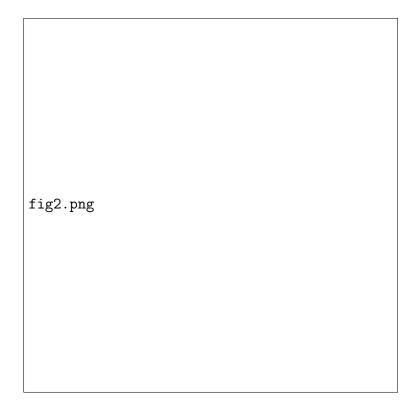
## Midterm for CSCI3160/ESTR3104

2:50-4:50pm, March 2, 2015.

- (10 points) Compare the following growth functions by filling the blanks using O, o, Ω, ω, or Θ. If both O and o are correct, use o; if both Ω and ω are correct, use ω; if both O and Ω are correct, use Θ. To distinguish O and o in handwriting, please write "Big-O" for O and "small-o" for o.
  - (a) 5n = (100n).
  - (b)  $0.1n = (100 \log n)$ .
  - (c)  $4^n = (10^6 n^3).$
  - (d)  $n^2 \log n = (n^2).$
  - (e)  $\log n = \underline{\qquad} (n \log n).$
- 2. (10 points) Consider the following computational task: Given an undirected graph G = (V, E) with unit edge weight, and two vertices  $s, t \in V$ , find the number of shortest paths from s to t. Can you give an algorithm with running time O(|E|)?
- 3. (10 points)
  - (a) (5 points) If we run Kruskal's algorithm on the following graph. What is the MST you get? (Write down the edges of the MST.)



- (b) (5 points) Prove that for any undirected graph, if all edge weights are distinct, then it has a unique minimum spanning tree.
- 4. (10 points) Consider the following LP.

min 
$$3x_1 + 4x_2$$
  
s.t.  $x_1 + 2x_2 \le 14$   
 $3x_1 - x_2 \ge 0$   
 $x_1 - x_2 \le 2$ 

- (a) (5 points) What is the optimal value?
- (b) (5 points) Transform it to equational form.
- (c) (5 bonus points) Write down the dual LP.
- 5. (10 points) Consider the Vertex Cover problem on trees: Given an undirected tree T = (V, E), find a set  $S \subseteq V$  of vertices with the minimum cardinality |S|, such that all edges are incident to at least one vertex  $v \in S$ . (That is, for any edge  $e = (u, v) \in E$ , either  $u \in S$  or  $v \in S$ , or both.) Design a polynomial-time algorithm for this problem using dynamic programming, prove its correctness and analyze its efficiency.

(Hint: For a node v in the tree, a subtree contains a vertex v and all vertices below it. The subproblems are the same Vertex Cover problem on subtrees. Solve these problems in a bottom-up manner, from leaves to the root. In the subproblem, you may need to consider two cases, depending on whether v is selected or not.) 6. (10 bonus points) (ELITE students must do this.)

Suppose that two people each take a random walk on a connected, undirected and non-bipartite graph G = (V, E). They start at the same time on different nodes, and each walks one step to a (uniformly) random neighbor at each time step. The two people meet each other if they are in the same vertex at some time step. The first time they meet is T, a random variable.

Prove that the expectation of T is at most  $O(|E|^2 \cdot |V|)$ .

You can use the following two facts, which you don't need to prove.

Fact 1. A graph is non-bipartite if and only if it does not have a cycle of odd length.

**Fact 2.** For any edge (u, v) on an undirected graph, if we start a random walk from u, then the expected time of reaching v is at most 2|E|.

(*Hint: Consider a larger graph with vertex set*  $V' = \{(u, v) : u, v \in V\}$ .)