CMSC5706 Topics in Theoretical Computer Science

Week 6: Algorithms for fair allocation

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Resource allocation

- General goals:
  - Maximize social welfare.
  - Fairness.
  - Stability.

Cake cutting

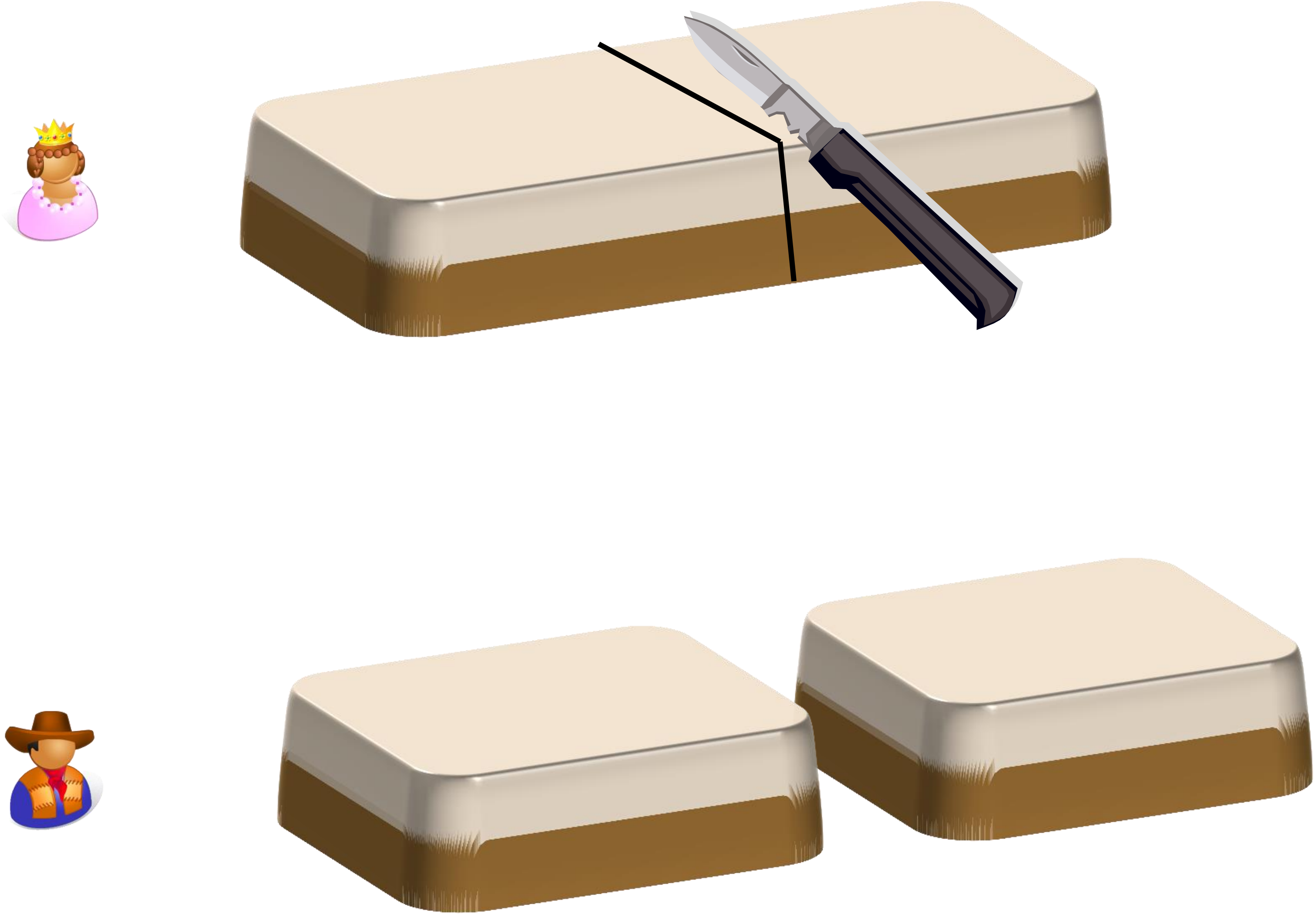
- Problem setting:
- One cake, n people (who want to split it).
- Each person might value different portions of the cake differently.
  - Some like strawberries, some like chocolate, …
  - Normalization: Each one values the whole cake as 1.
- This valuation info is private.
- Goal: divide the cake to make all people happy.

Cake cutting

- A cake cutting protocol is *fair* if each person gets $\geq 1/n$ fraction by her measure.
  - No matter how other people behave.
- A cake cutting protocol is *envy-free* if each person thinks that she gets the most by her measure.
- Envy-free ⇒ fair:
  - a_{ij}: how much person j gets in person i's measure.
  - Envy-free: $a_{ii} \geq a_{ij}, \forall j$ ⇒ fair: $a_{ii} \geq 1/n, \forall i$.

$n = 2$

- 1. Alice cuts the cake into two equal pieces
  - by her measure
- 2. Bob chooses a larger piece
  - by his measure
- 3. Alice takes the other piece



envy-free

- *Theorem*. The outcome is envy-free (and thus fair).
- Proof.
  - Alice: gets exactly half, no matter which piece Bob chooses.
  - Bob: gets at least half, no matter how Alice cuts the cake.

$n = 3$

- Stage 0: Player 1 divides into three equal pieces
  - according to his valuation.
- Player 2 trims the largest piece s.t. the remaining is the same as the second largest.
- The trimmed part is called Cake 2; the other form Cake 1.

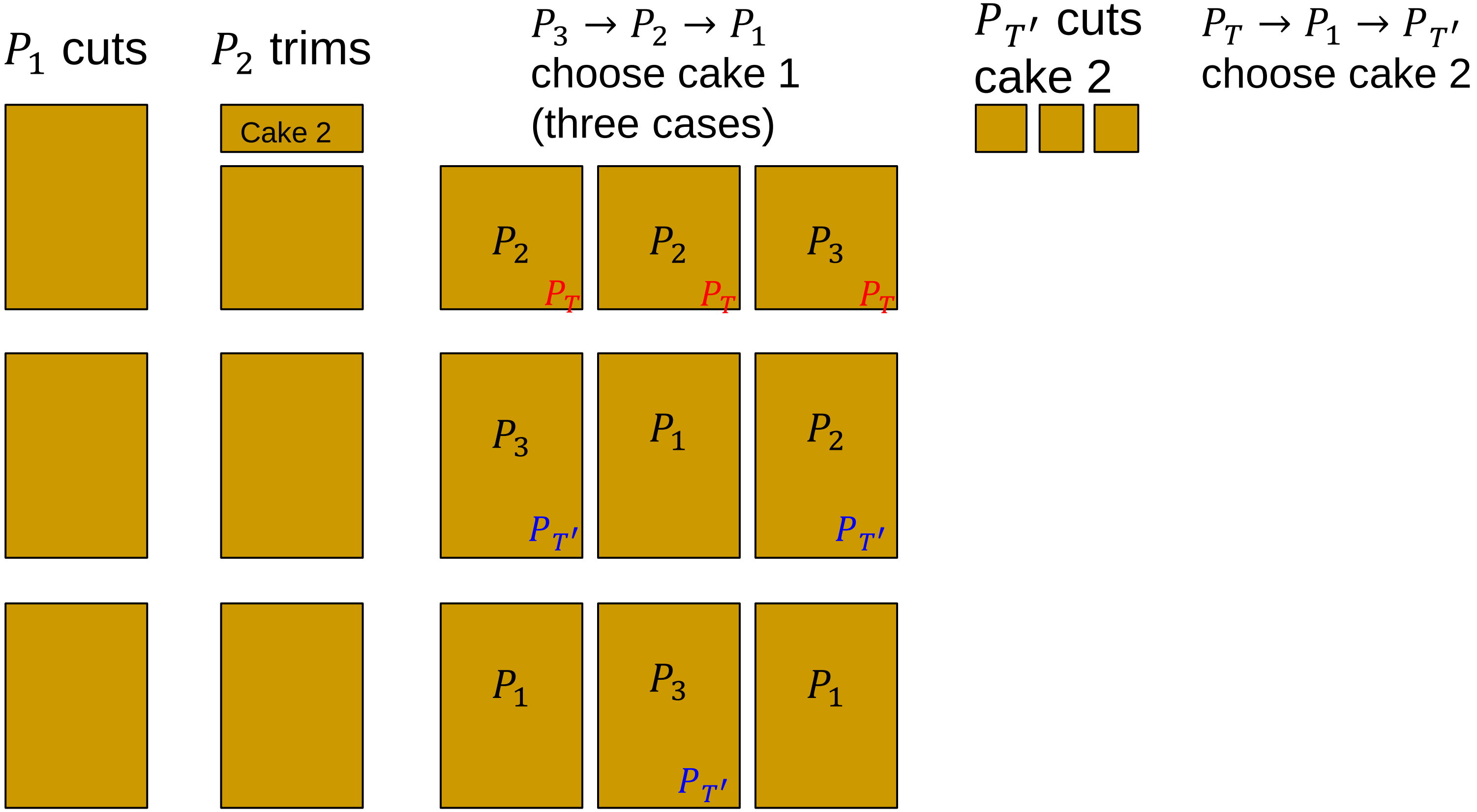
Stage 1: division of Cake 1

- Player 3 chooses the largest piece.
- If player 3 didn't choose the trimmed piece, player 2 chooses it.
- Otherwise, player 2 chooses one of the two remaining pieces.
- Either player 2 or player 3 receives the trimmed piece; call that player T
  - and the other player by T'.
- Player 1 chooses the remaining (untrimmed) piece

Stage 2 (division of Cake 2)

- T' divides Cake 2 into three equal pieces
  - according to his valuation.
- Players T, 1, and T' choose the pieces of Cake 2, in that order.

Whole process



P_1 cuts

P_2 trims

$P_3 \to P_2 \to P_1$ choose cake 1 (three cases)

$P_{T'}$ cuts cake 2

$P_T \to P_1 \to P_{T'}$ choose cake 2

Envy-freeness

- The division of Cake 1 is envy-free:
- Player 3 chooses first so he doesn’t envy others.
- Player 2 likes the trimmed piece and another piece equally, both better than the third piece. Player 2 is guaranteed to receive one of these two pieces, thus doesn’t envy others.
- Player 1 is indifferent judging the two untrimmed pieces and indeed receives an untrimmed piece.

Envy-freeness of Cake 2

- Player T goes first and hence does not envy the others.
- Player T' is indifferent weighing the three pieces of Cake 2, so he envies no one.
- Player 1 does not envy T': Player 1 chooses before T'
- Player 1 doesn't envy T: Even if T the whole Cake 2, it's just 1/3 according to Player 1's valuation.

General n?

- An algorithm using recursion.
- Suppose that the people are $P_1, \dots, P_n$.
- 1. Let $P_1, \dots, P_{n-1}$ divide the cake.
  - How? Recursively.
- 2. Now P_n comes.
  - Each of $P_1, \dots, P_{n-1}$ divides her share into n equal pieces.
  - P_n takes a largest piece from each of $P_1, \dots, P_{n-1}$.
- Let's try $n = 3$ on board.

Fairness

- *Theorem.* The protocol is fair.
- Proof.
  - For $P_1, \ldots, P_{n-1}$: each gets $\geq \frac{1}{n-1} \cdot \frac{n-1}{n} = \frac{1}{n}$.
  - P_n: gets $\geq \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{n} = \frac{1}{n}$
    - a_i: P_n's value of P_i's share in Step 1.
- Complexity? Let $T(n)$ be the number of pieces.
  - recursion: $T(n) = n \cdot T(n-1)$
    - Try a few examples for small n to convince yourself.
  - $T(1) = 1$, and $T(n) = n!$ for general n.

Moving Knife protocols

- Dubins-Spanier, 1961
  - Continuously move a knife from left to right.
- 1. A player yells out "STOP" as soon as knife has passed over $1/n$ of the cake
  - by her measure.
- 2. The player that yelled out is assigned that piece. (And she is out of the game; $n \leftarrow n - 1$.)
  - break tie arbitrarily
- 3. The procedure continues until all get a piece.

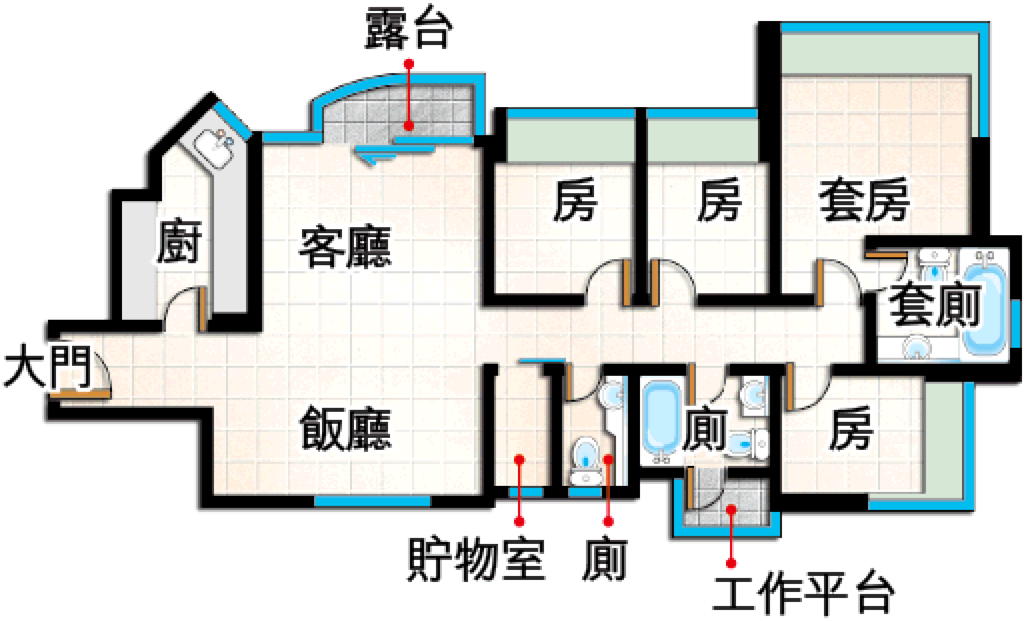
Fairness and complexity

- *Theorem.* The protocol is fair.
- Proof.
  - For the first who yells out: she gets $1/n$.
  - For the rest: each things that the remaining part has value at least $\frac{n-1}{n}$, and $n - 1$ people divide it.
    - Recursively: each gets $\frac{1}{n-1}\frac{n-1}{n} = \frac{1}{n}$.
- Complexity?
  - Only $n - 1$ cuts into n pieces.

Resource allocation

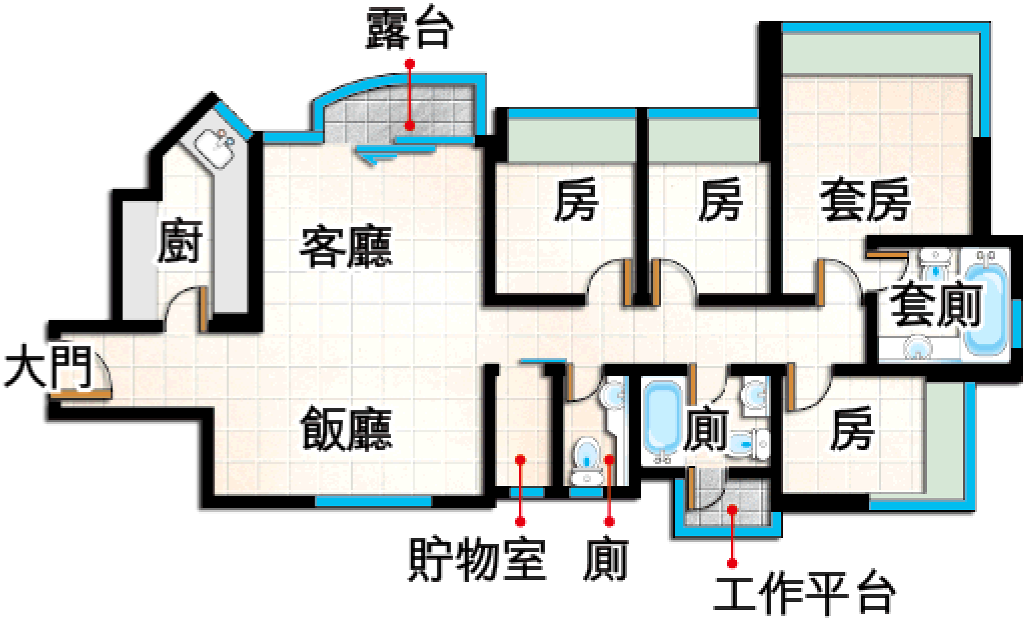
- The previous example of cake cutting is to allocate divisible resource.
- Similar examples include time, memory on a computer, etc.
- But sometimes resources are indivisible.
  - Pictures, cars, … in heritage.
  - Baby, house, … in a divorce

Assignment



- 4 students just came to HK and they found an apartment with 4 rooms.
  - Total rent for the apartment is c
- They need to decide
  - who lives in which room
  - and pays how much

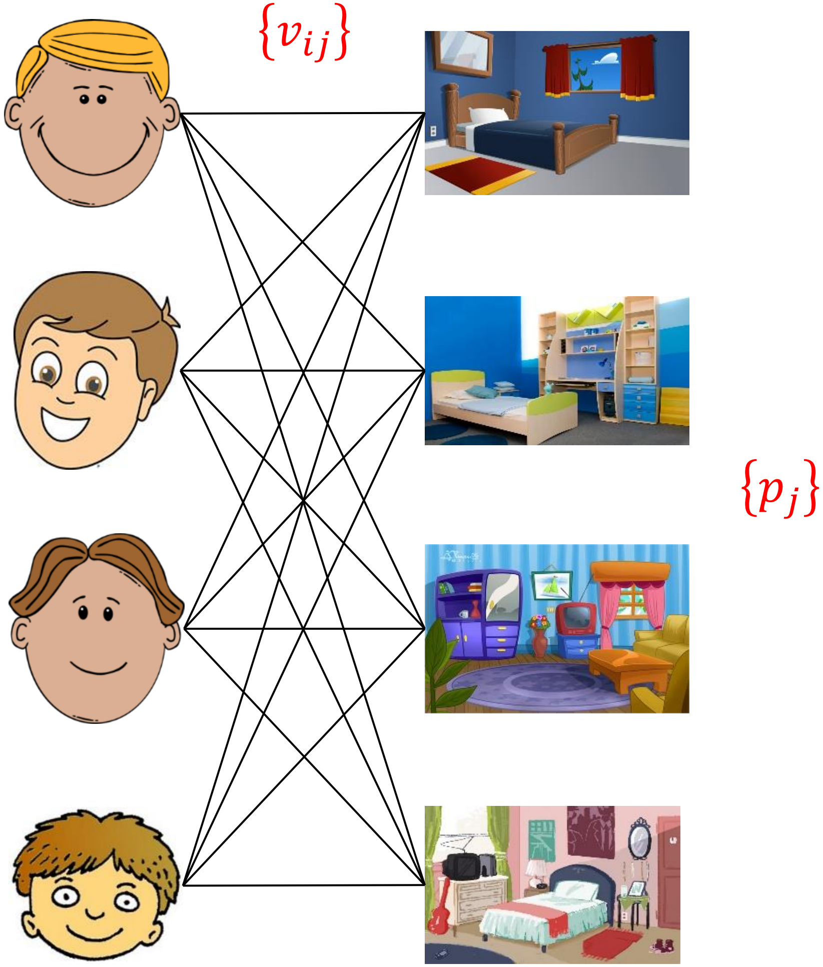
Assignment



- Note that each person has a different valuation of the four rooms.
  - Someone prefers a large room with private bathroom.
  - Someone prefers small room with low price.

General setup

- n people
- n items
- α_{ij}: person i's valuation of item j
- Solution: $(M, \{p_j\})$
  - M is a matching assigning item $M(i)$ to person i
  - p_j is the price for item j

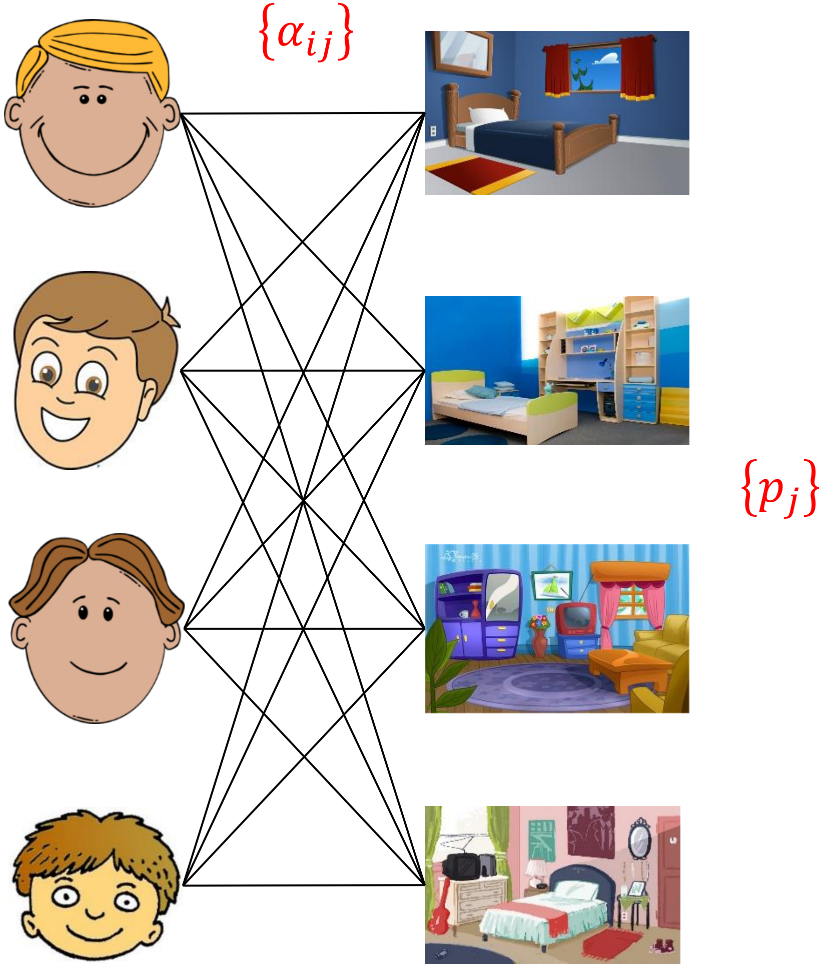


General setup

- Solution: $(M, \{p_j\})$
  - M is a matching assigning item $M(i)$ to person i
  - p_j is the price for item j
- Person i's utility:

$$u_i = \alpha_{ij} - p_j$$

where $j = M(i)$.



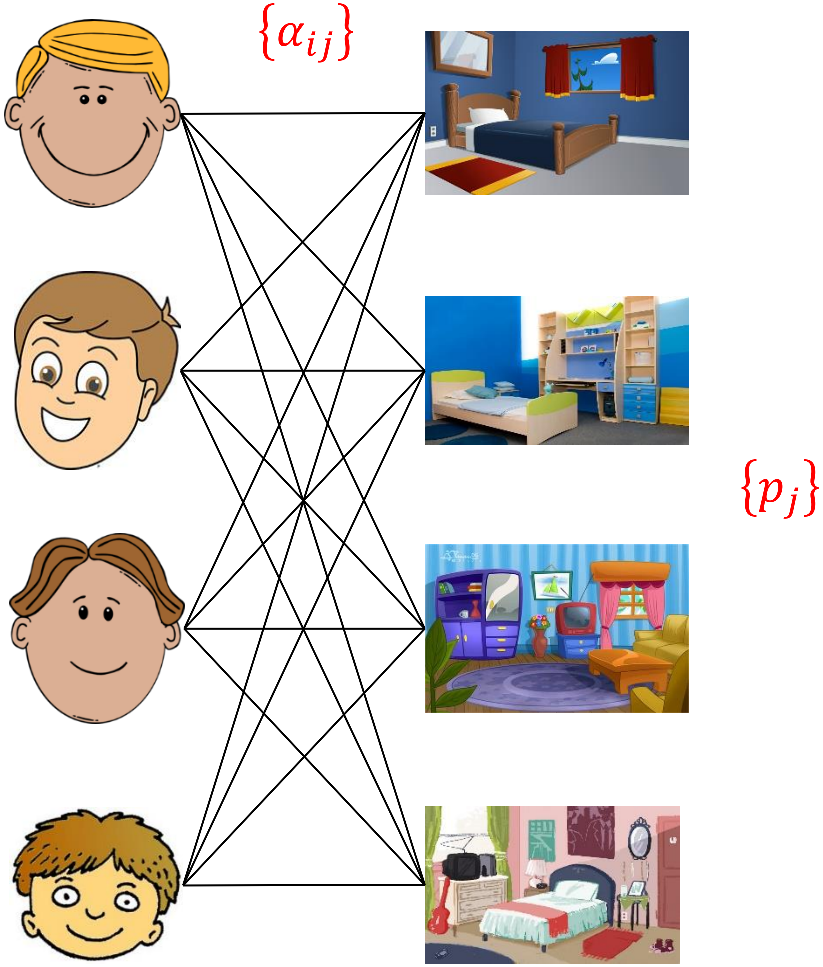
General setup

- Person i's utility:

  $$u_i = \alpha_{ij} - p_j$$

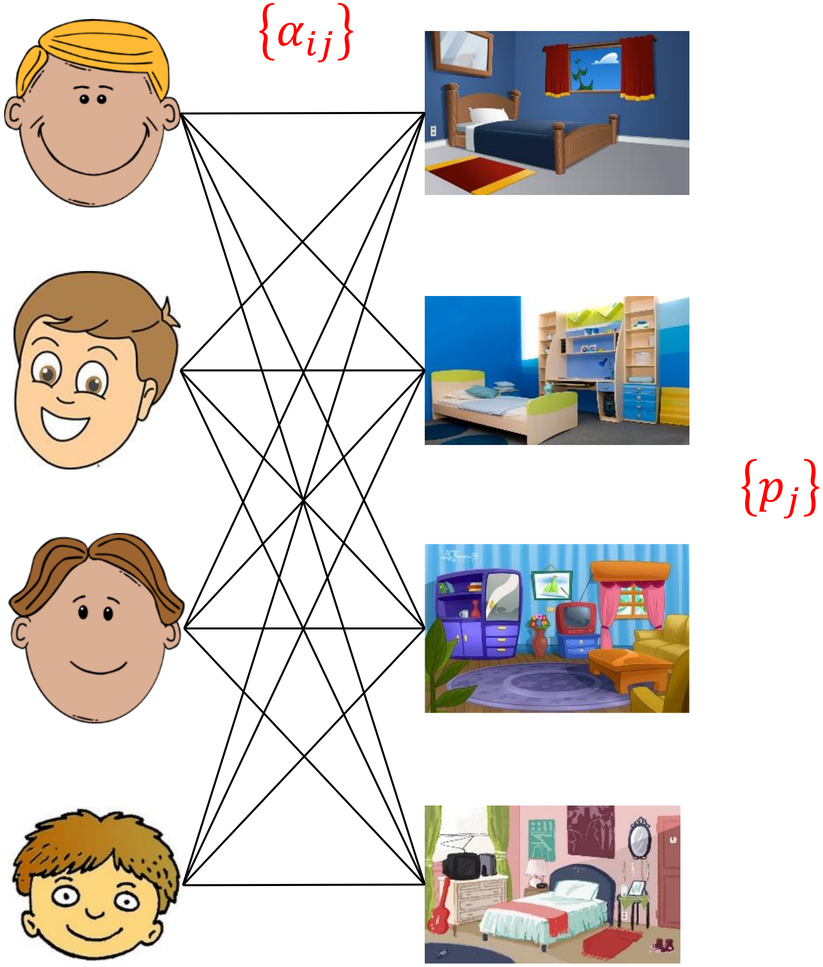
  where $j = M(i)$.
- The solution is *envy-free* if

  $u_i \geq \alpha_{ij'} - p_{j'}, \forall j'$
- Everyone is happy
  - and secretly thinks that all others are dumb ass!



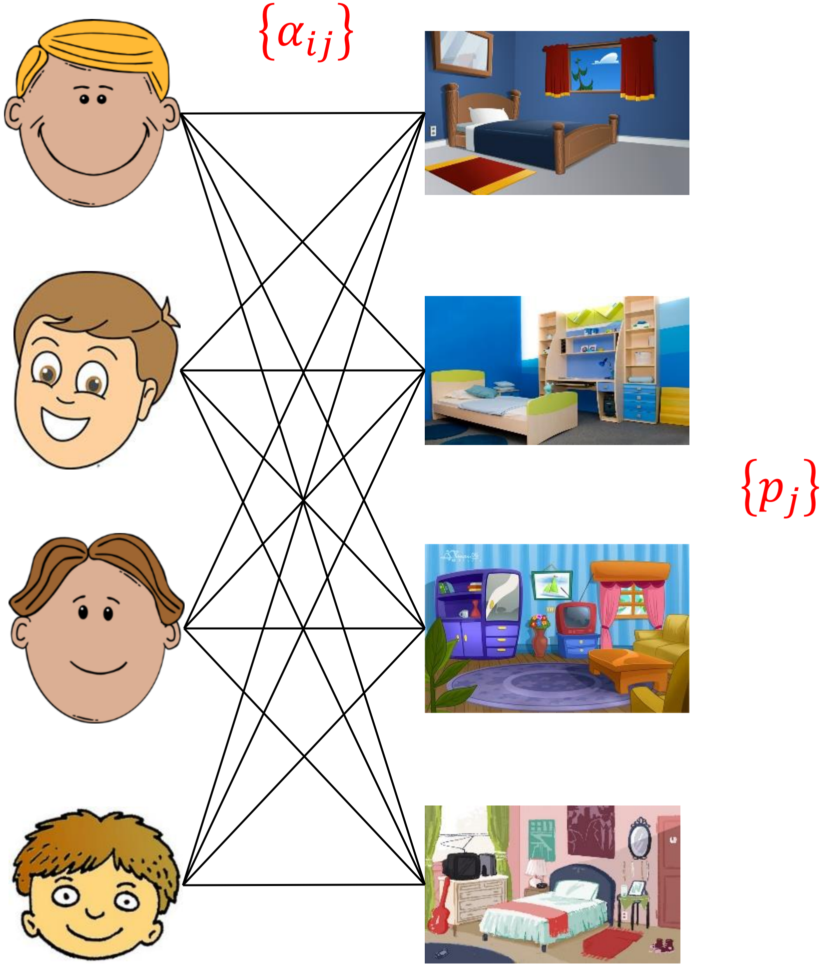
General setup

- *Question 1: Does there exist an envy-free solution?*
  - Sounds too good to be true.
- *Question 2: If there exists envy-free solutions, can we find one efficiently?*
  - Seems pretty hard…



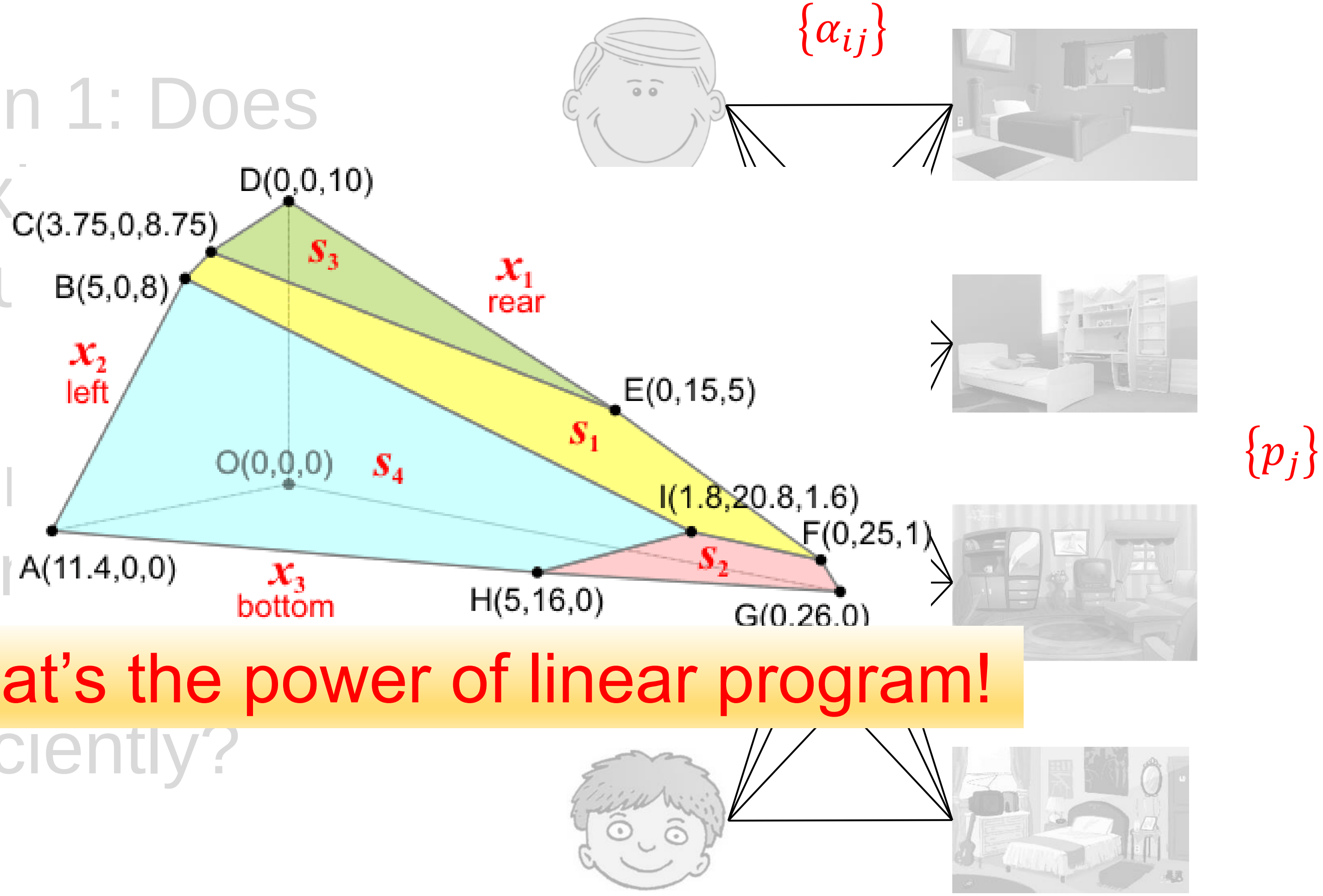
General setup

- *Question 1: Does there exist an envy-free solution?*
  - Yes!
- *Question 2: If there exists envy-free solutions, can we find one efficiently?*
  - Yes!



General setup

- Question 1: Does there ex free solu
  - □ Yes!
- Questio exists e solut one efficiently?
  - □ Yes!



That’s the power of linear program!

Item owner's utility

- Recall: If person i is assigned item j, then person i's utility is $u_i = \alpha_{ij} - p_j$.
- We can also think of item j has a utility of p_j
  - Item owner gets this money.
- Thus overall the pair (i, j) of agents get utility $u_i + p_j = \alpha_{ij}$.
- Social welfare: total utility of all agents.
  - $\sum_i \alpha_{ij}$, where $j = M(i)$.

LP

- Though the apartment is indivisible, let’s treat it as divisible for the moment.
- Let x_{ij} be the fraction of apartment j taken by person i.
- $\sum_j x_{ij} \leq 1$: each person takes at most 1 apartment.
- $\sum_i x_{ij} \leq 1$: the fractions sum up to 1.
- $x_{ij} \geq 0$.

LP

- Consider the following LP, which maximize the social welfare.
- max $\sum_{ij} \alpha_{ij} x_{ij}$

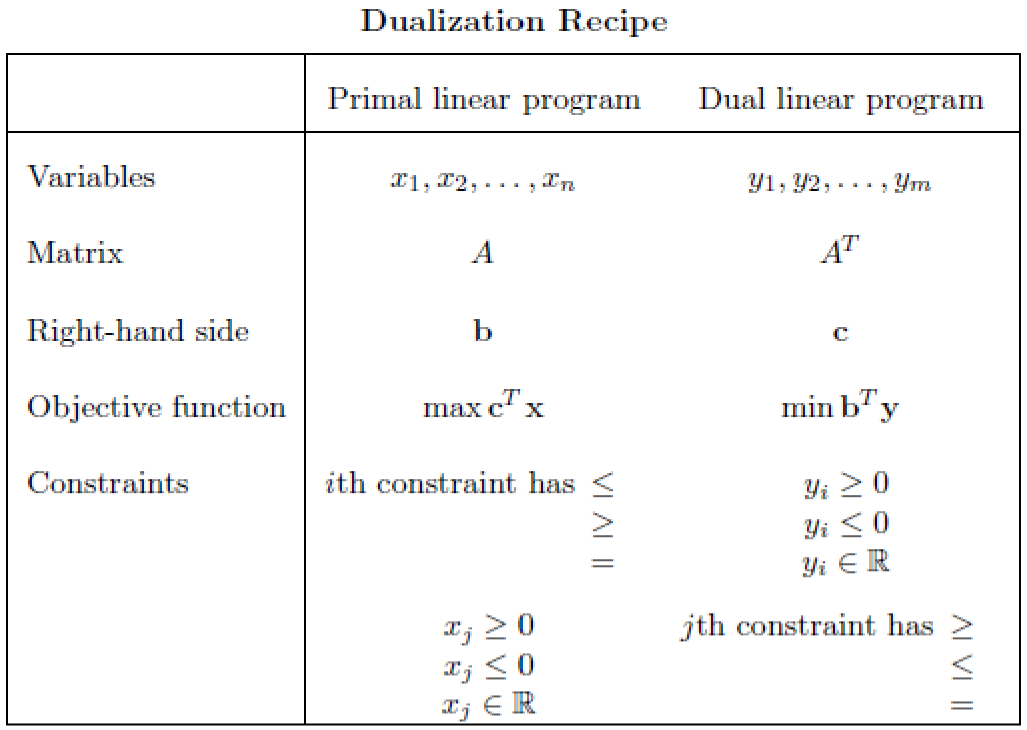
  s.t. $\sum_j x_{ij} \leq 1, \forall i$

  $\sum_i x_{ij} \leq 1, \forall j$

  $x_{ij} \geq 0, \forall i, j$
- *Issue*: If the optimal solution x to this LP is fractional, how to assign the indivisible items?

Surprise

- Good news: It’s not really an issue!
- *Theorem. The feasible region of the above LP is the convex hull of integral solutions x, where each $x_{ij} \in \{0,1\}$.*
- In particular, there exists an optimal $\{0,1\}$-solution.
- Next we show how to find it efficiently using duality.



Dualization Recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T \mathbf{x}$	$\min \mathbf{b}^T \mathbf{y}$
Constraints	ith constraint has $\leq$ $\geq$ $=$	$y_i \geq 0$ $y_i \leq 0$ $y_i \in \mathbb{R}$
	$x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	jth constraint has $\geq$ $\leq$ $=$

- Primal
  max $\boldsymbol{c}^T\boldsymbol{x}$
  s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$
  $\boldsymbol{x} \geq 0$

- Dual
  min $\boldsymbol{b}^T\boldsymbol{y}$
  s.t. $A^T\boldsymbol{y} \geq \boldsymbol{c}$
  $\boldsymbol{y} \geq 0$

- Primal

max $\boldsymbol{c}^T\boldsymbol{x}$

s.t. $A\boldsymbol{x} \le \boldsymbol{b}$

$\boldsymbol{x} \ge 0$

Dual

min $\boldsymbol{b}^T\boldsymbol{y}$

s.t. $A^T\boldsymbol{y} \ge \boldsymbol{c}$

$\boldsymbol{y} \ge 0$

- max $\sum_{ij} \alpha_{ij} x_{ij}$

s.t. $\sum_j x_{ij} \le 1, \forall i$

$\sum_i x_{ij} \le 1, \forall j$

$x_{ij} \ge 0, \forall i, j$

- min $\sum_i u_i + \sum_j p_j$

s.t. $u_i + p_j \ge \alpha_{ij}, \forall i, j$

$u_i \ge 0, \forall i$

$p_j \ge 0, \forall j$

$$A = \begin{bmatrix} 1\,1\,1 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 1\,1\,1 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 1\,1\,1 \\ 1\,0\,0 & 1\,0\,0 & 1\,0\,0 \\ 0\,1\,0 & 0\,1\,0 & 0\,1\,0 \\ 0\,0\,1 & 0\,0\,1 & 0\,0\,1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1\,0\,0 & 1\,0\,0 \\ 1\,0\,0 & 0\,1\,0 \\ 1\,0\,0 & 0\,0\,1 \\ 0\,1\,0 & 1\,0\,0 \\ 0\,1\,0 & 0\,1\,0 \\ 0\,1\,0 & 0\,0\,1 \\ 0\,0\,1 & 1\,0\,0 \\ 0\,0\,1 & 0\,1\,0 \\ 0\,0\,1 & 0\,0\,1 \end{bmatrix}$$

dual

- Primal
- max $\sum_{ij} \alpha_{ij} x_{ij}$

  s.t. $\sum_j x_{ij} \leq 1, \forall i$

  $\sum_i x_{ij} \leq 1, \forall j$

  $x_{ij} \geq 0, \forall i, j$

- Dual
- min $\sum_i u_i + \sum_j p_j$

  s.t. $u_i + p_j \geq \alpha_{ij}, \forall i, j$

  $u_i \geq 0, \forall i$

  $p_j \geq 0, \forall j$

- Dual
- $\min \sum_i u_i + \sum_j p_j$

  s.t. $u_i + p_j \geq \alpha_{ij}, \forall i, j$

  $u_i \geq 0, \forall i$

  $p_j \geq 0, \forall j$
- The condition has a meaning of envy-free:
- Suppose that u_i is utility, and p_j is price.
- If $u_i + p_j < \alpha_{ij}$, then person i would like to take item j.
  - since he then has utility $\alpha_{ij} - p_j > u_i$.

Complementary slackness

- Primal Dual
  max $\boldsymbol{c}^T\boldsymbol{x}$ min $\boldsymbol{b}^T\boldsymbol{y}$
  s.t. $A\boldsymbol{x} \le \boldsymbol{b}$ s.t. $A^T\boldsymbol{y} \ge \boldsymbol{c}$
  $\boldsymbol{x} \ge 0$ $\boldsymbol{y} \ge 0$
- Theorem. If $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal for Primal and Dual, respectively, then
  - $x_j^* > 0 \Rightarrow a_j \cdot \boldsymbol{y}^* = c_j$, where a_j is the j-th column of A
  - $y_i^* > 0 \Rightarrow a^i \cdot \boldsymbol{x} = b_i$, where a^i is the i-th row of A
- Proof. Note $\boldsymbol{c} \cdot \boldsymbol{x}^* \le A^T\boldsymbol{y}^* \cdot \boldsymbol{x}^* = \boldsymbol{y}^* \cdot A\boldsymbol{x}^* \le \boldsymbol{y}^* \cdot \boldsymbol{b}$.
- But by strong duality, $\boldsymbol{c} \cdot \boldsymbol{x}^* = \boldsymbol{b} \cdot \boldsymbol{y}^*$, thus equality holds.
- Thus if $x_j^* > 0$, the first (in)equality implies $a_j \cdot \boldsymbol{y}^* = c_j$.
- If $y_i^* > 0$, the second (in)equality implies$a^i \cdot \boldsymbol{x} = b_i$.

algorithm

- Complementary slackness here:

$$x_{ij} = 1 \Rightarrow u_i + p_j = \alpha_{ij}$$

- So to find an assignment, it is enough to
  - solve the dual, collect edges $E = \{(i,j): u_i + p_j = \alpha_{ij}\}$
  - find a perfect matching M in the graph $G = (P, Q, E)$.
  - define $x_{ij} = 1$ if and only if $(i,j) \in M$
- This x is a $\{0,1\}$ optimal solution to the primal.
  - $\sum_{ij} \alpha_{ij} x_{ij} = \sum_{(i,j):x_{ij}=1} \alpha_{ij} = \sum_{(i,j):x_{ij}=1} (u_i + p_j) = \sum_i u_i + \sum_j p_j$
- The utility and price are also given by u_i and p_j.
  - Dual variables coincide with utility and price.

Summary

- Resource allocation naturally arises in many applications.
- Main goal is to achieve high social welfare
- as well as fairness.
- Examples:
  - Divisible: cake cutting
  - Indivisible: assignment game