# Lecture 6. Formula complexity II

## From Khrapchenko’s bound to convex rectangle measures

In the last lecture, we proved a result by Khrapchenko for any $A⊆f^{-1}(1)$, $B⊆f^{-1}(0)$, define $S=\left\{\left(x,y\right):x\in A,y\in B,\left|x⊕y\right|=1\right\}$ and we have $L\left(f\right)\geq \frac{\left|S\right|^{2}}{\left|A\right|⋅\left|B\right|} $. (The last lecture considered the case that $A=f^{-1}(1)$ and $B=f^{-1}\left(0\right)$, but convince yourself that the lower bound holds even A and B are subsets of $f^{-1}(1)$ and $f^{-1}(0)$.) We then used it to prove that $L\left(Parity\_{n}\right)\geq n^{2}.$ We of course hope to prove stronger lower bounds. So a natural question is: Can we use the same technique to prove better lower bounds? In general, understanding the power and limit of a prevailing lower bound technique is an important task. For Khrapchenko’s method, unfortunately, the answer is pretty negative: $n^{2}$ is the best lower bound we can get.

**Exercise**. Show that for *A*, *B* and *S* as defined above, it always holds that $\frac{\left|S\right|^{2}}{\left|A\right|⋅\left|B\right|}\leq n^{2}$.

So it’s a bad news. But maybe we can use a similar technique? Well, that depends on the exact definition of “similar”. In this lecture, we’ll investigate the power of this type of technique along one specific direction.

Next we will

1. define a concept of convex rectangle measures,
2. show that Kharpchenko’s bound uses a convex rectangle measure,
3. show that all convex rectangle measures cannot give lower bounds better than $O(n^{2}).$

Fix a rectangle R. Consider a function $μ $on subrectangles of R. (As the name suggests, a set is a *subrectangle* of R if it’s a subset of R and it’s a rectangle.) The function is a *rectangle measure* if it satisfies

* *subadditive*: $μ\left(R\right)\leq μ\left(R\_{1}\right)+μ(R\_{2})$, where *R* is partitioned into the disjoint union of$R\_{1}$ and $R\_{2}$, which are both rectangles themselves,
* *normalized*: $μ\left(R'\right)\leq 1$ for any monochromatic subrectangle *R’*.

One can view Khrapchenko’s bound in the following way. Pick any rectangle $R=A×B⊆f^{-1}\left(1\right)×f^{-1}(0)$, and one gets a lower bound of $\frac{\left|S\right|^{2}}{\left|R\right|}$. (Here we used $|R|$ to denote the size of *R*, namely$\left|A\right|⋅|B|$, the number of entries in *R*.) Actually, from the proof of the theorem, we can see that what was actually showed is the following generalized fact:

**Theorem 1.1**. For any rectangle $R⊆f^{-1}\left(1\right)×f^{-1}\left(0\right)$ and any rectangle measure $μ$ on R, we have $L\left(f\right)\geq μ\left(R\right)$.

Now let’s define convexity. Suppose $R\_{1},…, R\_{k}$ are subrectangles of R, and $r\_{1},…, r\_{k}$ are real numbers in [0,1] called *weights*. The set $\{R\_{i,} r\_{i}:i\in \left[k\right]\}$ forms a *fractional partition* of R if for any element $e\in R$, $\sum\_{i:e\in R\_{i}}^{}r\_{i}=1$. Namely, each rectangle covers a fraction of each entry, and the collection of these covers exactly equals to R. We will use the notation $R=\sum\_{i}^{}r\_{i}R\_{i}$. A rectangle measure $μ$ on R is *convex* if

$$μ\left(R\right)\leq \sum\_{i=1}^{k}r\_{i}μ\left(R\_{i}\right)$$

(Note that it may be a bit misleading that the name of “convexity” suggest that $\sum\_{i}^{}r\_{i}=1$, but actually from $\sum\_{i}^{}r\_{i}|R\_{i}|=|R|$ we know that $\sum\_{i}^{}r\_{i}\geq 1$ in general.)

Now we proceed to the second step showing that Khrapchenko’s bound is a convex rectangle measure. Khrapchenko’s bound, when used on a fixed rectangle R, actually corresponds to the rectangle measure $μ\left(R'\right)=\frac{\left|S∩R^{'}\right|^{2}}{|R|}$. Now we prove that the function is a convex rectangle measure. In the proof of the original theorem in last lecture, we already see that $μ$ is a rectangle measure. (Review the proof if this is not clear to you.) Let’s prove the convexity. Suppose that $\{R\_{i,}r\_{i}:i\in \left[k\right]\}$ forms a fractional partition of R. Put $S\_{i}=S∩R\_{i}$. Since $R=\sum\_{i}^{}r\_{i}R\_{i} $, we have $|R|=\sum\_{i}^{}r\_{i}|R\_{i}|$. And by considering each entry, we can also see that $|S|=\sum\_{i}^{}r\_{i}|S\_{i}|$. Therefore, we have

$$μ\left(R\right)=\frac{\left|S\right|^{2}}{|R|}=\frac{\left(\sum\_{i}^{}r\_{i}\left|S\_{i}\right|\right)^{2}}{\sum\_{i}^{}r\_{i}\left|R\_{i}\right|}\leq \sum\_{i}^{}r\_{i}\frac{\left|S\_{i}\right|^{2}}{\left|R\_{i}\right|}=\sum\_{i}^{}r\_{i}μ\left(R\_{i}\right),$$

where the inequality is by Cauchy-Schwartz Inequality.

Finally, we show the most important result of this section: Convex rectangle measures cannot produce lower bounds better than quadratic. It was firstly proven in [KKN95] that convex rectangle measures can give lower bound no better than $4n^{2}$, later improved to about $\frac{9}{8}n^{2}$ in [HJKP10]. We’ll present the later result, from which you can see the interesting connection to parity function. (And interestingly, it is the $O\left(n^{2}\right)$ *upper* bound of the formula size of the parity function that is used to show this limit.)

**Theorem 1.2**. For any convex rectangle measure $μ$ and any rectangle $R⊆f^{-1}\left(1\right)×f^{-1}(0)$, it holds that $μ\left(R\right)\leq \frac{9}{8}\left(n^{2}+n\right).$

*Proof*. We will design a fractional partition as follows. For any $I⊆\left\{0,1\right\}^{n}$ and any $b\in \{0,1\}$, define rectangle $R\_{I,b}=Parity\_{I}^{-1}\left(b\right)×Parity\_{I}^{-1}\left(\overbar{b}\right)$. (Here $Parity\_{I}^{-1}\left(b\right)=\left\{x\in \left\{0,1\right\}^{n}:⊕\_{i\in I}x\_{i}=b\right\}$.) We claim that R has a fractional partition

$\{R∩R\_{I,b}:I⊆\left\{0,1\right\}^{n},b\in \left\{0,1\right\}\}$, each rectangle with weight $r\_{I,b}=2^{-\left(n-1\right)}$.

(Note that here the set is a multi-set, so some elements may appear more than once. But we count them differently. Or, an equivalent way to phrase this is that, if some rectangle appears k times, then its weight is $k2^{-\left(n-1\right)}.)$

Indeed, for any $\left(x,y\right)\in R$,

$$\left|\left\{\left(I,b\right):Parity\_{I}\left(x\right)=b,Parity\_{I}\left(y\right)=\overbar{b}\right\}\right|$$

$$=\left|\left\{I:Parity\_{I}\left(x\right)\ne Parity\_{I}\left(y\right)\right\}\right|$$

$$=\left|\left\{I:Parity\_{I}\left(x+y\right)\ne 0\right\}\right|$$

$$=2^{n-1},$$

for any $x\ne y$.

Next is the connection to the parity function. Note that rectangle $R\_{I,b}$ is actually the $g^{-1}\left(1\right)×g^{-1}\left(0\right)$ for $g=Parity or \overline{Parity}$. At the end of the last lecture, we showed that the formula leaf size of Parity is at most $O\left(n^{2}\right)$, and mentioned that the constant can be as small as 9/8. Since $μ\left(R\_{I,b}\right)$ is a lower bound of the leaf size, we know that $μ\left(R\_{I,b}\right)\leq \frac{9}{8}n^{2}.$ (It’s easy to see that negation doesn’t change the leaf size of a function: $L\left(g\right)=L(\overbar{g})$.)

Now we can bound $μ(R)$ from above:

$μ\left(R\right) \leq \sum\_{I,b}^{}r\_{I,b}μ\left(R\_{I,b}\right)$ // assumption of convexity of $μ$

$=2^{-\left(n-1\right)}\sum\_{I,b}^{}μ\left(R\_{I,b}\right)$ // each $r\_{I,b}=2^{-\left(n-1\right)}$

$\leq 2^{-\left(n-1\right)}\sum\_{I,b}^{}\frac{9}{8}\left|I\right|^{2}$ // above Fact.

$=\frac{9}{8}(n^{2}+n)$*.*

where the last step follows from some basic manipulation of binomial coefficients. $□$

## Monotone formula size

When we restrict the formula to be monotone, then we can get better lower bounds. How much better? Exponential. It’s probably a bit too technical to write down full details, but we can sketch an approach.

Recall that a rectangle$ A×B$ is 1-monochromatic if there exists an $i\in [n]$, s.t. for all $x\in A$and $y\in B$, it holds that $x\_{i}=1$ and $y\_{i}=0$. A rectangle $A×B$ is 0-monochromatic if there exists an $i\in [n]$, s.t. for all $x\in A$ and $y\in B$, it holds that $x\_{i}=0$ and $y\_{i}=1$. For monotone functions, the monotone leaf size $L\_{+}(f)$ is the minimal number of leaves of any monotone formula that computes *f*.

Consider a rectangle $R=A×B⊆f^{-1}\left(1\right)×f^{-1}(0)$. We associate with R a matrix $M\_{R}$. Formally, use a map $ϕ:R\rightarrow ϕ\left(R\right)=M\_{R}$. Suppose that $ϕ$ is linear, thus if R has a decomposition $R=\sum\_{i}^{}R\_{i}$ then so does $M\_{R}: M\_{R}=\sum\_{i}^{}M\_{R\_{i}}$. Consider all 1-monochromatic subrectangles R’, which corresponds to a submatrix $M\_{R'} $of $M\_{R}$. If we can bound $rank\left(M\_{R^{'}}\right)\leq r$ for all 1-monochromatic subrectangles R’, then we obtain a lower bound of $L\_{+}(f)$:

$L\_{+}\left(f\right)\geq rank\left(M\_{R}\right)/r$.

Actually, if $R=\sum\_{i=1}^{k}R\_{i}$ where $r=χ\_{+}(f)$ and each$R\_{i}$ is 1-monochromatic, we have $M\_{R}=\sum\_{i=1}^{r}M\_{R\_{i}}$. Thus $rank\left(M\_{R}\right)\leq \sum\_{i=1}^{k}rank\left(M\_{R\_{i}}\right)\leq kr$, thus

$$L\_{+}\left(f\right)\geq χ\_{+}\left(f\right)=k\geq \frac{rank\left(M\_{R}\right)}{r}.$$

In general it’s not easy to design the map $ϕ$ with those properties. One special case is the *local intersection* property. For that we need to mention concepts of 1-term and 0-term for monotone functions. For any x with $f(x)=1$, there is a *1-term* (of this value) defined as a minimal subset S of $\{i\in \left[n\right]:x\_{i}=1\}$ s.t. if we fix the subset, then no matter how we change the variables $x\_{j}$ outside S, the function value remains 1. If we have more than one min-1-certificate, take one with the minimum size. Similarly we can define the *0-term* as a subset S of $\{i\in \left[n\right]:x\_{i}=1\}$ with the minimum size s.t. fixing variables to be 0 on the subset guarantees the function value to be 0. Note that a simple but crucial property of certificates is that for any 1-term *a* and 0-term *b* we have $a∩b\ne ∅$. Indeed, if they are disjoint, then consider the input with all 1’s at positions in *a* and all 0’s at positions in b, then what are you gonna assign the function value? (No matter how you do it, it’ll violate the definition of one of the 0- and 1-term.)

Take a set $A⊆${1-terms} and $B⊆${0-terms}. A and B *locally intersect* if for any $b\in B$, there is a partition $b=b\_{0}∪b\_{1}$ so that for all $a\in A$, *a* intersects with exactly one of $b\_{0}$ and $b\_{1}$. Define a Disjointness matrix D of dimension $\left|A\right|×\left|B\right|$, with rows and columns indexed by *A* and *B*, respectively. The $(a,b)$-entry is defined as 1 if $a∩b\_{1}\ne ∅$ and 0 if $a∩b\_{0}\ne ∅$.

**Theorem 2.1**. If $A⊆${1-terms} and $B⊆${0-terms} locally intersect, then $L\_{+}\left(f\right)\geq rank\left(D\right).$

*Proof*. It’s enough to show that any 1-monochromatic subrectangle has has rank 1. Fix a 1-monochromatic subrectangle $R^{'}=A^{'}×B'$, with an index *i* s.t. any $a^{'}\in A'$ and $b^{'}\in B'$, $i\in a^{'}∩b^{'}. $Partition *B’* into $B\_{0}^{'}∪B\_{1}^{'}$, where $B\_{0}^{'}=\left\{b^{'}\in B^{'}:i\in b\_{0}^{'}\right\}$, and $B\_{1}^{'}=\left\{b^{'}\in B^{'}:i\in b\_{1}^{'}\right\}$. Now for any $a^{'}\in A'$ and $b^{'}\in B\_{0}^{'}$, since A and B locally intersect, a’ and b’ intersect at exactly one of the $b\_{0}^{'}$ and $b\_{1}^{'}$. But since $i\in a^{'}∩b^{'}$ and $i\in b\_{0}^{'}$, we know that $D\left(a’,b’\right)=0$. Similarly, for any $a^{'}\in A'$ and $b^{'}\in B\_{1}^{'}$, it holds that $D\left(a’,b’\right)=1$. So $rank\left(M\_{R^{'}}\right)=1$. $□$

## References

[HJKP10] P. Hrubes, S. Jukna, A. Kulikov, and P. Pudlak. **On convex complexity mesures**, *Theoretical Computer Science* 411, 1842–1854, 2010.

[KKN95] M. Karchmer, E. Kushilevitz, and N. Nisan: **Fractional covers and communication complexity**, *SIAM Journal on Discrete Mathematics* 8(1), 76–92, 1995.