A PSD LIFTING QUESTION OF LEE

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ABSTRACT. In this short note, we show a lower bound of $\tilde{\Omega}(\sqrt{d}\varepsilon)$ for the Lipschitz constant of a measurable map F from the d-dimensional unit sphere to the infinite-dimensional unit sphere, where F maps ε-almost orthogonal vectors to orthogonal vectors. Our lower bound has optimal dependence on d, answering an embedding question of Lee.

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid \|x\|_2 = 1\}$ be the $(d-1)$-dimensional unit sphere. In a blog post [Lee11], Lee asked the following question: Given a map $F : \mathbb{S}^{d-1} \to \mathbb{S}^\infty$ such that

$$|\langle u, v\rangle| \leqslant \varepsilon \implies \langle F(u), F(v)\rangle = 0 \qquad \forall u, v \in \mathbb{S}^{d-1},$$

what is the smallest possible Lipschitz constant $\|F\|_{\mathrm{Lip}}$. Recall that

$$\|F\|_{\mathrm{Lip}} \triangleq \sup_{u \neq v \in \mathbb{S}^{d-1}} \frac{\|F(u) - F(v)\|}{\|u - v\|}.$$

Lee's question was motivated by a possible approach to construct integrality gaps for Unique Games with large alphabet; i.e. an instance on n variables and alphabet size $(\log n)^{\omega(1)}$. Lee also constructed a map such that $\|F\|_{\mathrm{Lip}} \ll \sqrt{d}/(1-\varepsilon)$,[1] and asked whether the dependence on d is optimal. We answer that question completely here.

Theorem 1. *Let $F : \mathbb{S}^{d-1} \to \mathbb{S}^\infty$ be an "SDP solution" that maps almost orthogonal vectors to orthogonal ones, i.e. $\langle F(u), F(v)\rangle = 0$ whenever $|\langle u, v\rangle| \leqslant \varepsilon$. Then $\|F\|_{\mathrm{Lip}} \gg \sqrt{d}\varepsilon/\sqrt{\log(1/\varepsilon)}$.*

In other words, we rule out Lee's approach to large alphabet integrality gaps for Unique Games.

Our results were obtained in July 2011. Concurrently and independently, Barak, Gopalan, Håstad, Meka, Raghavendra, and Steurer constructed large alphabet integraly gap instances using another approach (derandomizing the noisy hypercube via Reed–Muller code) [BGH+12].

Proof of Theorem 1. Without loss of generality, assume $\langle F(u), F(v)\rangle$ depends only on $\langle u, v\rangle$ for $u, v \in \mathbb{S}^{d-1}$. This is done by symmetrizing F over all rotations, i.e. redefining $F(u)$ as the direct integral $F'(u) = \int^\oplus F(\rho u)\,\mathrm{d}\rho$ over all isometries $\rho \in \mathrm{O}(d)$. By symmetry,

$$\langle F'(u), F'(v)\rangle = \mathop{\mathbb{E}}_{p,q \in \mathbb{S}^{d-1}}[\langle F(p), F(q)\rangle \mid \langle p, q\rangle = \langle u, v\rangle].$$

Clearly, F' inherits orthogonality and Lipschitz bound from F.

Let $\langle F(u), F(v)\rangle = f(\langle u, v\rangle)$ for $u, v \in \mathbb{S}^{d-1}$. Such a function $f : [-1, 1] \to \mathbb{R}$ is positive definite on $\mathbb{S}^{d-1}$. By [Sch42], f has a series expansion

$$f(x) = \sum_{n \geqslant 0} a_n P_n(x) \qquad a_n \geqslant 0 \text{ for all } n, \sum_{n \geqslant 0} a_n < \infty. \tag{1}$$

Here, we adopt the normalization $P_n = C_n^{(d-2)/2}(x)/C_n^{(d-2)/2}(1)$, where $C_n^\lambda(x)$ are Gegenbauer polynomials (also known as ultraspherical polynomials). With this normalization, we have

$$|P_n(x)| \leqslant P_n(1) = 1 \tag{2}$$

[1] We use the notation $A \ll B$ to mean $A \leqslant CB$ for some universal constant C.

for any $n \in \mathbb{N}$ and any $-1 \leqslant x \leqslant 1$ [Gro96, Lemma 3.3.5]. Note that $\sum_n a_n = \sum_n a_n P_n(1) = f(1) = 1$, so the coefficients a_n are nonnegative weights that sum to one.

Roughly speaking, $\|F\|_{\mathrm{Lip}}^2 \geqslant f'(1)$ because

$$\|F\|_{\mathrm{Lip}}^2 = \sup_{u,v\in\mathbb{S}^{d-1}} \frac{\|F(u)-F(v)\|^2}{\|u-v\|^2} = \sup_{u,v\in\mathbb{S}^{d-1}} \frac{f(1)-f(\langle u,v\rangle)}{1-\langle u,v\rangle}.$$

For the second eqaulity, we have used $\|F(u)-F(v)\|^2 = \|F(u)\|^2 + \|F(v)\|^2 - 2\langle F(u),F(v)\rangle = 2(f(1)-f(\langle u,v\rangle))$ for the numerator (and a similar relation for the denominator).

We will need two claims (proven later).

Claim 2.

$$\sup_{-1\leqslant x<1} \frac{f(1)-f(x)}{1-x} \geqslant \sum_{n\geqslant 0} a_n P_n'(1), \tag{3}$$

Claim 3. *For $t = \beta d$ with $\beta \asymp \varepsilon^2/\log(1/\varepsilon)$,*

$$\sum_{0\leqslant n<t} a_n = o(1).$$

Together with the derivative formula $P_n'(1) = n(n+d-2)/(d-1)$ [Gro96, Lemma 3.3.9], our lower bound follows because with our choice of t in Claim 3,

$$\begin{aligned}\|F\|_{\mathrm{Lip}}^2 &\geqslant \sum_{n\geqslant 0} a_n P_n'(1) \geqslant \sum_{n\geqslant t} a_t P_t'(1)\\ &= (1-o(1))P_t'(1) \gg t.\end{aligned}$$

□

It remains to prove the two claims.

Proof of Claim 3. We need the relation

$$\mathop{\mathbb{E}}_{u,v\in\mathbb{S}^{d-1}} f(\langle u,v\rangle)^2 = \sum_{n\geqslant 0} \frac{a_n^2}{N(d,n)}, \tag{4}$$

where $N(d,n) = \frac{2n+d-2}{n+d-2}\binom{n+d-2}{d-2}$ is the dimension of the space of spherical harmonics of degree n. (Equation (4) is proved using standard othorgonality relations among the P_n's, namely $\mathbb{E}_{v\in\mathbb{S}^{d-1}} P_m(\langle u,v\rangle)P_n(\langle u,v\rangle) = \delta_{mn}/N(d,n)$ for any $u \in \mathbb{S}^{d-1}$.)

Orthogonality of F and Lévy concentration shows that the LHS of (4) is at most $2\exp(-d\varepsilon^2/2)$. Thus the first t levels have squared weight

$$\sum_{0\leqslant n<t} a_n^2 \leqslant 2\exp(-d\varepsilon^2/2)N(d,t). \tag{5}$$

With our choice of t, we have $N(d,t) \ll \binom{n+d-2}{d-2} = 2^{(1+o(1))H(\beta/(1+\beta))(1+\beta)d}$, where $H(\cdot)$ is the binary entropy function. Further $H(\beta/(1+\beta)) \asymp \varepsilon^2$, so the RHS of (5) is at most $\exp(-\Omega(d\varepsilon^2))$. It then follows from Cauchy–Schwarz that

$$\left(\sum_{0\leqslant n<t} a_n\right)^2 \leqslant \sum_{0\leqslant n<t} a_n^2 \cdot t \leqslant \exp(-\Omega(d\varepsilon^2)) \cdot \frac{d\varepsilon^2}{\log(1/\varepsilon)},$$

which tends to zero as $d\varepsilon^2 \to \infty$. □

Proof of (3). Fix any $m \in \mathbb{N}$. We have for any $-1 \leqslant x < 1$,

$$\frac{f(1)-f(x)}{1-x} \geqslant \sum_{n\leqslant m} a_n \frac{P_n(1)-P_n(x)}{1-x},$$

since $a_n \geqslant 0$ and $P_n(1) \geqslant P_n(x)$ for all n (by (1) and (2)). Letting $x \to 1$, we get

$$\sup_{-1 \leqslant x < 1} \frac{f(1) - f(x)}{1 - x} \geqslant \sum_{n \leqslant m} a_n P_n'(1).$$

Now take $m \to \infty$, and the claim follows. $\square$

References

[BGH+12] Boaz Barak, Parikshit Gopalan, Johan Håstad, Raghu Meka, Prasad Raghavendra, and David Steurer. Making long code shorter. In *FOCS*, 2012. Preliminary version at http://arxiv.org/abs/1111.0405.

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