp(m')$ since w' and m' are paired up in the stable matching f.
- If $w <_{m'} w'$, m' should have proposed to w'. But now m' is with w, so m' has been dumped by w'. Impossible.
- Hence $w >_{m'} w'$. Contradiction to fact that f is stable. □

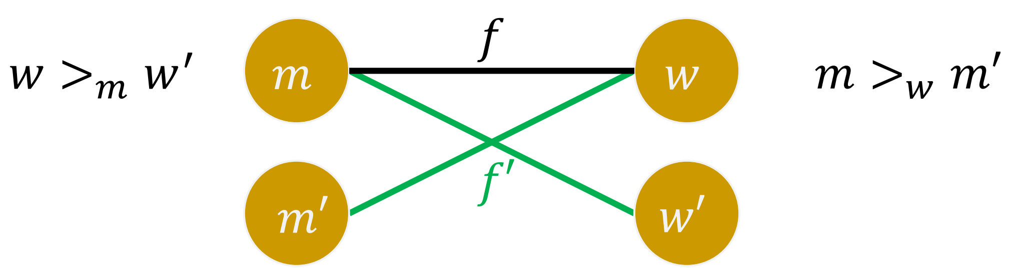
How about women?

- Recall: $best(m)$ is the best woman matched to m in all possible stable matchings.
- GS algorithm matches all men m to $best(m)$.
- $worst(w)$ is the worst man matched to w in all possible stable matchings.

- Theorem. GS algorithm matches all women w to $worst(w)$.

Proof

- By the last theorem, each m is matched to $w = best(m)$ when GS(men propose) gives f.
- We'll show that $m = worst(w)$.
- Suppose there is a stable matching f' in which w is matched to an even worse $m' <_w m$.
- Consider m's partner in f'; call her w'.
- $w >_m w'$, because $w = f(m) = best(m)$.
- Then (m, w) is a blocking pair in f'. Contradiction!



Who should propose?

- Thus if men propose, then
- in each man's eyes:
  - His engaged women get worse and worse.
  - But finally he gets the best possible. (The best that avoids a later divorce.)
- in each woman's eyes:
  - Her engaged men get better and better.
  - But finally she gets the worst possible. (The worst that avoids a later divorce.)

Next: Lower bounds

- Recall: Gale-Shapley algorithm runs in time $O(n^2)$ in the worst case.
- *Question: Can we improve this?*

- Note: An input has $O(n^2 \log n)$ bits, so even reading the input needs this much time.
- So the above question should be asked in certain random access model.

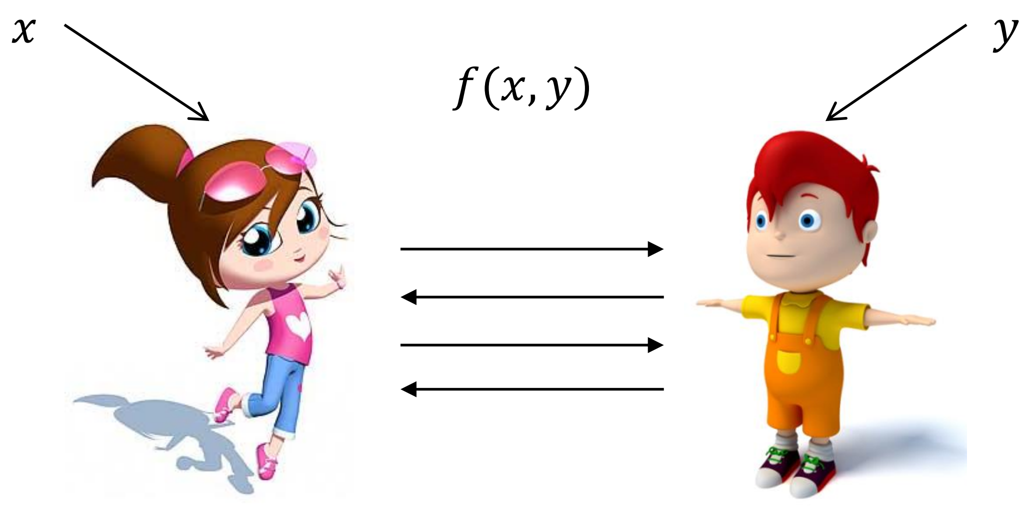
Query

- For example, such queries
  - What's woman w's ranking of man m?
  - Which man does woman w rank at place k?
  - Who does woman w prefer, m or m'?
  - …
- The above examples are on women's preferences. Similarly we can have queries on men's preferences.
- Some queries need $\log n$ bits to answer, some need only 1 bit.
  - The latter is called Boolean queries.

Simulation by communication

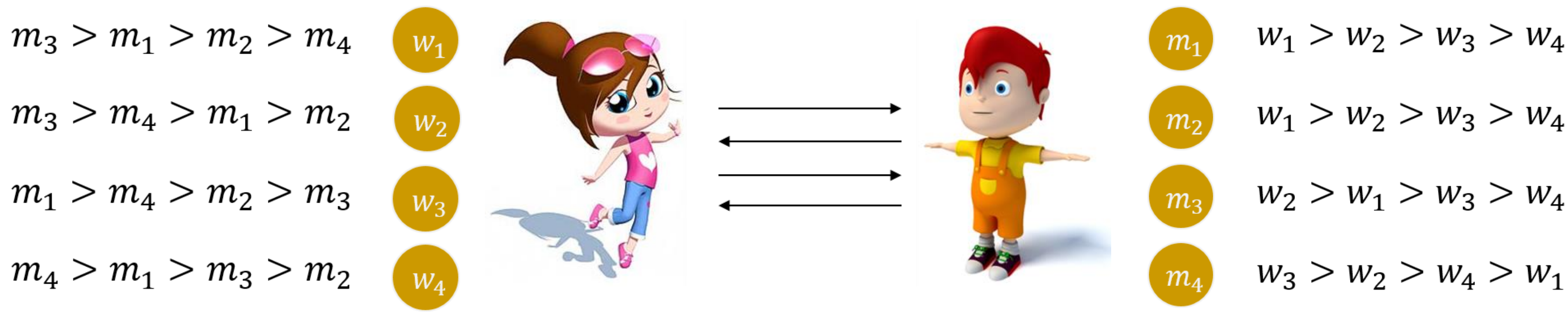
- Observation: Communication can simulate all these queries.

Recall: Communication complexity



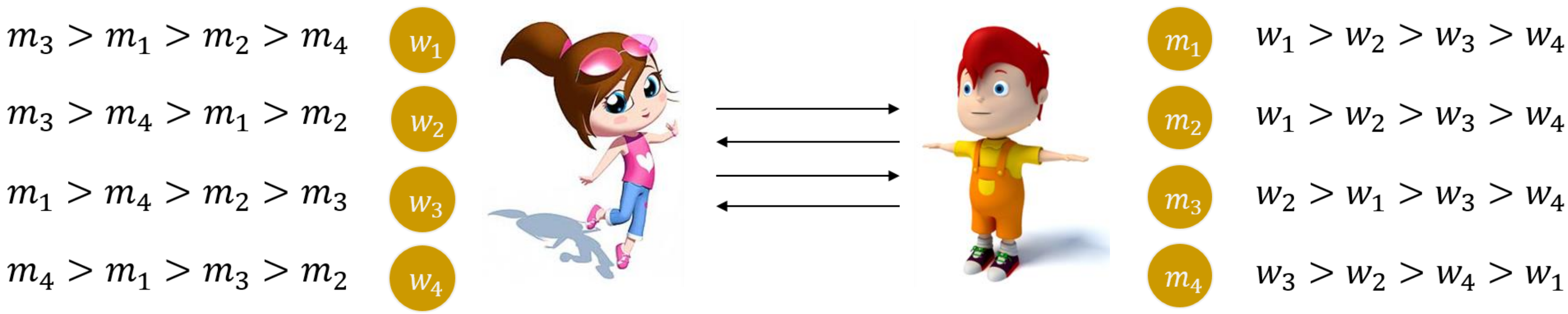
- Two parties, Alice and Bob, jointly compute a function f on input (x, y).
  - x known only to Alice and y only to Bob.
- Communication complexity: how many bits are needed to be exchanged?

communication setting



- Suppose that Alice has all women's preference lists,
- and Bob has all men's preference lists.
- Then any aforementioned query can be simulated by communication.

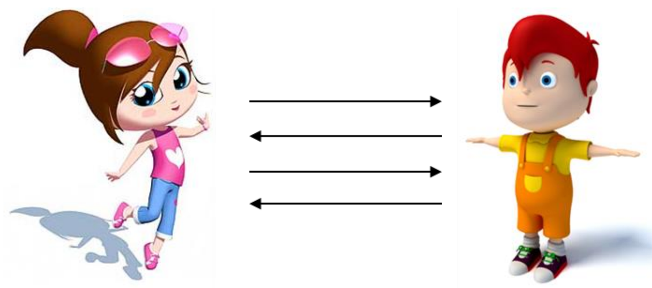
Algorithm to protocol



- *Fact. Any algorithm using k queries of b-bit answer can be made into a communication protocol using kb communication bits.*
- Method: Both Alice and Bob run the algorithm. Whenever they need to make a query, the one who has the answer tells the other.

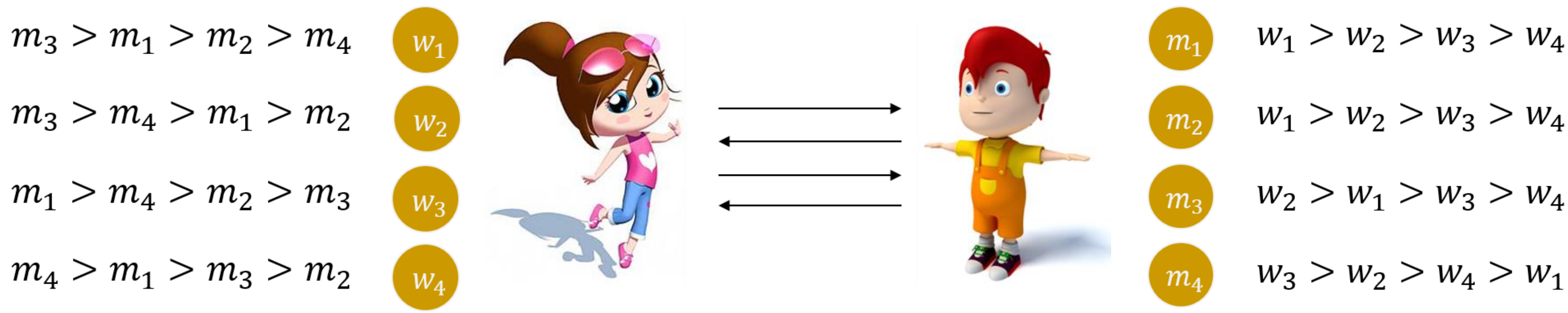
Algorithm to protocol

$m_3 > m_1 > m_2 > m_4$	w_1		m_1	$w_1 > w_2 > w_3 > w_4$
$m_3 > m_4 > m_1 > m_2$	w_2		m_2	$w_1 > w_2 > w_3 > w_4$
$m_1 > m_4 > m_2 > m_3$	w_3		m_3	$w_2 > w_1 > w_3 > w_4$
$m_4 > m_1 > m_3 > m_2$	w_4		m_4	$w_3 > w_2 > w_4 > w_1$



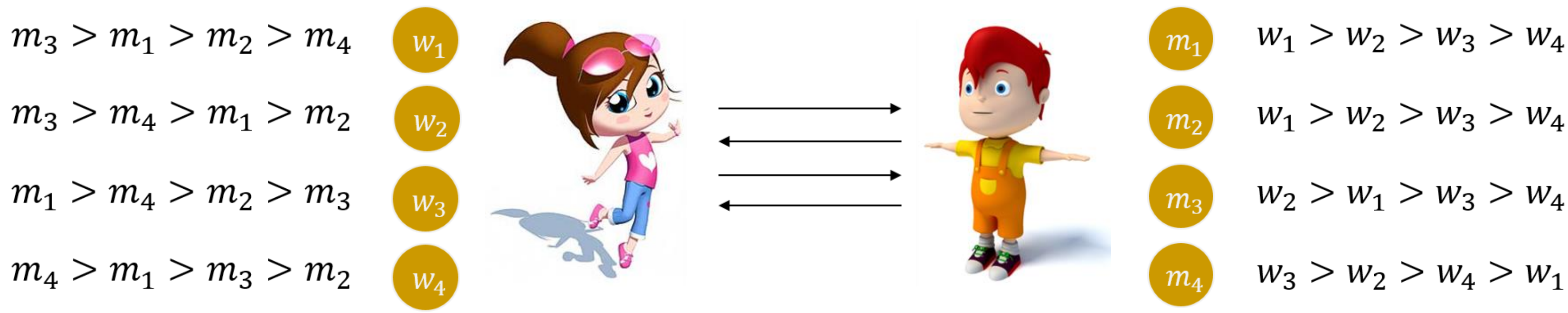
- E.g. consider query "What's woman w's ranking of man m?"
- Alice has the answer
  - since she owns all women's preference lists
- So Alice sends the answer to Bob, who then also knows the answer to continue the algorithm.

Lower bounds



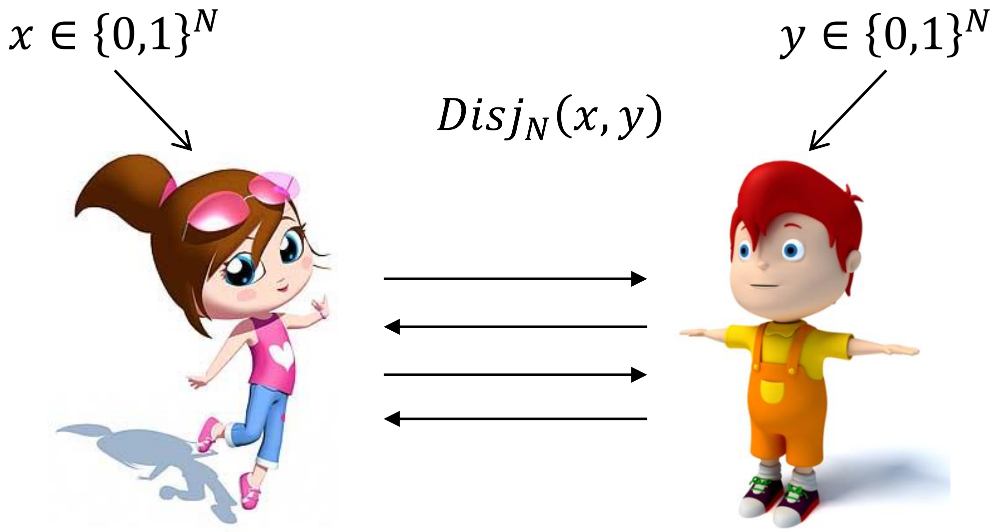
- *Theorem. Any protocol to find a stable matching needs $\Omega(n^2)$ communication bits.*
- *Theorem. Any protocol verifying whether a given matching is stable needs $\Omega(n^2)$ communication bits.*
- Together with the query-communication relation, we know that it takes $\Omega(n^2/t)$ queries if each query has a t-bit answer.
  - In particular, both tasks need $\Omega(n^2)$ Boolean queries.

Lower bounds



- *Theorem. Any protocol to find a stable matching needs $\Omega(n^2)$ communication bits.*
- *Theorem. Any protocol verifying whether a given matching is stable needs $\Omega(n^2)$ communication bits.*
- Method: Reduce the problem to a well-known problem called Disjointness.

Recall: Communication complexity



$$\mathrm{Disj}_N(x, y) = \begin{cases} 0 & \text{if } \exists i \text{ s.t. } x_i = y_i = 1 \\ 1 & \text{otherwise} \end{cases}.$$

- *Theorem. Any protocol solving $Disj_N$ problem needs $\Omega(N)$ communication bits.*
  - *even for randomized protocols.*

Reduction to verification

- For two strings x and y both of $n(n-1)$ bits,
  - as input of Disj_N, where $N = n(n-1)$
- we map them to instance of Stable Matching
- For w_i: $\left(m_j: x_{ij} = 1\right) m_i (m_j: x_{ij} = 0)$
- For m_j: $\left(w_i: y_{ij} = 1\right) w_j (w_i: y_{ij} = 0)$
- Matching $\mu_{id} = \{(1,1), \ldots, (n,n)\}$.
- μ_{id} is unstable $\Leftrightarrow \exists(i,j),\ x_{ij} = 1$ and $y_{ij} = 1$

  $\Leftrightarrow Disj_N(x,y) = 0$

Finding

- The lower bound for finding a stable matching is similar, but a bit more technically involved.
- Omitted here.

Summary for Stable Matching

- A bipartite matching is stable if no block pair exists.
- Gale-Shapley algorithm finds a stable matching by at most n^2 iterations.
  - This $\Omega(n^2)$ complexity is necessary.
- Whichever side proposes finally get their best possible.