CMSC5706 Topics in Theoretical Computer Science

Week 3: Streaming and Sketching

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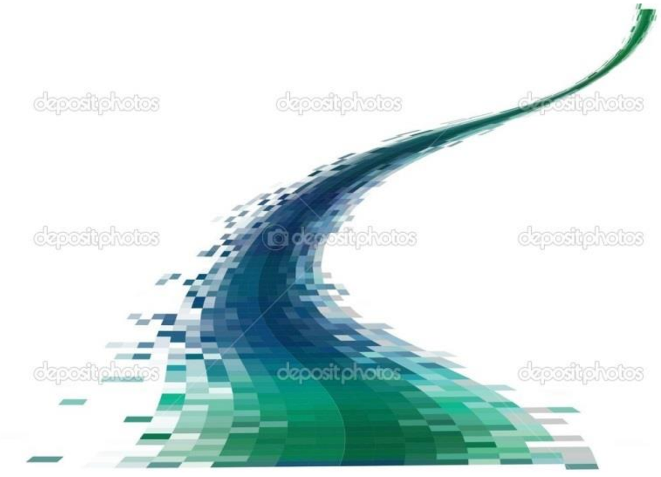
Map

- Motivations and model
- Problem 1: Missing numbers
- Problem 2: Count-Min sketch
- Lower bounds
  - Communication complexity

Motivations



- Big mass of data.
- Data comes as a stream.
  - Cannot see future data.
- Relatively small space. “sketch”
  - Cannot store past data
- Need to process each item fast.
  - Quick update time.
- Examples: Phone calls, Internet packets, satellite pictures, …

Problem 1: Missing numbers

- A set of numbers $S = \{1,2, \dots, n\}$
- $n - 1$ of them come in a stream $x_1, x_2, \dots, x_{n-1}$; one number is missing.

$$3, 25, 6, 19, 1, 10, \dots$$

- Task: identify which one is missing.
  - Using small space.

A simple algorithm

- Maintain the sum of the input numbers.

- $sum = 0$
- **for** $i = 1$ to $n - 1$

  $sum = sum + x_i$
- **return** $\frac{n(n+1)}{2} - sum$

Space complexity

- sum is at most $\frac{n(n+1)}{2}$ during the algorithm.
- Thus it takes at most $\log_2 \frac{n(n+1)}{2} = O(\log_2 n)$ bits to write it down.

- Space complexity: $O(\log_2 n)$.
- Much smaller than storing the whole stream, which takes at least $O(n \log n)$.

More complicated

- Now the task gets harder.
- $n - 2$ of them come in a stream $x_1, x_2, \dots, x_{n-2}$, two numbers are missing.

$$3, 25, 6, 19, 1, 10, \dots$$

- Task: identify which two are missing.
  - Using small space.

First try

- Maintain the sum and product of the input numbers.

- $sum = 0; product = 1$
- **for** $i = 1$ to $n - 2$

  $sum = sum + x_i$

  $product = product \cdot x_i$
- $a = \frac{n(n+1)}{2} - sum,\ \ b = n!/product$
- solve equations $x + y = a, x \cdot y = b$
- **return** (x, y)

Problem and solution

- Issue: $product$ is at least $(n-2)!$
- Thus even writing down the number needs $\log_2(n-2)! = \Theta(n \log n)$ bits.
  - Too much compared to $O(\log n)$ before.

- How to do?

Improvement

- Note that we don't need to maintain product.
- We can maintain anything, as long as finally we can reconstruct the solution from the stored results.
- One summary that is much smaller than product: sum of squares.
- Recall: $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Improvement

- Maintain the sum and sum of squares of the input numbers.

- $sum = 0; sos = 0$
- **for** $i = 1$ to $n - 2$

  $sum = sum + x_i$

  $sos = sos + x_i^2$
- $a = \frac{n(n+1)}{2} - sum,\ b = \frac{n(n+1)(2n+1)}{6} - sos$
- solve equations $x + y = a, x^2 + y^2 = b$.
- **return** (x, y)

Space complexity

- sos is at most $\frac{n(n+1)(2n+1)}{6}$ during the algorithm.
- Thus it takes at most $\log_2 \frac{n(n+1)(2n+1)}{6} =$ $O(\log_2 n)$ bits to write it down.
- Space complexity: $O(\log_2 n)$.

Further question

- Now assume that the numbers are from an arbitrary set $S = \{s_1, s_2, \ldots, s_n\}$.
- $n - k$ of them come in a stream $x_1, x_2, \ldots, x_{n-k}$; k numbers are missing.

- Task: identify which k are missing.
  - Using small space.

First try

- Maintain $\sum_i x_i, \sum_i x_i^2, \ldots, \sum_i x_i^k$ of the input numbers.

- $sum_1 = 0; sum_2 = 0; \ldots; sum_k = 0$
- **for** $i = 1$ **to** $n - k$

  for $d = 1$ **to** k

  $sum_d = sum_d + x_i^d$

- solve system of equations

$$\begin{aligned} \textstyle\sum_i y_i &= \textstyle\sum_{i=1}^n s_i - sum_1 \\ \textstyle\sum_i y_i^2 &= \textstyle\sum_{i=1}^n s_i^2 - sum_2 \\ &\vdots \\ \textstyle\sum_i y_i^k &= \textstyle\sum_{i=1}^n s_i^k - sum_k \end{aligned}$$

- **return** $(y_1, \ldots, y_k)$

Space complexity

- sum_d is at most $O(n^d)$ during the algorithm.
- Thus it takes at most $\mathrm{O}(k \log_2 n^k) = O(k^2 \log_2 n)$ bits to write it down.

- Space complexity:$O(k^2 \log_2 n)$.

Problem 2: high frequency estimation

- Consider an array $F[1..n]$ of size n.
- Items like $(i_1, +), (i_2, -), \ldots, (i_T, +)$ come in a stream.

$$(3, +), (3, +), (2, +), (3, -), \ldots$$

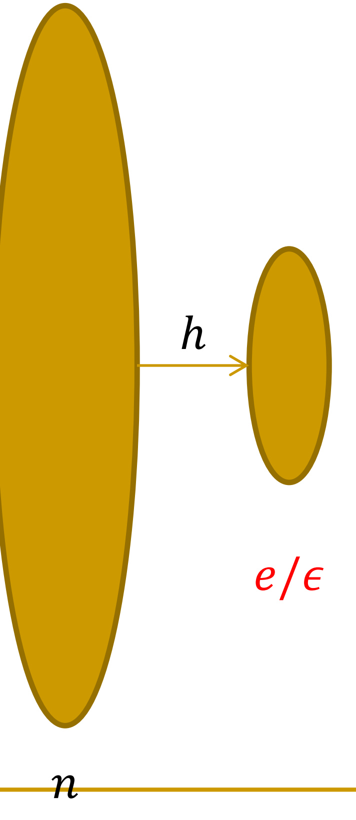
- $F[i]++$ when $(i, +)$ comes, and $F[i]--$ when $(i, -)$ comes
  - Assumption: $F[i] \geq 0$ all the time.

- Task: Answer queries like "what is $F[18]$"?

Approximation and error

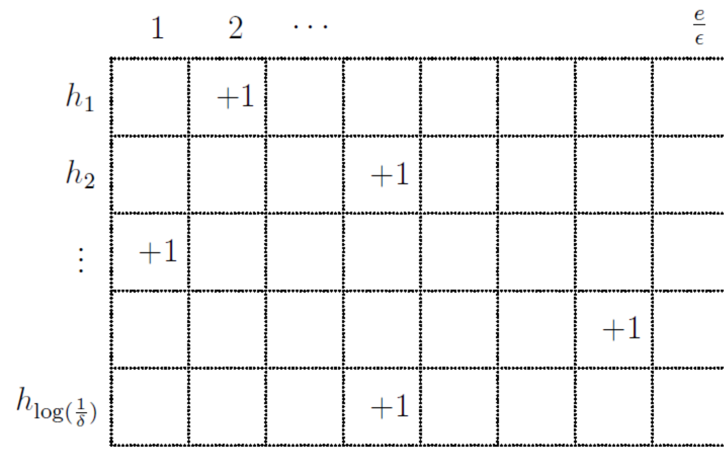
- Unlike the previous algorithm, here deterministic algorithm needs a lot of space.
- But if we allow
  - approximation: only estimate $F[i]$ up to certain precision
  - error: algorithm fails with some small probability
- then we'll have an efficient randomized algorithm.

- Pick $\log(1/\delta)$ hash functions $h_j: [n] \to [e/\epsilon]$
  - uniformly at random from a family of pairwise independent hash functions.
- $e/\epsilon \ll n$, so it's space efficient.
- For each $i \in [n]$, different h_j's map it to different "buckets".
- Idea: only maintain counters for buckets.



Algorithm

- **for** $j = 1$ **to** $\log(1/\delta)$
  for $d = 1$ **to** e/ϵ
  $count(j,k) = 0$
- **for** $t = 1$ **to** T
  if item t is $(i, +/-)$
  for $j = 1$ **to** $\log(1/\delta)$
  $\text{count}\big(j, h_j(i)\big)$ ++/ − −
- On query $F[i]$: **return** $F'[i] = \min_j \text{count}\big(j, h_j(i)\big)$



Guarantee

- At any time of query:
- Define $\|F\| = \sum_i F[i]$

- Theorem.
  - $F'[i] \geq F[i]$
  - $F'[i] \leq F[i] + \epsilon\|F\|$ with probability $\geq 1 - \delta$.

Analysis

- $F'[i] \geq F[i]$ is easy:
- Any time when $F[i]$ increases by 1, we increase $\text{count}\left(j, h_j(i)\right)$ for each j.
- Thus $\min_j \text{count}\left(j, h_j(i)\right)$ also increases by 1.
- Thus we never miss any increment.

Analysis

- Next: $F'[i] \leq F[i] + \epsilon \|F\|$ with prob. $\geq 1 - \delta$.
- X_{ji}: the contribution of items other than i to $\text{count}\left(\text{j}, h_j(i)\right)$.
- Claim. $\mathbf{E}\left[X_{ji}\right] = \frac{\epsilon}{e}(\|F\| - F[i]) \leq \frac{\epsilon}{e}\|F\|$.
- Proof. For each fixed item $i' \neq i$, the probability of $h_j(i') = h_j(i)$ is ϵ/e.
- There are $\|F\| - F[i]$ many items $i' \neq i$ (counting multiplicity), thus $\mathbf{E}\left[X_{ji}\right] = \frac{\epsilon}{e}(\|F\| - F[i])$.

- $\Pr[F'[i] > F[i] + \epsilon\|F\|] = \Pr\left[F[i] + X_{ji} > F[i] + \epsilon\|F\|, \forall j\right]$
  - $F'[i] = F[i] + X_{ji}$ by definition
  - $\min_j \operatorname{count}\left(j, h_j(i)\right) > F[i] + \epsilon\|F\| \Leftrightarrow F[i] + X_{ji} > F[i] + \epsilon\|F\|, \forall j$
- $\Pr\left[F[i] + X_{ji} > F[i] + \epsilon\|F\|, \forall j\right] = \Pr\left[F[i] + X_{ji} > F[i] + \epsilon\|F\|\right]^{\log 1/\delta}$ because different h_j's are independently chosen.

- $\Pr\left[F[i] + X_{ji} > F[i] + \epsilon\|F\|\right] = \Pr\left[X_{ji} > \epsilon\|F\|\right]$
- Recall: $\mathbf{E}\left[X_{ji}\right] = \frac{\epsilon}{e}(\|F\| - F[i]) \leq \frac{\epsilon}{e}\|F\|$
- By Markov's inequality,

$$\Pr\left[X_{ji} > \epsilon\|F\|\right] \leq \Pr\left[X_{ji} > e\mathbf{E}\left[X_{ji}\right]\right] < 1/e$$

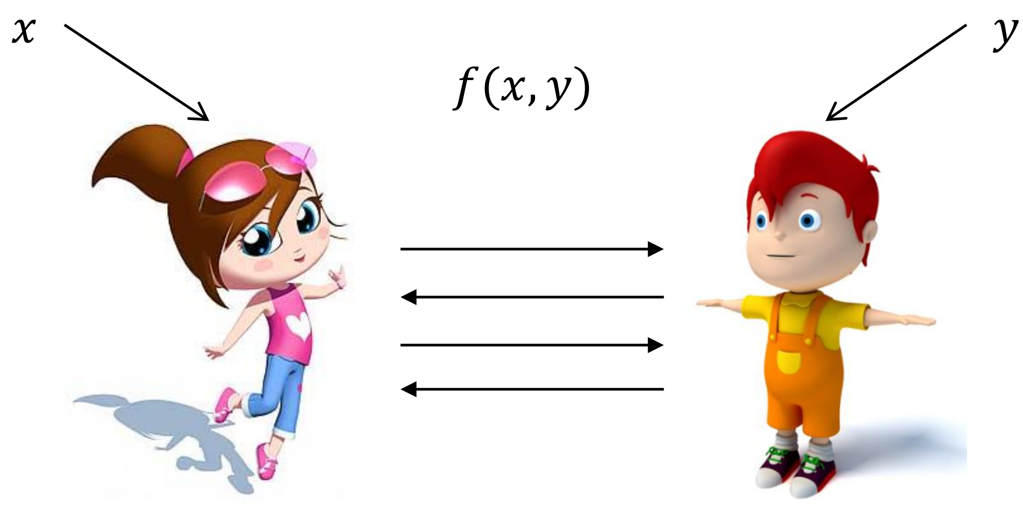
- Putting everything together,

$$\Pr[F'[i] > F[i] + \epsilon\|F\|] \leq \left(\frac{1}{e}\right)^{\log\frac{1}{\delta}} = \delta$$

Lower bounds

- Theorem. In order to estimate $F[i]$ within an error of $\epsilon||F||$ with probability 2/3, one needs to use $\Omega\left(\frac{1}{\epsilon}\right)$ space.
- Proof. We will use one-way communication complexity.

Communication complexity



- Two parties, Alice and Bob, jointly compute a function f on input (x, y).
  - x known only to Alice and y only to Bob.
- Communication complexity: how many bits are needed to be exchanged?

One-way communication complexity

$x \in \{0,1\}^n$

$f(x,i) = x_i$

$i \in [n]$

- Theorem. Index function needs $\Omega(n)$ communication bits.
  - even for randomized protocols.

Lower bound

- Theorem. In order to estimate $F[i]$ within an error of $\epsilon \|F\|$ with probability 2/3, one needs to use $\Omega\left(\frac{1}{\epsilon}\right)$ space.
- Proof. Given an Index problem input (x, i), with $n = 1/2\epsilon$.
- Let F be: $F[i] = 2x_i$ for $i = 1, \ldots, n$, and $F[0] = 2 \cdot |\{i \in [n]: x_i = 0\}|$.
- $\|F\| = 2n = 1/\epsilon$. Thus $\epsilon \|F\| = 1$.

- If one can estimate $F[i]$ within error $\epsilon\|F\| = 1$ using space s, then
- Alice can use this way to transmit the space to Bob.
  - communication: s bits.
- Bob then gets $F'[i]$ which differ from $F[i]$ by 1.
- Bob can then determine whether $x_i = 0$ or $x_i = 1$.
- Thus the communication lower bound implies $s = \Omega(n) = \Omega\left(\frac{1}{\epsilon}\right)$, as desired.

One thing left

- Pairwise independent hash family
- A family of functions $H = \{h | h : N \to M\}$ is pairwise independent if the following two conditions hold when we pick $h \in H$ uniformly at random:
  - $\forall x \in N$, the random variable $h(x)$ is uniformly distributed in M
  - $\forall x_1 \neq x_2 \in N$, the random variables $h(x_1)$ and $h(x_2)$ are independent,.

- Note that the condition is equivalent to the following.

- For any two different $x_1 \neq x_2 \in N$, and any $y_1, y_2 \in M$, it holds that

$$\mathbf{Pr}_{h \in H}[h(x_1) = y_1 \; and \; h(x_2) = y_2] = 1/|M|^2$$

Construction

- There is an easy construction of the pairwise independent hash function family.
- Let p be a prime, and define
$$h_{a,b}(x) = (ax + b)\ mod\ p$$
- Define family
$$H = \{h_{a,b} : 0 \le a, b \le p - 1\}$$
- Theorem. H is a family of pairwise independent hash functions.

- It is enough to show that $\mathbf{Pr}_{h\in H}[h(x_1) = y_1 \; and \; h(x_2) = y_2] = \; 1/p^2$
- For any $x_1 \neq x_2$, y_1 and y_2, there is a unique pair (a, b) s.t. $h_{a,b}(x_1) = y_1 \; and \; h_{a,b}(x_2) = y_2$.
- Indeed, this is just

$$ax_1 + b = y_1 \; mod \; p$$
$$ax_2 + b = y_2 \; mod \; p$$

- which has a unique solution for (a, b) because $\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} \neq 0$ due to $x_1 \neq x_2$.