Midterm for CSCI3160/ESTR3104

2:50-4:50pm, March 2, 2015.

1. (10 points) Compare the following growth functions by filling the blanks using O, o, Ω, ω, or Θ. If both O and o are correct, use o; if both Ω and ω are correct, use ω; if both O and Ω are correct, use Θ. To distinguish O and o in handwriting, please write "Big-O" for O and "small-o" for o.

   (a) $5n =$ __________$(100n)$.

   (b) $0.1n =$ __________$(100 \log n)$.

   (c) $4^n =$ __________$(10^6 n^3)$.

   (d) $n^2 \log n =$ __________(n^2).

   (e) $\log n =$ __________$(n \log n)$.

2. (10 points) Consider the following computational task: Given an undirected graph $G = (V, E)$ with unit edge weight, and two vertices $s, t \in V$, find the number of shortest paths from s to t. Can you give an algorithm with running time $O(|E|)$?

3. (10 points)

   (a) (5 points) If we run Kruskal's algorithm on the following graph. What is the MST you get? (Write down the edges of the MST.)

fig2.png

(b) (5 points) Prove that for any undirected graph, if all edge weights are distinct, then it has a unique minimum spanning tree.

4. (10 points) Consider the following LP.

$$\begin{aligned} \min \quad & 3x_1 + 4x_2 \\ s.t. \quad & x_1 + 2x_2 \leq 14 \\ & 3x_1 - x_2 \geq 0 \\ & x_1 - x_2 \leq 2 \end{aligned}$$

(a) (5 points) What is the optimal value?

(b) (5 points) Transform it to equational form.

(c) (5 bonus points) Write down the dual LP.

5. (10 points) Consider the Vertex Cover problem on trees: Given an undirected tree $T = (V, E)$, find a set $S \subseteq V$ of vertices with the minimum cardinality $|S|$, such that all edges are incident to at least one vertex $v \in S$. (That is, for any edge $e = (u, v) \in E$, either $u \in S$ or $v \in S$, or both.) Design a polynomial-time algorithm for this problem using dynamic programming, prove its correctness and analyze its efficiency.

(*Hint: For a node v in the tree, a subtree contains a vertex v and all vertices below it. The subproblems are the same Vertex Cover problem on subtrees. Solve these problems in a bottom-up manner, from leaves to the root. In the subproblem, you may need to consider two cases, depending on whether v is selected or not.*)

6. (10 bonus points) (*ELITE students must do this.*)

   Suppose that two people each take a random walk on a connected, undirected and non-bipartite graph $G = (V, E)$. They start at the same time on different nodes, and each walks one step to a (uniformly) random neighbor at each time step. The two people meet each other if they are in the same vertex at some time step. The first time they meet is T, a random variable.

   Prove that the expectation of T is at most $O(|E|^2 \cdot |V|)$.

   You can use the following two facts, which you don't need to prove.

   Fact 1. *A graph is non-bipartite if and only if it does not have a cycle of odd length.*

   Fact 2. *For any edge (u, v) on an undirected graph, if we start a random walk from u, then the expected time of reaching v is at most $2|E|$.*

   (*Hint: Consider a larger graph with vertex set $V' = \{(u, v) : u, v \in V\}$.*)