CMSC5706 Topics in Theoretical Computer Science

Week 12: Quantum computing

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Roadmap

- Intro to math model of quantum mechanics
- Review of quantum algorithms
- The power of quantum computers.
- Quantum games.

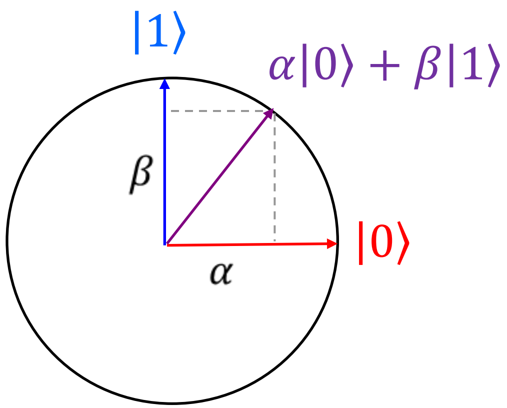
Postulate 1: States

- *State space: Every isolated physical system corresponds to a unit vector in a complex vector space.*
  - Unit vector: ℓ_2-norm is 1.
- Such states are called pure states.
- We use a weird "ket" notation $|\cdot\rangle$ to denote such a state.

Ket notation

- Mathematically, $|\cdot\rangle$ is a column vector.
- And $\langle\cdot|$ is a row vector.
- $\langle\psi|\phi\rangle$ is the inner product between the vectors $|\phi\rangle$ and $|\psi\rangle$.
- $\langle\psi|M|\psi\rangle$ is just the quadratic form $\psi^T M\psi$.

- A quantum bit, or qubit, is a state of the form
$$\alpha|0\rangle + \beta|1\rangle$$
where $\alpha, \beta \in \mathbb{C}$ are called amplitudes, satisfying that $|\alpha|^2 + |\beta|^2 = 1$.
- So a qubit can sit anywhere between 0 and 1 (on the unit circle).
- We say that the state is in superposition of $|0\rangle$ and $|1\rangle$.



A quantum bit (qubit)

Postulate 2: operation

- *Evolution: The evolution of a closed quantum system is described by a unitary transformation.*
- That is, if a system is in state $|\psi_1\rangle$ at time t_1, and in state $|\psi_2\rangle$ at time t_2, then there is a unitary transformation U s.t.

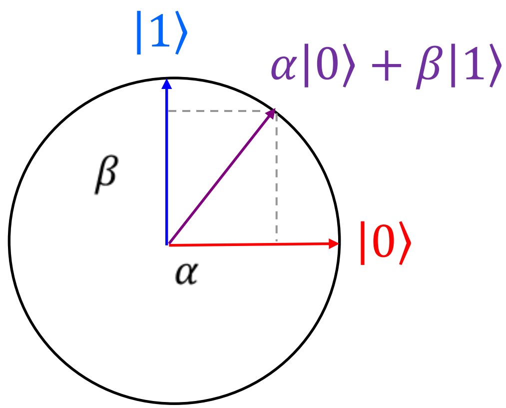
$$|\psi_2\rangle = U|\psi_1\rangle.$$

- Unitary transformation: $U^\dagger = U^{-1}$
  - Recall: $U^\dagger = (U^T)^*$, transpose + complex conjugate
  - You can think of it as a rotation operation.

Postulate 3: measurement

- *Measurement: We can only observe a quantum system by measuring it.*
- The outcome of the measurement is random.
- And the system is changed by the measurement.

- If we measure qubit $\alpha|0\rangle + \beta|1\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$, then outcome "0" occurs with prob. $|\alpha|^2$, and outcome "1" occurs with prob. $|\beta|^2$.
- The system becomes $|0\rangle$ if outcome "0" occurs, and becomes $|1\rangle$ if outcome "1" occurs.
  - The system collapses.



A quantum bit (qubit)

Measurement on general states

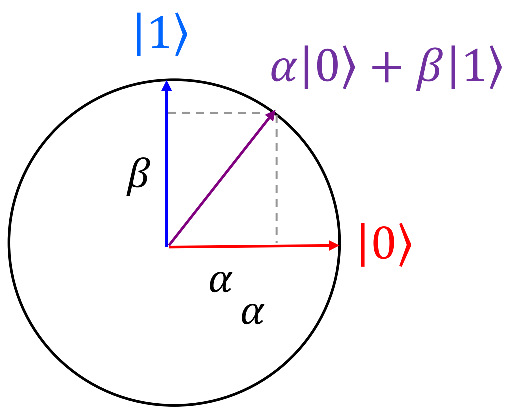
- In general, an orthogonal measurement of a d-dim state is given by an orthonormal basis $\{|\psi_1\rangle, \ldots |\psi_d\rangle\}$.
- If we measure state $|\phi\rangle$ in basis $\{|\psi_1\rangle, \ldots |\psi_d\rangle\}$, then outcome $i \in \{1, \ldots, d\}$ occurs with prob. $|\langle\phi|\psi_i\rangle|^2$.
- The system collapses to $|\psi_i\rangle$ if outcome i occurs.

Postulate 4: composition

- Composition: The state of the joint system (S_1, S_2), where S_1 is in state $|\psi_1\rangle$ and S_2 in $|\psi_2\rangle$, is $|\psi_1\rangle \otimes |\psi_2\rangle$.
- $\otimes$: tensor product of vectors.
  - $(a_1, a_2) \otimes (b_1, b_2, b_3) = (a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_1, a_2 b_2, a_2 b_3)$.
  - $\dim(|\psi_1\rangle \otimes |\psi_2\rangle) = \dim(|\psi_1\rangle) \cdot \dim(|\psi_2\rangle)$
  - $\text{size}(|\psi_1\rangle \otimes |\psi_2\rangle) = \text{size}(|\psi_1\rangle) + \text{size}(|\psi_2\rangle)$
    - size: number of qubits.
- Notation: $|0\rangle^{\otimes n} = |0\rangle \otimes \cdots \otimes |0\rangle$, n times.

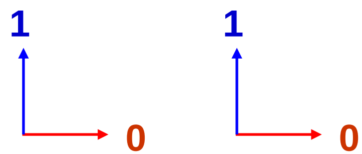
Quantum mechanics in one slide

Physics		Math
Physical System	$\Leftrightarrow$	Unit Vector
Evolution	$\Leftrightarrow$	Unitary Matrix
Measurement	$\Leftrightarrow$	Projection
Composition	$\Leftrightarrow$	Tensor Product

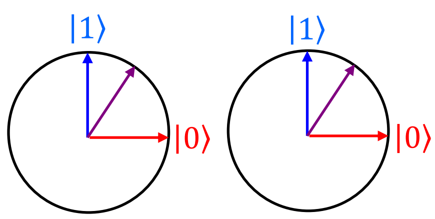


A quantum bit
(qubit)

Classical:



Quantum:



State space for 2 bits:
combinations $\{00,01,10,11\}$

State space for 2 qubits:
space $\text{span}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

Density matrix

- If a system is in state $|\psi_1\rangle$ with probability p_1, and in state $|\psi_2\rangle$ with probability p_2, then the system is in a mixed state.
- The mixed state is represented as a density matrix

$$\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|.$$

- In general, if a system is in state $|\psi_i\rangle$ with probability p_i, then the mixed state is

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$$

- For pure state $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi|$.

Density matrix

- Fact. A matrix ρ is a density matrix of some mixed quantum state iff
  - ρ is positive semi-definite (psd)
  - $\mathrm{Tr}(\rho) = 1$.
- Recall:
  - A matrix M is psd if all its eigenvalues are nonnegative. Equivalently, if $\langle v|M|v\rangle \geq 0, \forall v$.
  - The trace of a matrix M is $\mathrm{Tr}(M) = \sum_i M_{ii}$.

Postulates on mixed states

- Unitary operation U: $\rho \mapsto U\rho U^\dagger$
  - For pure state $\rho = |\phi\rangle\langle\phi|$, it becomes $U\rho U^\dagger = U|\phi\rangle\langle\phi|U^\dagger = |\phi'\rangle\langle\phi'|$ where $|\phi'\rangle = U|\phi\rangle$.
- Orthogonal measurement $\{|\psi_1\rangle, \dots |\psi_d\rangle\}$: outcome i occurs with probability $\left|\langle\psi|\rho|\psi\rangle\right|^2$, and the system collapses to $\rho' = |\psi_i\rangle\langle\psi_i|$.
  - For pure state $\rho = |\phi\rangle\langle\phi|$, the probability is $|\langle\phi|\psi_i\rangle|^2$, and the collapsed state is $|\psi_i\rangle$.
- If we measure $\rho \in \mathbb{C}^{d\times d}$ in the computational basis $\{|1\rangle, |2\rangle, \dots, |d\rangle\}$, then $\Pr[\text{outcome } i \text{ occurs}] = \rho_{ii}$, the i-th diagonal entry of ρ.

- Composition of ρ_1 and ρ_2 is just $\rho_1 \otimes \rho_2$
  - For pure state $\rho_1 = |\phi_1\rangle\langle\phi_1|$ and $\rho_2 = |\phi_2\rangle\langle\phi_2|$, the joint state is

$$|\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2| = (|\phi_1\rangle \otimes |\phi_2\rangle)(\langle\phi_1| \otimes \langle\phi_2|)$$

- Recall tensor product of matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

- For operation, measurement and composition, these formulas for mixed states are all consistent to what we learned for pure states.
- So the formulas for mixed states extend those for pure states.

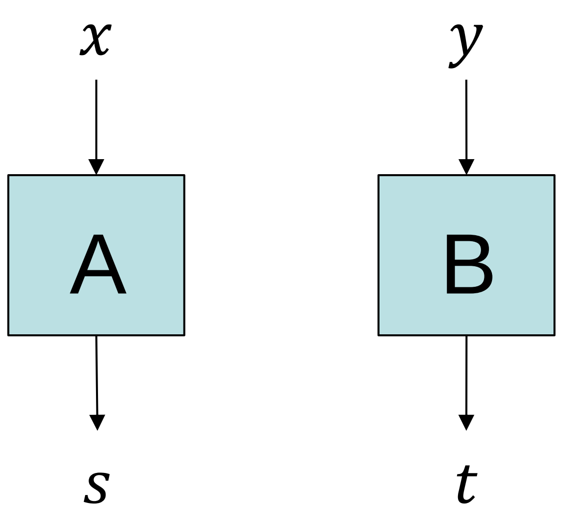
entanglement

- Consider the following EPR pair state in a 2-qubit system: $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$
- It's in superposition of $|00\rangle$ and $|11\rangle$.
- There is no classical counterpart of this.
- *Question: Is it really different than the classical correlation*

$$\begin{cases} 00 & \text{with prob. } 1/2 \\ 11 & \text{with prob. } 1/2 \end{cases} \quad ?$$

CHSH non-local game

- $x \in_R \{0,1\}$, $y \in_R \{0,1\}$
- Goal: A outputs s and B outputs t s.t.
$$s \oplus t = x \cdot y$$
- Value = largest $\Pr[s \oplus t = x \cdot y]$.
- Classical value: $3/4 = 0.75$.
  - Even with shared randomness.
- Quantum value: $\frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85$
  - By sharing an EPR pair.



$$s \oplus t = x \cdot y\,?$$

Roadmap

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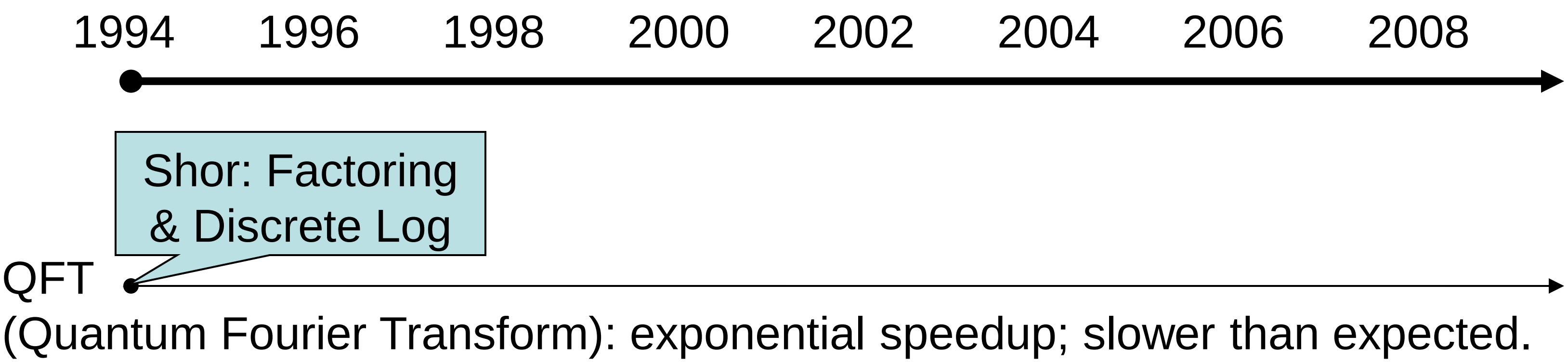
- Review of quantum algorithms
- The power of quantum computers.
- Quantum games.

Areas in quantum computing

- Quantum algorithms
- Quantum complexity
- Quantum cryptography
- Quantum error correction
- Quantum information theory
- Others: Quantum game theory / control / …

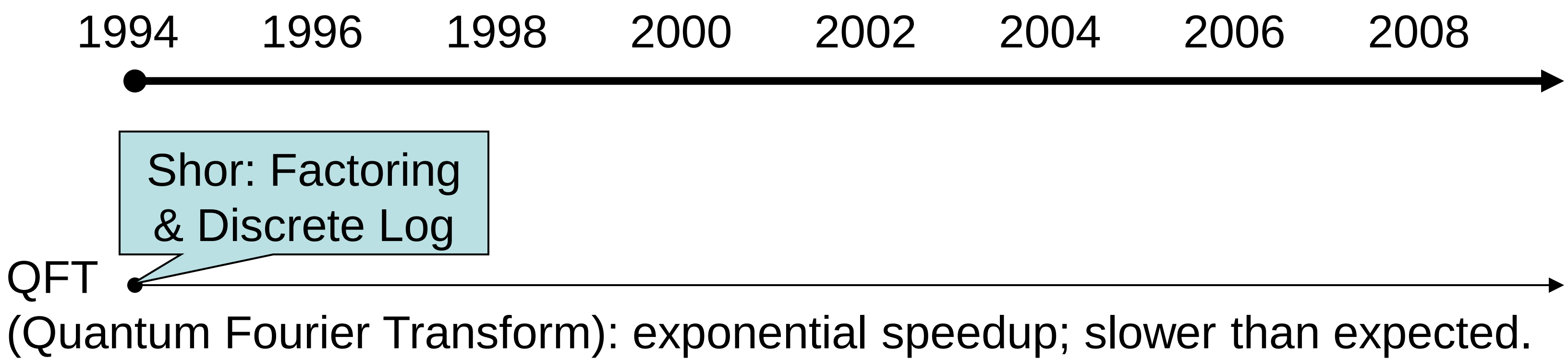
Area 1: Quantum Algorithms



- Factoring: Given an n-bit number, factor it (into product of two numbers).
- Classical (best known) : $O\left(2^{n^{1/3}}\right)$
- Quantum*1: $O(n^2)$

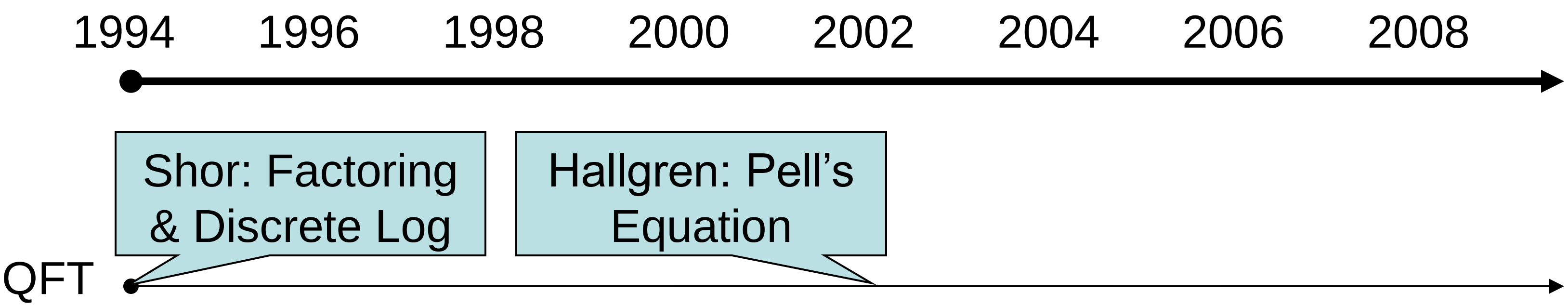
*1: P. Shor. *STOC*'93, *SIAM Journal on Computing*, 1997.

Area 1: Quantum Algorithms



- Implication of fast algorithm for Factoring
  - Theoretical: Church-Turing thesis
  - Practical: Breaking RSA-based cryptosystems

Area 1: Quantum Algorithms

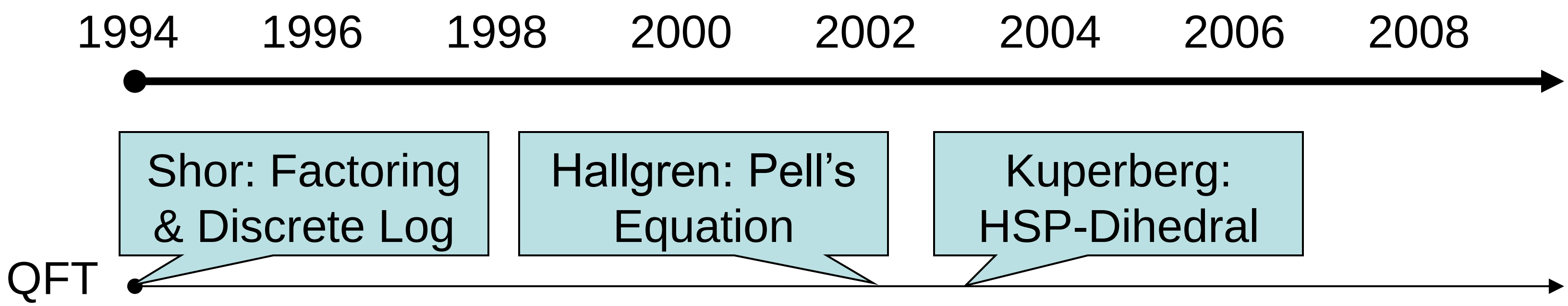


(Quantum Fourier Transform): exponential speedup; slower than expected.

- Pell's Equation: $x^2 - ny^2 = 1$.
- Problem: Given n, find solutions (x, y) to the above equation.
- Classical (best known):
  - $\sim 2^{\sqrt{\log n}}$ (assuming the generalized Riemann hypothesis)
  - $\sim n^{1/4}$ (no assumptions)
- Quantum*1: $poly(\log n)$.

*1: S. Hallgren. *STOC*'02. *Journal of the ACM*, 2007.

Area 1: Quantum Algorithms

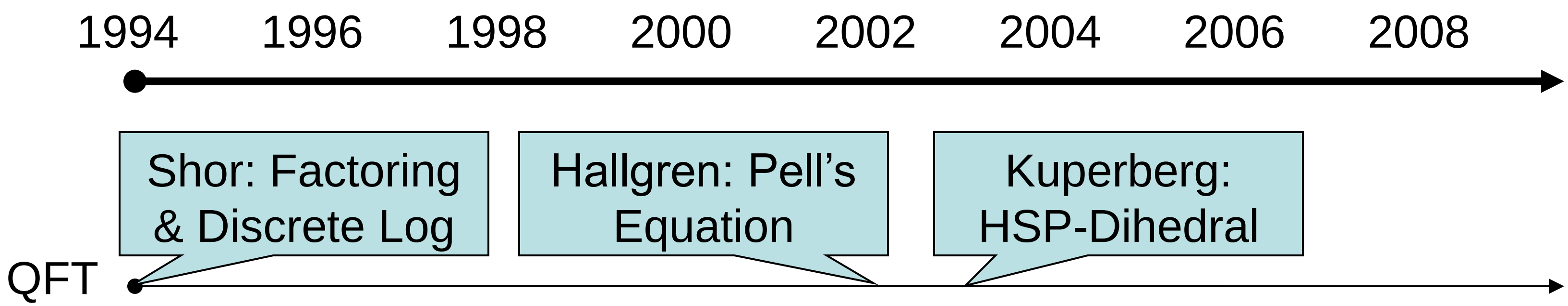


(Quantum Fourier Transform): exponential speedup; slower than expected.

- Hidden Subgroup Problem (HSP): Given a function f on a group G, which has a hidden subgroup H, s.t. f is
  - constant on each coset aH,
  - distinct on different cosets.

  Task: find the hidden H.
- Factoring, Pell's Equation both reduce to it.
- Efficient quantum algorithms are known for Abelian groups.
- Main question: HSP for non-Abelian groups?

Area 1: Quantum Algorithms

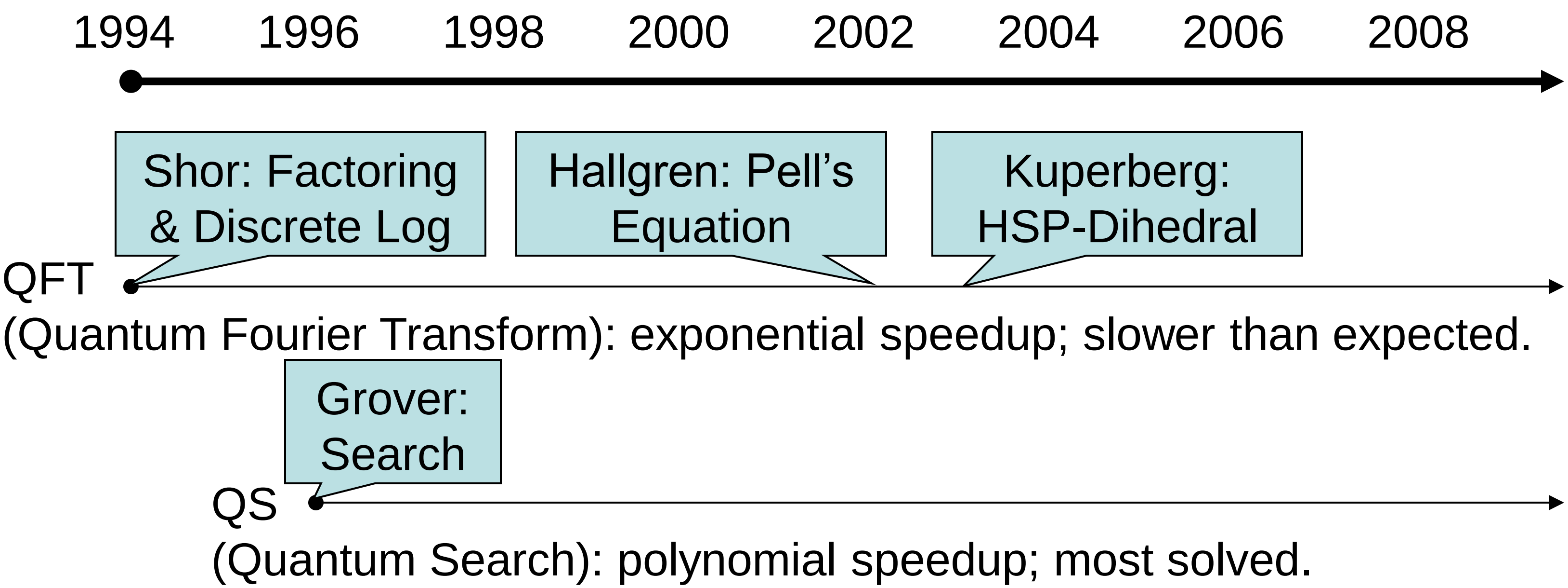


(Quantum Fourier Transform): exponential speedup; slower than expected.

- Two biggest cases:
  - HSP for symmetric group S_n: Graph Isomorphism reduces to it.
  - HSP for dihedral group D_n: Shortest Lattice Vector reduces to it.
- HSP(D_n):
  - Classical (best known): $2^{\log|G|}$
  - Quantum*1: $2^{O(\sqrt{}\log|G|)}$ $2^{O\left(\sqrt{\log|G|}\right)}$

*1: G. Kuperberg. arXiv:quant-ph/0302112, 2003.

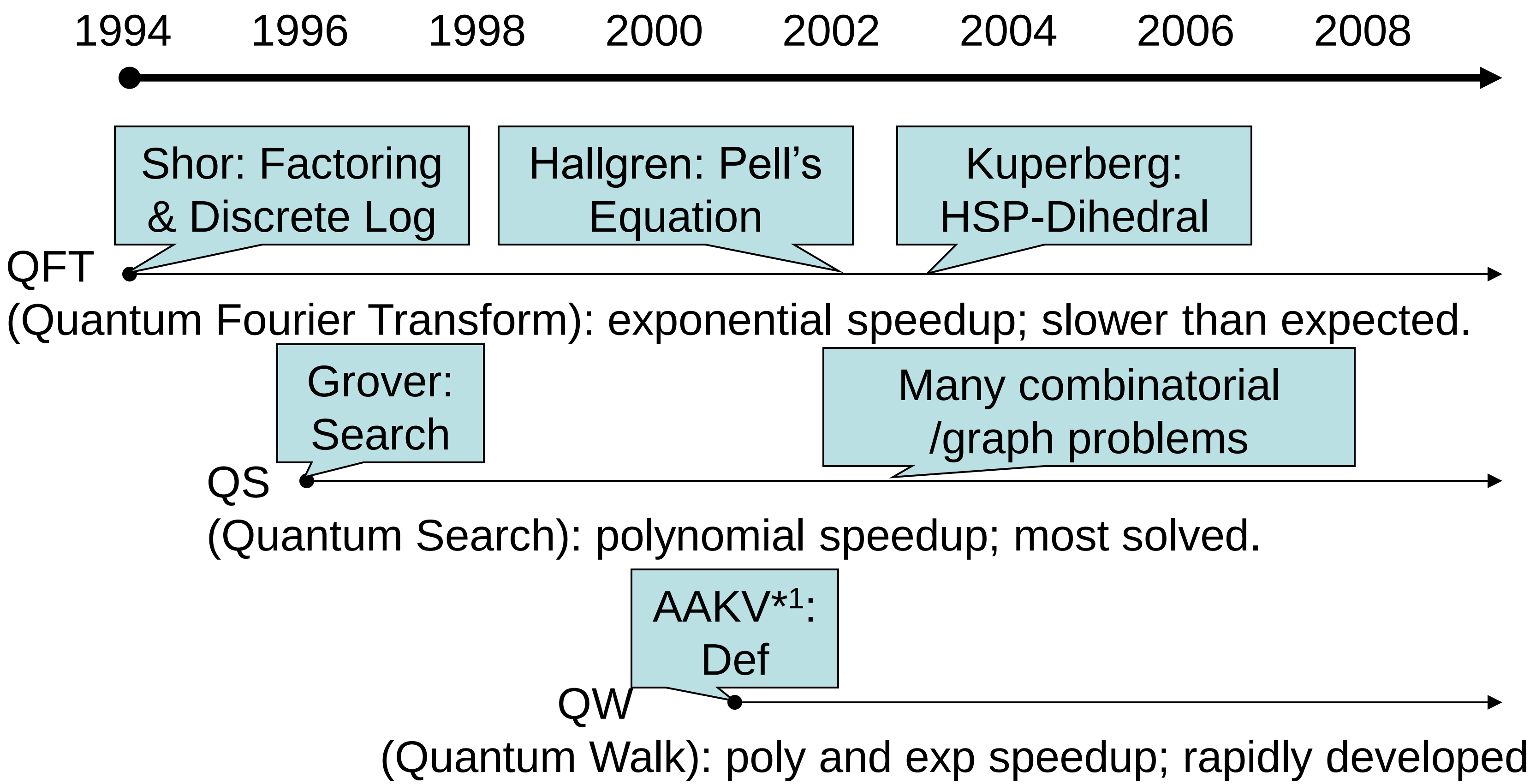
Area 1: Quantum Algorithms



- Given n bits $x_1, \ldots, x_n$, find an i with $x_i = 1$.
- Classical: $\Theta(n)$
- Quantum*1: $\Theta(\sqrt{n})$

*1: L. Grover. *Physical Review Letters*, 1997.

Area 1: Quantum Algorithms



*1: D. Aharonov, A. Ambainis, J. Kempe, U. Vazirani. *STOC'01*

Area 1: Quantum Algorithms

1994 1996 1998 2000 2002 2004 2006 2008

- Classical random walk on graphs: starting from some vertex, repeatedly go to a random neighbor
  - Many algorithmic applications
- Quantum walk on graphs: even definition is non-trivial.
  - For instance: classical random walk converges to a stationary distribution, but quantum walk doesn't since unitary is reversible.

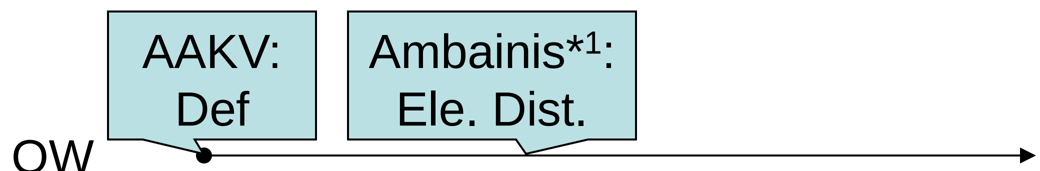
AAKV*1: Def

QW (Quantum Walk): poly and exp speedup; rapidly developed.

*1: D. Aharonov, A. Ambainis, J. Kempe, U. Vazirani. *STOC'01*

Area 1: Quantum Algorithms

- Element Distinctness: Given n integers, decide whether they are the all distinct.
- Classical: $\Theta(n)$
- Quantum: $\Theta\left(n^{2/3}\right)$

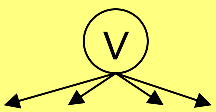


(Quantum Walk): poly and exp speedup; rapidly developed.

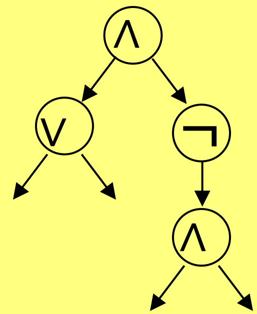
*1: A. Ambainis, *FOCS'04*

Area 1: Quantum Algorithms

1994 1996 1998 2000 2002 2004 2006 2008



Grover's search:
OR function



general formula
by {AND-OR-NOT}

- Classical: $\Theta(n)$
- Quantum: ~ $\Theta(\sqrt{n})$
- apply QW on the formula graph with weight carefully designed for inductions to work.

AAKV: Def

Ambainis: Ele. Dist.

ACRSZ*1: Formula Evaluation

QW
(Quantum Walk): poly and exp speedup; rapidly developed.

*1: A. Ambainis, A. Childs, B. Reichardt, R. Spalek, S. Zhang. *FOCS'07*

Roadmap

- Intro to math model of quantum mechanics
- Review of quantum algorithms
- The power of quantum computers.
- Quantum games.

Power of quantum computing

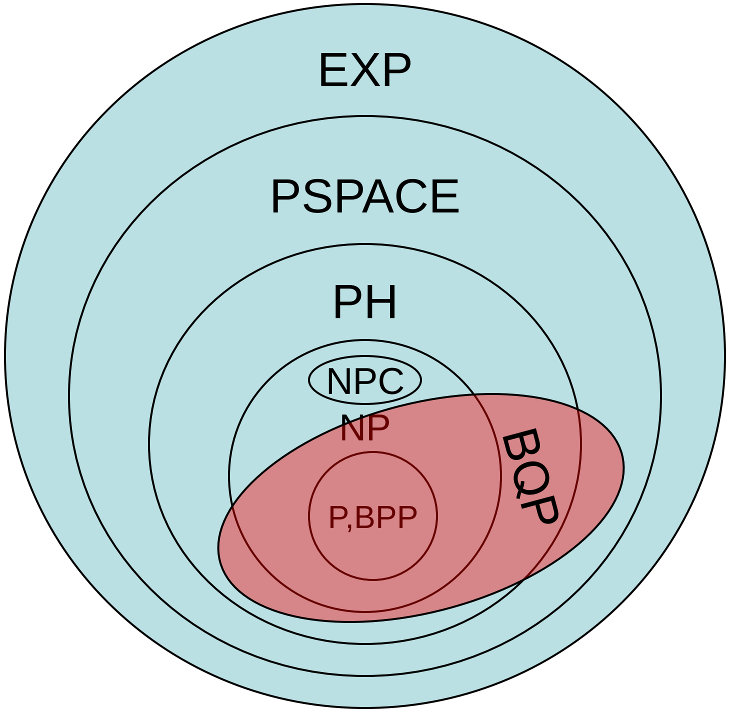
- *Question: How powerful is quantum computer?*
- P: problems solvable in polynomial time
  - One characterization of efficient computation
- BPP: problems solvable in probabilistic polynomial time w/ a small error tolerated
  - Another characterization of efficient computation
- BQP: problems solvable in polynomial time by a quantum computer w/ a small error tolerated
  - Yet another characterization of efficient computation, if you believe large-scale quantum mechanics.

Classical upper bound of BQP

- Central in complexity theory: comparison of different modes of computation
- How to compare classical and quantum efficient computation?
- Quantum is more powerful: BPP ⊆ BQP
- An upper bound (of quantum by classical)
- [Thm*[1]] BQP ⊆ PSPACE
  - PSPACE: problems solvable in polynomial space.
  - Believed to be much larger than NP.

*1: Bernstein, Vazirani. *STOC'93, SIAM J. on Computing, 1997*

Where does BQP sit in?



- BQP contains BPP and P.
- But it probably doesn't contain all NP.
- Yet it's possible to be outside PH.
- It's position may be weird.

Roadmap

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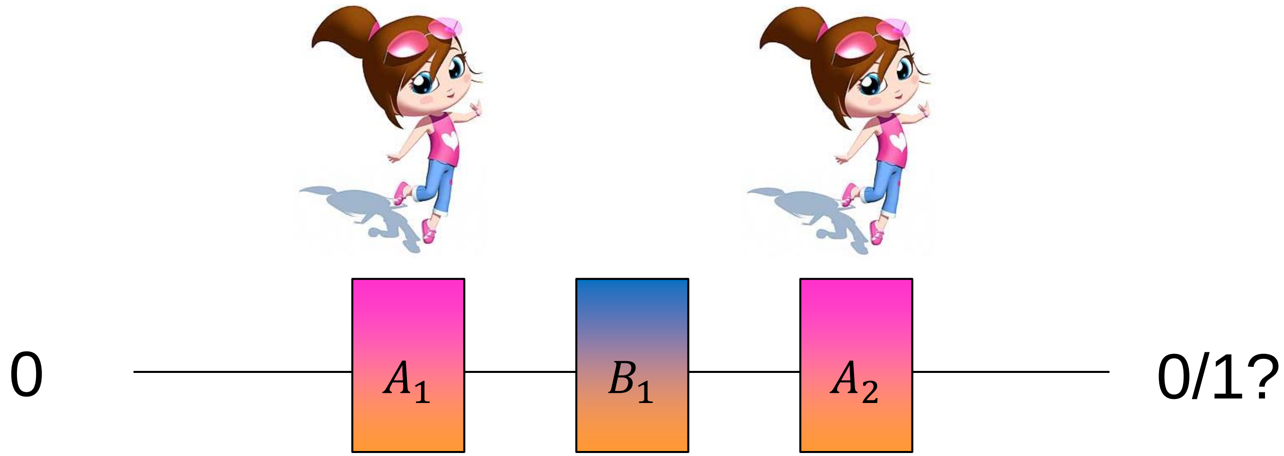
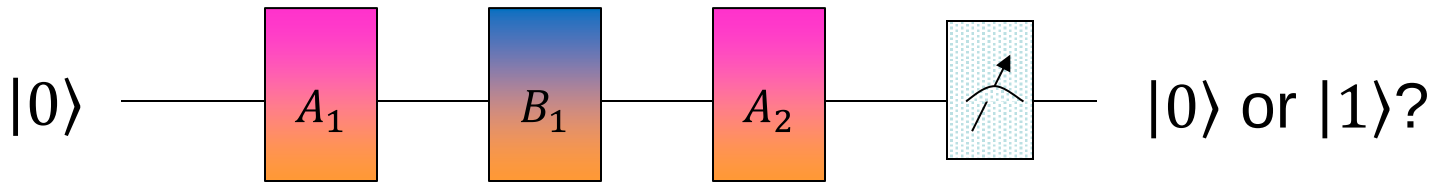
Meyer’s game: classical



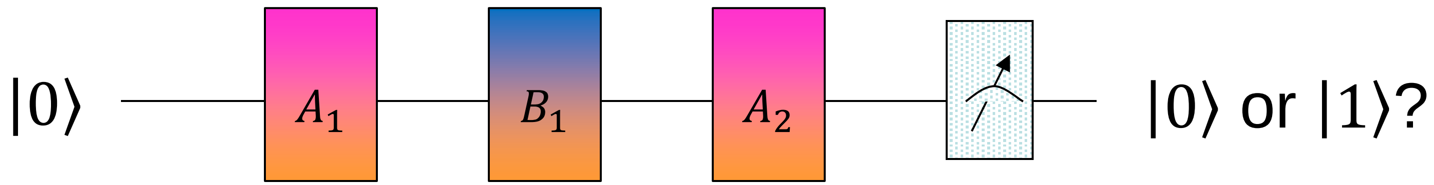
- Actions A_i, B_i:
  to flip or not to flip
- Alice’s Goal: 0. Bob’s Goal: 1.
- A Nash equilibrium: A_i, B_i flip with half prob.
  – Then each wins with half prob.

Meyer's game: quantum



- Bob remains classical: B_1 is either $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (Swap $|0\rangle$ and $|1\rangle$) or identity (doing nothing).
- Alice is quantum: A_i can be any 1-qubit operation.
- Alice's Goal: $|0\rangle$. Bob's Goal: $|1\rangle$.
- Now Alice can win for sure by applying a Hadamard gate. $A_1: |0\rangle \rightarrow |+\rangle$. $A_2: |+\rangle \rightarrow |0\rangle$.

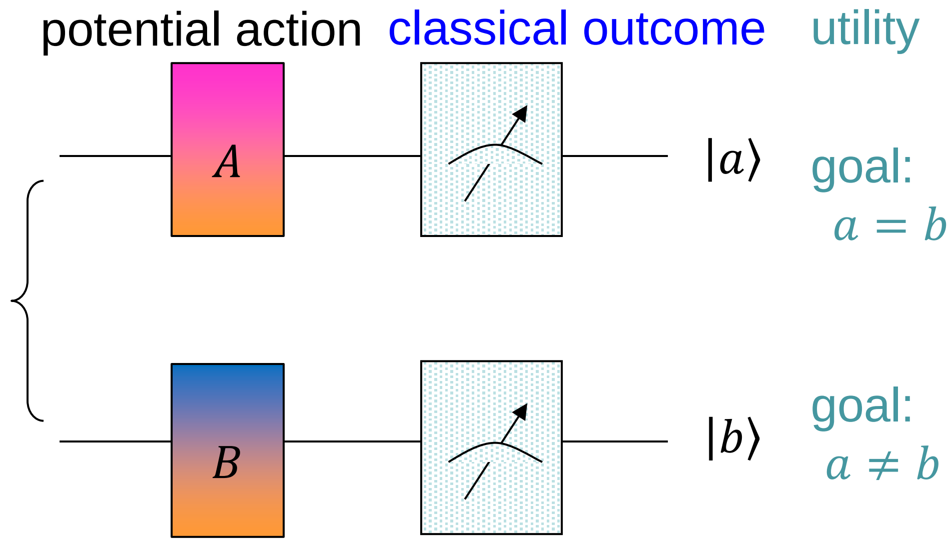
Meyer's game: fairness issue



- Despite the quantum advantage, there is clear a fairness issue.
  - Alice has two actions.
  - And the actions are in a fixed order of $A_1 \to B_1 \to A_2$.
- *Question: Can quantum advantage still exist in a more fair setting?*
- For fairness: each player makes just one action, simultaneously.
  - This is nothing but strategic games!

Quantization*[1] of strategic game: Penny Matching

$$|\varphi\rangle = \frac{|+\rangle|0\rangle + |-\rangle|1\rangle}{\sqrt{2}}$$



- $|\varphi\rangle$ is an equilibrium if both players are classical,
  - Each wins with half prob.
- If Alice turns to quantum: $A = H$ turns $|\varphi\rangle$ into $\frac{|0\rangle|0\rangle+|1\rangle|1\rangle}{\sqrt{2}}$. Then she wins for sure!
- *Message: quantum player has a huge advantage when playing against a classical player.*

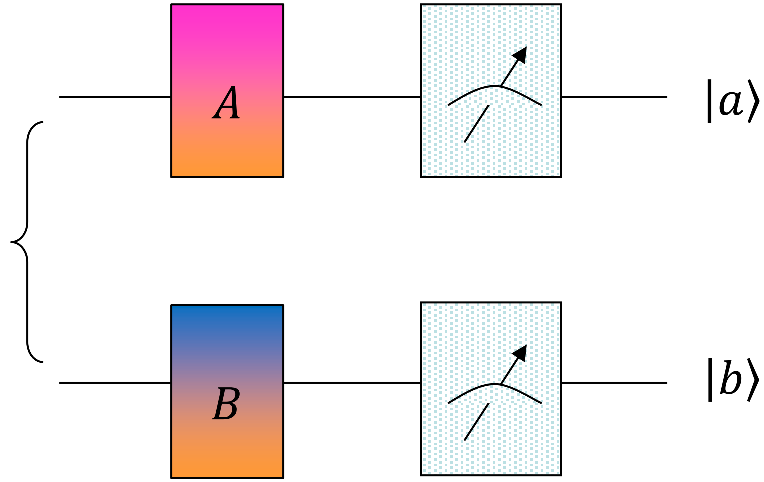
*1. Zu, Wang, Chang, Wei, Zhang, Duan, NJP, 2012.

Quantization of strategic game: Penny Matching

potential action | classical outcome | utility

$$|\varphi\rangle = \frac{|+\rangle|0\rangle + |-\rangle|1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle|+\rangle + |1\rangle|-\rangle}{\sqrt{2}}$$



goal: $a = b$

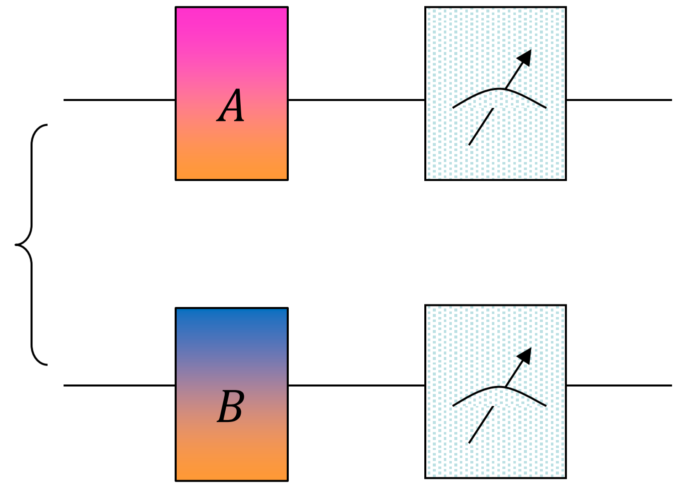
goal: $a \neq b$

- State is symmetric, so it doesn't matter who takes which qubit.
- We can also let the classical player Bob to choose the target goal.
  - If Bob wants $a = b$, then Alice applies XH.

Quantum advantage in strategic games

$$|\varphi\rangle = \frac{|+\rangle|0\rangle + |-\rangle|1\rangle}{\sqrt{2}}$$

Entangled.
Necessary?



No! $\rho = \frac{1}{4}\begin{pmatrix} |+\rangle\langle+| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |+\rangle\langle+| \\ +|-\rangle\langle-| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |-\rangle\langle-| \end{pmatrix}$

No entanglement.
But has discord.

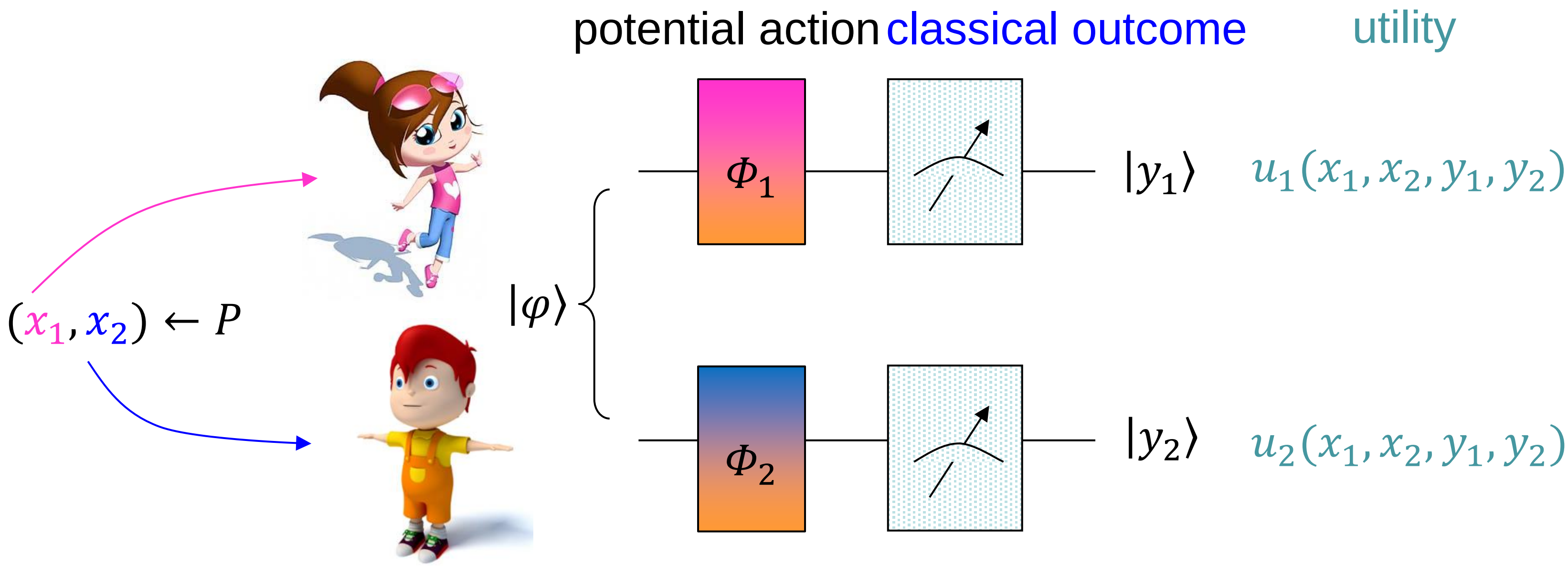
- ρ is an equilibrium if both players are classical, each winning with prob. = ½
- If Alice uses quantum, $A = H$ increases her winning prob. to ¾.
- *Question*[*2]*: Is discord necessary?*
  - Yes, if each player's part (of the shared state) is a qubit,
  - No, if each player's part (of the shared state) has dimension 3 or more.

*2. Wei, Zhang, TAMC, 2014.

Games between quantum players

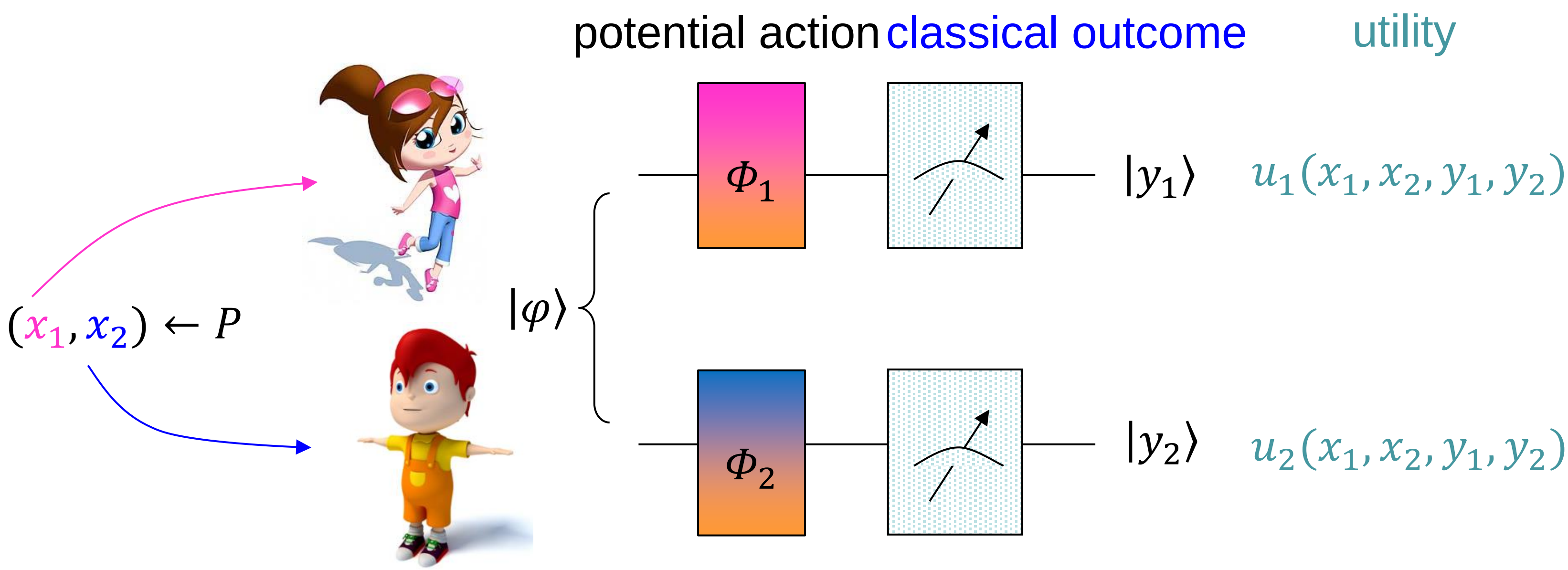
- After these examples, Bob realizes that he should use quantum computers as well.
- *Question: Any advantage when both players are quantum?*
- Previous correspondence results imply a negative answer for complete information games.
- But quantum advantage exists for Bayesian games!

Quantum Bayesian games



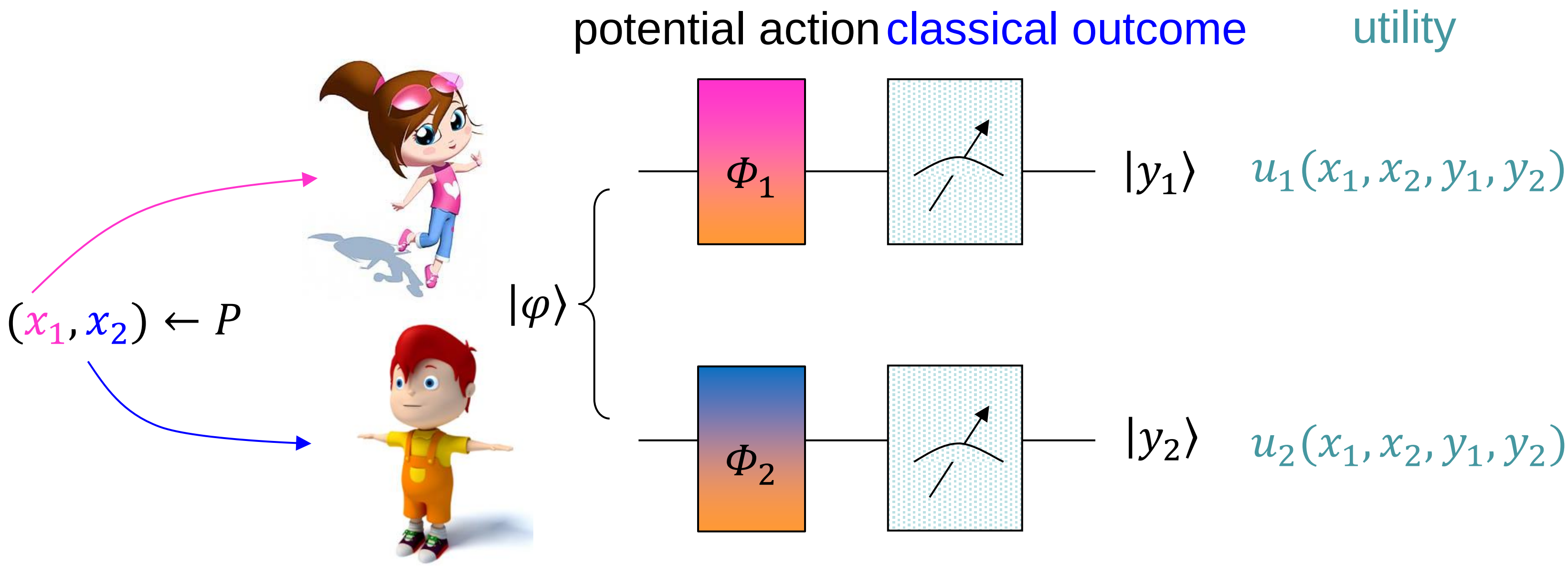
- Each player i has a private input/type x_i.
  - x_i is known to Player i only.
  - The joint input is drawn from some distribution P.
- Each player i can potentially apply some operation Φ_i.
- A measurement in the computational basis gives output $|y_i\rangle$ for Player i, who receives utility $u_i(x_1, x_2, y_1, y_2)$.

Quantum Bayesian games



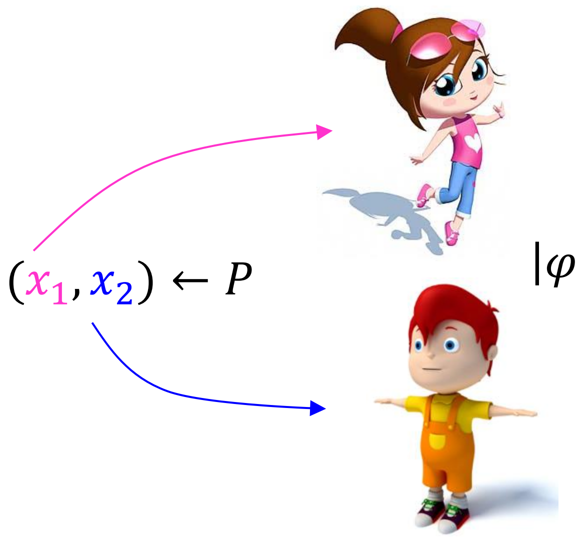
- Classical state $|\varphi\rangle = (r_1, r_2) \leftarrow$ distribution Q.
- Classical strategy $\Phi_i = c_i(x_i, r_i)$.
- Classical payoff
$$\mathbf{E}[u_i] = \mathbf{E}_{x \leftarrow P, r \leftarrow Q}[u_i(x, c_1(x_1, r_1), c_2(x_2, r_2))]$$
- (Q, c_1, c_2) is equilibrium if no player can gain a higher payoff by changing her strategy unilaterally.

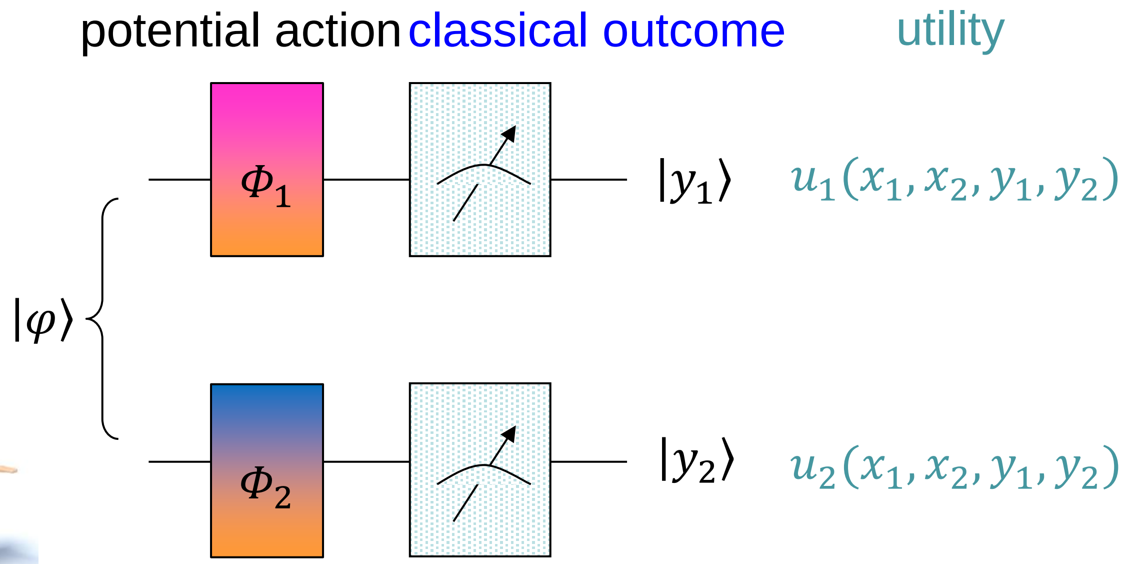
Quantum Bayesian games



- Quantum strategy $\Phi_1 = \{E_{x_1}^{y_1} : E_{x_1}^{y_1} \succcurlyeq 0, \sum_{y_1} E_{x_1}^{y_1} = I\}$, $\Phi_2 = \{F_{x_2}^{y_2} : F_{x_2}^{y_2} \succcurlyeq 0, \sum_{y_2} F_{x_2}^{y_2} = I\}$.
- Quantum payoff
$$\mathbf{E}[u_i] = \mathbf{E}_{x \leftarrow P}\left[\langle\varphi\middle|E_{x_1}^{y_1} \otimes F_{x_2}^{y_2}\middle|\varphi\rangle \cdot u_i(x, y)\right]$$
- $(|\varphi\rangle, \Phi_1, \Phi_2)$ is equilibrium if no player can gain a higher payoff by changing her strategy unilaterally.

Quantum Bayesian games





	$y_B = 0$	$y_B = 1$
$y_A = 0$	(1,1/2)	(0,0)
$y_A = 1$	(0,0)	(1/2,1)

Table I: $x_A \wedge x_B = 0$

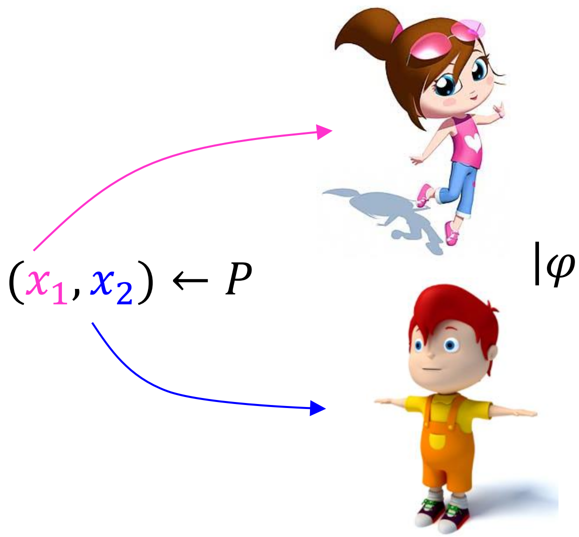
	$y_B = 0$	$y_B = 1$
$y_A = 0$	(0,0)	(3/4,3/4)
$y_A = 1$	(3/4,3/4)	(0,0)

Table II: $x_A \wedge x_B = 1$

- A game*[1] combining Battle of the Sexes and CHSH.
- The players need to coordinate like in CHSH, except when $x_1 = x_2 = 1$, in which case they need to anti-coordinate.
- In Table I, they have conflicting interest.

*1. Pappa, Kumar, Lawson, Santha, Zhang, Diamanti, Kerenidis, *PRL*, 2015.

Quantum Bayesian games



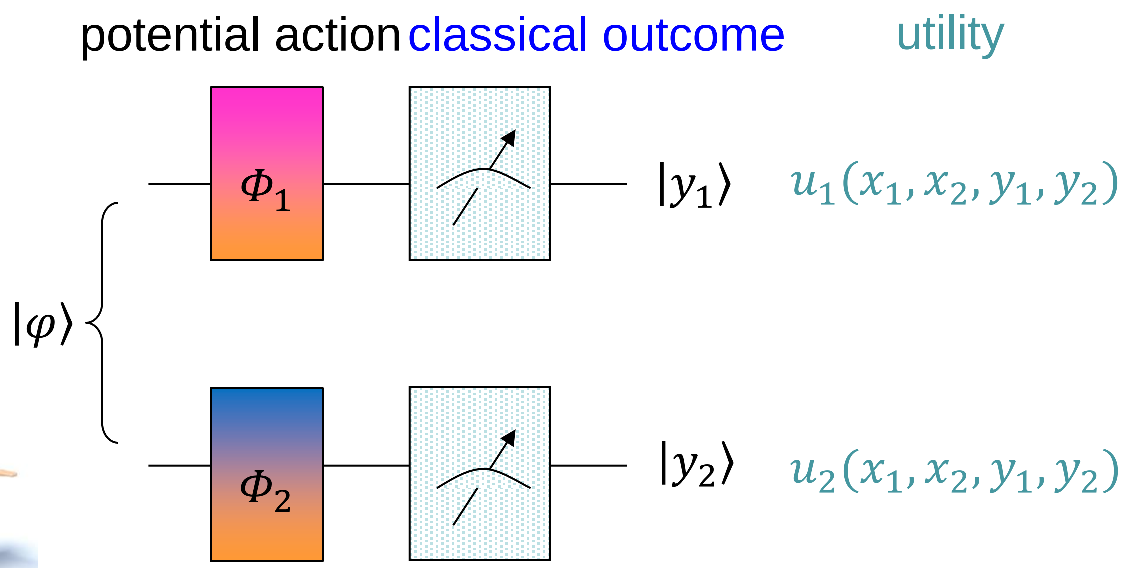


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Table II: $x_A \wedge x_B = 1$

- P is uniform.
- Classical: $u_1 + u_2 \le 9/8$.
  And ∃ a fair equilibrium with
  $u_1 = u_2 = 9/16 = 0.5625.$
- Quantum: ∃ a fair equilibrium with
  $u_1 = u_2 = (3/4)\cos^2(\pi/8) \approx 0.64$

*1. Pappa, Kumar, Lawson, Santha, Zhang, Diamanti, Kerenidis, *PRL*, 2015.

Viewed as non-locality

- Traditional quantum non-local games exhibit quantum advantages when the two players have the common goal.
  - CHSH, GHZ, Magic Square Game, Hidden Matching Game, Brunner-Linden game.
- Now the two players have conflicting interests.
- Quantum advantages still exist.
- *Message: If both players play quantum strategies in an equilibrium, they can also have advantage over both being classical.*

Summary

- Quantum algorithms: offer huge speedup for certain computational problems.
- Quantum entanglement:
  - A distinctive feature of quantum mechanics.
  - Proof that our world is quantum mechanical.
- Quantum games: quantum players can have big advantages.