$(k+1)$-cores have k-factors

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Abstract

We prove that almost surely the first non-empty $(k+1)$-core to arise during the random graph process will have a k-factor or will be k-factor-critical. Thus the threshold for the appearance of a k-regular subgraph is at most the threshold for the $(k+1)$-core. This improves a result of Pralat, Verstraete and Wormald [5] and proves a conjecture of Bollobas, Kim and Verstraete [3].

1 Introduction

This paper concerns k-regular subgraphs of random graphs. A natural starting point for such a study is with the k-core; i.e. the unique maximal subgraph with minimum degree at least k. Pittel, Spencer and Wormald[4] determined the threshold $c_k = k + \sqrt{k \log k} + o(\sqrt{k})$ for the appearance of a non-empty k-core in $G_{n,p=c/n}$, the random graph with n vertices where each of the $\binom{n}{2}$ possible edges appears independently with probability p. So for $c < c_k$, a.s.[1] $G_{n,p=c/n}$ has no non-empty k-core and hence a.s. has no k-regular subgraph. In [3], Bollobás, Kim and Verstraete studied the threshold for the appearance of a 3-regular subgraph, and determined that it is strictly larger than c_3. They also conjectured that the threshold for a k-regular subgraph is strictly larger than c_k for all $k \geq 4$. Pretti and Weigt[6] used some statistical physics techniques to predict the opposite: for every $k \geq 4$, the threshold for the appearance of a k-regular subgraph is c_k. In other words, for every $c > c_k$. a.s. the k-core contains a k-regular subgraph. Those conflicting conjectures remain unresolved.

Bollobás, Kim and Verstraete also conjectured that if $c > c_{k+1}$ then a.s. the $(k+1)$-core of $G_{n,p=c/n}$ has a k-regular subgraph (see Conjecture 1.3 from [3]). We prove that conjecture here for k sufficiently large. They proved that $G_{n,p=c/n}$ a.s. contains a k-regular subgraph if $c > \rho_k n$ for a specific function $\rho_k = 4k + o(k)$; note that $\rho_k \approx 4c_k$.

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[1]A property holds *almost surely (a.s.)* if it holds with probability tending to 1 as $\lim_{n\to\infty}$.

A *k-factor* of a graph G is a spanning k-regular subgraph; note that if G has a k-factor, then $k \times |V(G)|$ must be even. G is said to be *k-factor-critical* if for every $v \in V(G)$, $G - v$ has a k-factor. Suppose $c_{k+2} < c < c_{k+2} + 10\sqrt{k \log k}$ and let C denote the $(k+2)$-core of $G_{n,p=c/n}$. Pralat, Verstraete and Wormald[5] proved that if k is sufficiently large then a.s.: (i) if $k \times |V(C)|$ is even then C contains a k-factor; (ii) if $k \times |V(C)|$ is odd then C is k-factor-critical. We extend this result to the $(k+1)$-core:

Theorem 1.1 *There is an absolute constant k_0 such that for all $k \geq k_0$, and for any $c_{k+1} < c < c_{k+1} + 10\sqrt{k \log k}$ a.s. the $(k+1)$-core, K, of $G_{n,p=c/n}$ satisfies:*

(a) if $k \times |V(K)|$ is even then K has a k-factor;

(b) if $k \times |V(K)|$ is odd then K is k-factor-critical.

This result is best possible (for large k) in that, as observed in [5], for every $c > c_k$ a.s. the k-core of $G_{n,p=c/n}$ neither contains a k-factor nor is k-factor-critical, because it a.s. contains many vertices of degree greater than k whose neighbours all have degree exactly k.

By monotonicity, Theorem 1.1 implies that for any $c > c_{k+1}$, a.s. the $(k+1)$-core of $G_{n,p=c/n}$ contains a k-regular subgraph, although for very large c we do not guarantee an actual k-factor. This proves the aforementioned conjecture from [3]. It also establishes that the threshold for the appearance of a k-regular subgraph is at most the threshold for the appearance of a $(k+1)$-core. [3] remarked that perhaps a.s. the $(k+1)$-core of the random graph will contain a k-factor (so long as its size times k is even); Theorem 1.1 confirms this for large k.

Our proof makes use of Tutte's f-factor Theorem[7] (see also Exercise 3.3.29 of [8]). We state it here, in terms of k-factors; Tutte's actual statement applies to more general factors. For $X, Y \subset V(G)$, we use $\lambda(X, Y)$ to denote the number of edges with one endpoint in X and the other in Y. And we use $q(X, Y)$ to denote the number of components Q of $G - (X \cup Y)$ such that $k|Q|$ and $\lambda(Q, Y)$ have different parities.

Theorem 1.2 (Tutte[7]) *A graph G has a k-factor iff for every pair of disjoint sets $R, W \subset V(G)$,*

$$k|R| \geq q(R, W) + k|W| - \sum_{v \in W} \deg_{G-R}(v).$$

Rearranging, we see that the condition of Theorem 1.2 is equivalent to:

$$\sum_{v \in W} \deg_G(v) + k|R| \geq q(R, W) + k|W| + \lambda(R, W). \tag{1}$$

To prove Theorem 1.1(a), we will prove that K satisfies a stronger condition. Using $\omega(H)$ to denote the number of components of a subgraph H, we will show that for every pair of disjoint sets $S, T \subset V(K)$ with $S \cup T \neq \emptyset$,

$$\sum_{v \in T} \deg_K(v) + k|S| \geq \omega(K - S \cup T) + k|T| + \lambda(S, T). \tag{2}$$

By Theorem 1.2, with $R := S, W := T$ this will suffice to prove part (a), since $\omega(K - R \cup W) \geq q(R, W)$.

For part (b), it would suffice to prove that for every pair of disjoint sets $S, T \subset V(K)$ with $|S| \geq 1$ and $|S \cup T| \geq 2$, we have:

$$\sum_{v \in T} \deg_K(v) + k|S| \geq \omega(K - S \cup T) + k|T| + \lambda(S, T) + k. \tag{3}$$

It is straightforward to show that if S, T satisfy (3) then for any $x \in S$, (2) holds upon substituting $K := K - x, S := S - x$ (the quick argument appears in the proof of Corollary 2 of [5]). Thus, if (3) were to hold for all S, T with $|S| \geq 1$ and $|S \cup T| \geq 2$ then this would establish part (b). This was indeed the case in [5]. Unfortunately there are some cases in our setting where (3) does not hold, so we need to instead focus directly on (1).

To see why our setting is a bit more delicate, consider a vertex x whose neighbours all have degree $k+1$ in K. In $K - x$, they all have degree k, and this forces all of their edges into any k-factor. It is easy to verify that $S = \{x\}$ and $T = N(x)$ will violate (3); equivalently, $S = \emptyset$ and $T = N(x)$ will violate (2) when K is replaced by $K - x$. Fortunately $R = \emptyset$ and $W = N(x)$ does not violate (1), with $G = K - x$.

Our proof follows the same outline as that of [5]. Their proof covered four separate cases for the sizes of S, T. In Case 1, we require a somewhat different argument for the setting of this paper. Case 2 is where the main new ideas of this paper are required. Their arguments for Cases 3 and 4 apply to the setting of this paper, so we didn't need any new ideas there; we combine them into our Case 3. The reader who is already familiar with [5] may want to skip directly to Case 2 (in particular, Case 2b).

We close this introduction by noting that our main theorem extends to $G_{n,M}$, a model that permits a somewhat stronger statement. The *random graph process* begins with n vertices and no edges, and then repeatedly adds an edge chosen uniformly at random from amongst those edges not yet present. $G_{n,M}$ is the graph obtained after M steps.

Theorem 1.3 *There is an absolute constant k_0 such that for all $k \geq k_0$, a.s. K, the first non-empty $(k+1)$-core to arise during the random graph process, satisfies:*

(a) if $k \times |V(K)|$ is even then K has a k-factor;

(b) if $k \times |V(K)|$ is odd then K is k-factor-critical.

1.1 Preliminaries

We will make use of the following lemmas from [5] concerning the structure of K. (Actually, their lemmas were stated a bit differently in that they were in terms of the k-core. But it is straightforward to adapt their proofs to obtain the statements below.)

Lemma 1.4 *(Lemma 2 of [5].) There is a constant $\gamma > 0$ (independent of k) such that a.s. for every set $X \subset V(K)$ of at most $\frac{1}{2}|V(K)|$ vertices, we have:*

$$\lambda(X, K - X) \geq \gamma(k+1)|X|.$$

For the remainder of the paper, we use γ to denote the constant from Lemma 1.4. We define:

$$s(n) = \log n/(2ec \log\log n).$$

A standard first moment argument nearly identical to the proof of Lemma 3 of [5] yields:

Lemma 1.5 *For any constant $c > 0$, a.s. every subset Y of the vertices of $G_{n,p=c/n}$ with $|Y| \leq 4s(n)$ has at most $|Y|$ edges.*

Lemma 4 of [5] says:

Lemma 1.6 *If k is sufficiently large then: a.s. for every subset $Y \subseteq V(K)$ with $|Y| \leq s(n)$, $K - Y$ contains a component with more than $|V(K)| - 2s(n)$ vertices.*

Proof: Let X be the union of the vertex sets of some components of $K - Y$, such that $|X| > s(n)$. We'll show that if the a.s. properties from Lemmas 1.4 and 1.5 hold then $|X| > \frac{1}{2}|V(K)|$; this implies the lemma.

Cosnsider any $Z \subset X$ where $|Z| = |Y|$. Thus $|Y \cup Z| \leq 2s(n)$ and so by Lemma 1.5 we can assume $\lambda(Y, Z) \leq |Y \cup Z| = 2|Z|$. Averaging over all such $Z \subseteq X$ yields $\lambda(Y, X) \leq 2|X| < \gamma k|X|$, for k sufficiently large (since γ does not depend on k). Since $\lambda(X, K - X) = \lambda(Y, X)$, the a.s. property of Lemma 1.4 implies $|X| > \frac{1}{2}|V(K)|$ as required. □

We often use the following well-known bound which follows easily from Stirling's Inequality:

$$\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b.$$

And finally, recall that [4] established $c_k = k + \sqrt{k \log k} + o(\sqrt{k})$ and that the hypothesis of Theorem 1.1 requires $c < c_k + 10\sqrt{k \log k}$. Thus, for k sufficiently large, we have:

$$c < 2k.$$

2 Proof of Theorem 1.1

We will consider three cases for the sizes of S, T from (2) and (3). Recall that $s(n) = \log n/(2ec \log\log n)$.

Case 1: $|S| + |T| \leq s(n)$.

The proof of this case is similar to that from [5]. Let $\omega(K - (S \cup T)) = \ell + 1$. By Lemma 1.6, a.s. K is such that the sizes of $C_1, ..., C_\ell$, the ℓ smallest components of $K - (S \cup T)$, must total less than $2s(n)$. So Lemma 1.5 implies that a.s. the subgraph X induced by $S \cup T \cup C_1 \cup ... \cup C_\ell$ has no more edges than vertices. Let X' be the graph obtained by contracting each C_i into the single vertex c_i. Since C_i has at most one cycle (by Lemma 1.5) and every vertex of C_i has degree at least $k+1$ in X, it follows that $\deg(c_i) \geq k+1$. Since each c_i is only adjacent to vertices in $S \cup T$ we have $|E(X')| \geq (k+1)\ell + \lambda(S,T)$. Since X has no more edges than vertices and since each C_i is connected, $|E(X')| \leq |V(X')| = |S| + |T| + \ell$. Therefore:

$$|T| + k|S| \geq k\ell + \lambda(S,T) + (k-1)|S| = \omega(K - (S \cup T)) + \lambda(S,T) + (k-1)(|S| + \ell) - 1.$$

Since every vertex in T has degree at least $k+1$, this implies (2) for $|S| + \ell \geq 1$ and (3) for $|S| + \ell \geq 2$ (and $k \geq 3$).

If $|S| = \ell = 0$ and $S \cup T \neq \emptyset$ then we must have $|T| \geq 1, \omega(K - (S \cup T)) = 1$ and $\lambda(S,T) = 0$, and so (2) holds.

We aren't required to prove (3) for $|S| = 0$. So we have only failed to prove (3) for the case $|S| = 1, \ell = 0$; in fact, (3) does not a.s. hold in this case. Proving (3) is required only to prove Theorem 1.1(b); i.e. to establish that if $k|K|$ is odd then $K - x$ has a k-factor for every $x \in V(K)$. We will establish that by showing directly that (1) holds for $G = K - x$. The fact that (3) holds for K when $|S| \geq 2$ or $|S| = 1, \ell \geq 1$ implies that (1) holds for $G = K - x$ whenever $|R| \geq 1$ and whenever $|R| = 0$ and $(K - x) - W$ has more than one component (recall the discussion following the statement of (3)). So we can assume $R = \emptyset$ and $(K - x) - W$ has at most one component. Then (1) becomes:

$$\sum_{v \in W} \deg_{K-x}(v) \geq q(\emptyset, W) + k|W|.$$

K has minimum degree at least $k+1$ and so $K - x$ has minimum degree at least k. Since $(K - x) - W$ has at most one component, $q(\emptyset, W) \leq 1$. So (1) holds if there at least one $v \in W$ with $\deg_{K-x}(v) \geq k+1$. Let Q be the only component of $K - x - W$. If every $v \in W$ has $\deg_{K-x}(v) = k$ then $\lambda(Q, W) = k|W| - 2E(W)$ which has the same parity as $k|Q|$ since $|Q| + |W| = |K| - 1$ and $k|K|$ is odd (as we are in Theorem 1.1(b)). Thus, $q(\emptyset, W) = 0$ and so (1) holds.

This proves that a.s. for every S, T satisfying Case 1, (6) holds for S, T and (1) holds for $R := S - x, W := T$ with $G := K - x$.

To specify Case 2, we fix an absolute constant ϵ_0, independent of k, chosen so that $\epsilon_0 < \frac{\gamma^2}{10^5}$ (recall γ from Lemma 1.4).

Case 2: $s(n) \leq |S| + |T| \leq \epsilon_0 n$

We use the following two technical bounds, which are very much like bounds found in [5]. We defer the proofs until Section 3.

A.s. for every disjoint pair of sets X, Y with $|X| \geq \frac{1}{200}|Y|$ and $|Y| \leq \epsilon_0 n$ we have:

$$\lambda(X,Y) \leq \frac{1}{2}\gamma k|X|. \tag{4}$$

A.s. for every disjoint pair of sets S, T with $s(n) \leq |S| + |T| \leq \epsilon_0 n$ we have:

$$\lambda(S,T) < \frac{101}{100}|T| + \frac{k}{2}|S|. \tag{5}$$

We use (4) to bound $\omega(K - S \cup T)$. Let X be the set of vertices in all components of $K - S \cup T$ that have size at most $\frac{1}{2}|V(K)|$. By applying Lemma 1.4 to each component of X, we have $\lambda(X, S \cup T) \geq \gamma(k+1)|X|$. Therefore, letting $Y = S \cup T$ and recalling that, in Case 2, $|Y| \leq \epsilon_0 n$, (4) is violated unless $|X| < \frac{1}{200}|S \cup T|$. Since $\omega(K - S \cup T) \leq |X| + 1$, this implies that a.s. K is such that for every S, T in Case 2 we have:

$$\omega(K - S \cup T) < \frac{1}{200}(|S| + |T|) + 1 < \frac{1}{100}(|S| + |T|). \tag{6}$$

Case 2a: $|T| \leq 20k|S|$.

(5) and (6) imply that a.s. every pair S, T with $s(n) \leq |S| + |T| \leq \epsilon_0 n$ and $|T| \leq 20k|S|$ satisfies:

$$\omega(K - S \cup T) + \lambda(S,T) < \frac{1}{100}(|T| + |S|) + \frac{101}{100}|T| + \frac{k}{2}|S| = \frac{102}{100}|T| + (\frac{k}{2} + \frac{1}{100})|S| < |T| + k|S| - k,$$

where the last inequality uses $|T| \leq 20k|S|$.

This implies that a.s. (2) and (3) hold for every S, T satisfying Case 2a.

Case 2b: $|T| > 20k|S|$.

Note that, since $|S| + |T| \leq \epsilon_0 n$, we have $|S| \leq \frac{\epsilon_0}{20k} n$.

This case contains most of the new ideas for this paper. To prove (2) and (3), it would suffice to show $\omega(K - S \cup T) + \lambda(S,T) \leq |T| + k|S| - k$. Above, we saw that (5) and (6) yield $\omega(K - S \cup T) + \lambda(S,T) \leq \frac{102}{100}|T| + (\frac{k}{2} + \frac{1}{100})|S|$, which is less than $|T| + k|S| - k$ if T is a lot smaller than S, eg. in Case 2a. Throughout Case 2, that bound clearly yields $\omega(K - S \cup T) + \lambda(S,T) \leq 2|T| + k|S| - k$, which would suffice for (2) and (3) if K were the $(k+2)$-core. So the analysis above sufficed to cover all of Case 2 in [5].

It is natural to try and tighten the proof of (5) to obtain: $\lambda(S,T) < |T| + \frac{k}{2}|S|$. Unfortunately, this approach fails - the proof of (5) uses a first moment calculation, and the $\binom{n}{|T|}$ term in that calculation is far too big. But instead of bounding $\lambda(S,T)$, we can bound $\lambda(S,N(S))$. The advantage of replacing T by $N(S)$ is that the choice of the vertices in S determines $N(S)$ and so the $\binom{n}{|T|}$ term is replaced by 1. We will obtain:

A.s. for every set $S \subset V(K)$ with $|S| \le \frac{\epsilon_0}{20k}n$ we have:

$$\lambda(S,N(S)) \le |N(S)| + \frac{k}{4}|S|. \tag{7}$$

This yields that a.s. for every disjoint pair of sets S,T as in Case 2b, we have:

$$\lambda(S,T) \le \lambda(S,N(S)) - |N(S)\backslash T| \le |N(S) \cap T| + \frac{k}{4}|S|. \tag{8}$$

We will also show a bound similar to (4):

A.s. for every pair of disjoint sets $S,X \subset V(K)$ with $|S| \le \frac{\epsilon_0}{20k}n$ and $|X| \ge |S|$ we have:

$$\lambda(X,(S \cup N(S))\backslash X) \le \frac{1}{2}\gamma k|X|. \tag{9}$$

The proofs of (7) and (9) appear in Section 3.

Next, we will bound $\omega(K - S \cup T)$. Consider any pair of sets S,T with sizes as in Case 2b. First, we note that if $S = \emptyset$ then $|T| \ge s(n)$ and (6) implies that:

$$\omega(K - S \cup T) + k|T| + \lambda(S,T) \le \frac{1}{100}(|S|+|T|) + kT + \lambda(S,T) = \frac{1}{100}|T| + k|T| < \sum_{v \in T} \deg_K(v),$$

and so (2) holds. (We can also show that (3) holds, but it is not required to hold when $S = \emptyset$.) Thus, we will assume $|S| \ge 1$.

Recall that we defined X to be the set of vertices in all components of $K - S \cup T$ of size at most $\frac{1}{2}|V(K)|$ and so $|X| \ge \omega(K - S \cup T) - 1$. Recall also that in Case 2b we have $|S| \le \frac{\epsilon_0}{20k}n$. If $|X| \ge \max(\frac{1}{200}|T\backslash N(S)|, |S|)$ then (4) with $Y = T\backslash N(S)$ and (9) imply:

$$\lambda(X, S \cup T) = \lambda(X, T\backslash N(S)) + \lambda(X, S \cup (T \cap N(S))) \le \lambda(X, T\backslash N(S)) + \lambda(X, (S \cup N(S))\backslash X) \le \gamma k|X|,$$

which contradicts Lemma 1.4 unless $X = 0$, since $\lambda(X, K - X) = \lambda(X, S \cup T)$. Since we can assume $|S| \ge 1$, this implies $|X| < \max(\frac{1}{200}|T\backslash N(S)|, |S|)$, which again since $|S| \ge 1$, implies $|X| \le |S| + \frac{1}{200}|T\backslash N(S)| - 1$. Therefore

$$\omega(K - S \cup T) \le |X| + 1 \le |S| + \frac{1}{200}|T\backslash N(S)|.$$

This, along with (8) implies

$$\begin{aligned}\omega(K-S\cup T)+\lambda(S,T) &\leq |S|+\frac{1}{200}|T\backslash N(S)|+|T\cap N(S)|+\frac{k}{4}|S| \\ &= k|S|+|T|-\frac{199}{200}|T\backslash N(S)|-\left(\frac{3k}{4}-1\right)|S|.\end{aligned}$$

This yields (2). It also implies (3) if $|S|\geq 2$ and so $\left(\frac{3k}{4}-1\right)|S|>k$. When $|S|=1$, we can trivially strengthen (8) to $\lambda(S,T)=|N(S)\cap T|$. That improves the above bound to

$$\omega(K-S\cup T)+\lambda(S,T)\leq k|S|+|T|-\frac{199}{200}|T\backslash N(S)|-k+1,$$

which implies (3) if at least one $v\in T$ has $\deg_K(v)\geq k+2$ or if $|T\backslash N(S)|\geq 1$.

So the only remaining case is where $|S|=1, T\subseteq N(S)$ and every vertex in T has degree $k+1$. Above, we proved that $|X|<\max(\frac{1}{200}|T\backslash N(S)|,|S|)$ and so, in this case, $|X|=0$. We work directly with (1), proving that it holds for $R:=\emptyset, W:=T, G=K-x$ with the same argument that was used in Case 1.

This proves that a.s., for every S,T satisfying Case 2b, (2) holds for S,T and (1) holds for $R:=S-x, W:=T$ with $G=K-x$.

Case 3: $|S|+|T|\geq\epsilon_0 n$

This is covered by Cases 3 and 4 from [5]. The proofs from that paper also apply to the setting of this paper (after a straightforward adjustment of some of the constants).

In particular, if $|T|<\frac{1}{10}\epsilon_0 n$ then $|S|>\frac{9}{10}\epsilon_0 n$. The same analysis as in Case 3 of [5] shows that a.s. every such S,T satisfies $\lambda(S,T)\leq\frac{3}{4}k|S|$. Indeed, they use a straightforward bound on the tail of the degree sequence to show that a.s. $G_{n,p=c/n}$ is such that $\sum\deg(v)$ over all $v\in T$ with $\deg(v)>\frac{3}{2}c$ must be less than $\epsilon_0 n$, and trivially, $\sum\deg(v)$ over all $v\in T$ with $\deg(v)\leq\frac{3}{2}c$ is at most $\frac{3}{2}c|T|<\frac{3}{20}c\epsilon_0 n$. So, using $c<2k$ and $|S|>\frac{9}{10}\epsilon_0 n$, we obtain:

$$\lambda(S,T)\leq\sum_{v\in T}\deg(v)<\epsilon_0 n+\frac{3}{20}c\epsilon_0 n<\frac{1}{5}c\epsilon_0 n<\frac{3}{4}k|S|.$$

Since $\sum_{v\in T}d(v)\geq(k+1)|T|$ and $\omega(G-(S\cup T))<n<\frac{1}{4}k|S|-1$ for $k>\frac{8}{\epsilon_0}$, (2) and (3) both hold.

If $|T|\geq\frac{1}{10}\epsilon_0 n$ then the same argument that yielded (18) from [5] (the only difference is a trivial reworking of a few constants) yields that there exists $\epsilon>0$ such that a.s. $\lambda(S,T)\leq k|S|+(1-\epsilon)\sqrt{k\log k}|T|$ for every such S,T. The degree sequence analysis preceding (18) in [5] (after replacing ϵ by $\frac{\epsilon}{2}$) yields that for k sufficiently large, we a.s. have $\sum_{v\in T}d(v)>(k+(1-\frac{\epsilon}{2}\sqrt{k\log k}))|T|$ for every such T. Since $\omega(G-(S\cup T))+1<n<\frac{\epsilon}{2}\sqrt{k\log k}|T|-k$ for $k>4/(\epsilon\epsilon_0)^2$, this yields (2) and (3).

Remark: It is in this final step that we require $c \leq c_{k+1} + 10\sqrt{k \log k} < k + 12\sqrt{k \log k}$. Replacing 10 by any other constant would suffice.

Therefore, a.s. (2) and (3) hold for every S, T in Case 3.

Proof of Theorem 1.1 We have proved that (2) holds for every S, T, which implies that (1) holds for every R, W when $G := K$. This establishes Theorem 1.1(a). We have proved that (2) holds for all but a few cases of S, T; as described in the introduction, this implies that (1) holds when $R := S - x, W := T$ and $G := K - x$. For those few remaining cases, we showed directly that (1) holds. Thus (1) holds for all R, W when $G := K - x$; this establishes Theorem 1.1(b).

□

We close this section by presenting the adaptation of our arguments to the $G_{n,M}$ model.

Proof of Theorem 1.3 It suffices to prove that all of the a.s. statements from our proof also hold when K is the first non-empty $(k+1)$-core to arise during the random graph process. Specifically, these statements are: Lemmas 1.4, 1.5, 1.6, (4), (5), (7) and (9) and the bound on $\lambda(S, T)$ corresponding to (18) from [5], as well as the degree sequence analysis from Case 3. All but Lemma 1.4 were proven to hold for the entire graph $G_{n,p=c/n}$ when $c < 2k$, rather than just for the k-core. Each of these properties are monotone (Lemma 1.5 is preserved under the addition of edges, the others are preserved under the deletion of edges), and so Theorem 2.2 of [2] implies that they all hold a.s. for $G_{n,M=\frac{1}{2}cn}$ for any $c < 2k$. This implies that they will a.s. hold for the first $(k+1)$-core to arise. Lemma 1.4 is Lemma 2 from [5] which, in turn, follows from Lemma 5.3 of [1]. That last lemma was proven for random graphs on a fixed degree sequence, whose degrees all lie between 3 and $n^{0.02}$. It is well known that the first $(k+1)$-core to arise is uniformly random on its degree sequence (see eg. [4]), and those degrees lie between $k+1 > 3$ and the maximum degree of $G_{n,M}$ which is a.s. less than $\log n << n^{0.02}$. It follows that Lemma 1.4 also holds when K is the first non-empty $(k+1)$-core to arise during the random graph process. The remainder of the proof is identical to that of Theorem 1.1. □

3 The remaining details

Here we provide the proofs of some of the technical statements from Case 2. Rather than working with the $(k+1)$-core K directly, we will actually prove that the statements hold over the entire graph $G_{n,p=c/n}$.

We begin with equations (4) and (5) from Case 2a.

A.s. for every disjoint pair of sets X, Y with $|X| \geq \frac{1}{200}|Y|$ and $|Y| \leq \epsilon_0 n$ we have:

$$\lambda(X,Y) \le \frac{1}{2}\gamma k|X|. \qquad (4)$$

Proof of (4): Clearly (4) holds for $X = \emptyset$, so we can assume $|X| \ge 1$.

Let $xn = |X|$, and $yn = |Y|$. For any fixed x, y, the expected number of sets X, Y in $G_{n,p=c/n}$ that violate (4) is at most:

$$\begin{aligned}
& \binom{n}{yn}\binom{n}{xn}\binom{(yn)(xn)}{\frac{1}{2}\gamma kxn}\left(\frac{c}{n}\right)^{\frac{1}{2}\gamma kxn} \\
< \; & \left(\frac{e}{y}\right)^{yn}\left(\frac{e}{x}\right)^{xn}\left(\frac{exyn^2c}{\frac{1}{2}\gamma kxn^2}\right)^{\frac{1}{2}\gamma kxn} \\
< \; & \left(\frac{e}{y/200}\right)^{201xn}\left(\frac{4ey}{\gamma}\right)^{\frac{1}{2}\gamma kxn} \qquad \text{since } x > \frac{y}{200}, \frac{e}{y/200} > 1 \text{ and } c < 2k \\
< \; & \left(\frac{3200e^3y}{\gamma^2}\right)^{\frac{1}{4}\gamma kxn} \qquad \text{if } k \text{ is large enough that } 201 < \frac{1}{4}\gamma k \\
< \; & \left(\frac{1}{2}\right)^{xn} \qquad \text{since } y \le \epsilon_0 < \frac{1}{2}\left(\frac{\gamma^2}{3200e^3}\right) \text{ and } \frac{1}{4}\gamma k > 1.
\end{aligned}$$

For each fixed x, there are at most $200xn$ choices for y, since $s(n) < |Y| \le 200|X|$. Therefore, summing over all x, y we find that the expected number of pairs X, Y violating (4) with $|X| \ge \log n$ is less than:

$$\sum_{|X| \ge \log n} 200|X|\left(\frac{1}{2}\right)^{|X|} = o(1).$$

For $|X| < \log n$ we have $|Y| < 200\log n$; i.e. $y < \frac{200\log n}{n}$. Thus $\left(\frac{3200e^2y}{\gamma^2}\right)^{\frac{1}{4}\gamma kxn} < \frac{1}{n^3}$ (since we can assume $xn = |X| \ge 1$ and we can choose k such that $\frac{1}{4}\gamma k \ge 4$). There are fewer than n^2 choices for x, y and so the expected number of pairs X, Y with $|X| < \log n$ that violate (6) is $o(1)$. $\Box$

A.s. for every disjoint pair of sets S, T with $s(n) \le |S| + |T| \le \epsilon_0 n$ we have:

$$\lambda(S,T) < \frac{101}{100}|T| + \frac{k}{2}|S|. \qquad (5)$$

Proof of (5): Let $\sigma n = |S|$ and $\tau n = |T|$. For any choice of σ, τ, the expected number of such sets S, T in $G_{n,p=c/n}$ violating (5) is at most:

$$\binom{n}{\sigma n}\binom{n}{\tau n}\binom{(\sigma n)(\tau n)}{\frac{101}{100}\tau n + \frac{k}{2}\sigma n}\left(\frac{c}{n}\right)^{\frac{101}{100}\tau n + \frac{k}{2}\sigma n} < \left(\frac{e}{\sigma}\right)^{\sigma n}\left(\frac{e}{\tau}\right)^{\tau n}\left(\frac{e\sigma\tau n^2 c}{(\frac{101}{100}\tau n + \frac{k}{2}\sigma n)n}\right)^{\frac{101}{100}\tau n + \frac{k}{2}\sigma n}$$

$$= \left(\frac{e}{\sigma}\right)^{\sigma n}\left(\frac{e}{\tau}\right)^{\tau n}\left(\frac{e\sigma\tau c}{\frac{101}{100}\tau+\frac{k}{2}\sigma}\right)^{\frac{101}{100}\tau n+\frac{k}{2}\sigma n}.$$

Since $c < 2k$ and $\tau < \epsilon_0 < (16e^3)^{-100}$, we have:

$$\frac{e\sigma\tau c}{\frac{101}{100}\tau+\frac{k}{2}\sigma} < \frac{e\sigma\tau c}{\frac{k}{2}\sigma} < 4e\tau < \left(\frac{\tau}{2e}\right)^{\frac{100}{101}}.$$

Furthermore, if $\sigma > e^{-k/3}$ then for k sufficiently large we have:

$$\frac{e\sigma\tau c}{\frac{101}{100}\tau+\frac{k}{2}\sigma} < \left(\frac{\tau}{2e}\right)^{\frac{100}{101}} < e^{-1} < \left(\frac{\sigma}{2e}\right)^{\frac{2}{k}},$$

while if $\sigma \le e^{-k/3}$ then for k sufficiently large we have:

$$\frac{e\sigma\tau c}{\frac{101}{100}\tau+\frac{k}{2}\sigma} < \frac{e\sigma\tau c}{\frac{101}{100}\tau} < ec\sigma^{1/2}\sigma^{1/2} < e(2k)e^{-k/6}\sigma^{1/2} < \sigma^{1/2} < \left(\frac{\sigma}{2e}\right)^{\frac{2}{k}}.$$

This implies that the expected number of pairs S,T with $|S| = \sigma n, |T| = \tau n$ is at most

$$\left(\frac{e}{\sigma}\right)^{\sigma n}\left(\frac{e}{\tau}\right)^{\tau n}\left(\frac{\tau}{2e}\right)^{\frac{100}{101}\frac{101}{100}\tau n}\left(\frac{\sigma}{2e}\right)^{\frac{2}{k}\frac{k}{2}\sigma n} = \left(\frac{1}{2}\right)^{(\sigma+\tau)n}.$$

For each choice of $y = |S| + |T|$, there are y choices for $|S|, |T|$. So the expected number of sets S, T violating (5) is at most:

$$\sum_{y=s(n)}^{n} y(\frac{1}{2})^y = o(1).$$

□

Now we turn to the results required for Case 2b. We begin with a technical lemma. Note that in Case 2b, we have $|S| + 20k|S| \le |S| + |T| \le \epsilon_0 n$ and so $|S| < \frac{\epsilon_0}{20k}n$.

Lemma 3.1 *A.s. every set S in $G_{n,p=c/n}$ of size at most $\frac{\epsilon_0}{20k}n$ satisfies:*

(a) $\left(\frac{25|S\cup N(S)|}{\gamma n}\right)^{\frac{1}{2}\gamma k}\frac{n^2}{|S|^2} < \frac{1}{2e^2}$.

(b) $\left(\frac{25|N(S)|}{n}\right)^{k/4}\frac{n}{|S|} < \frac{1}{2e}$.

Proof Note that if (a) holds then $\frac{25|S\cup N(S)|}{\gamma n} < 1$ and so

$$\left(\frac{25|N(S)|}{n}\right)^{k/4}\frac{n}{|S|} < \left(\frac{25|S\cup N(S)|}{\gamma n}\right)^{k/4}\frac{n}{|S|} < \left(\frac{25|S\cup N(S)|}{\gamma n}\right)^{\frac{1}{2}\gamma k}\frac{n^2}{|S|^2}\frac{|S|}{n} < \frac{1}{2e^2}\times\frac{\epsilon_0}{20k} < \frac{1}{2e};$$

i.e. (a) implies (b). So we will prove (a).

Let S^* be the $|S|$ vertices of largest degree in $G_{n,p}$ and let D be the sum of their degrees. Clearly $|N(S)| \leq D$. For $i \geq 0$, the expected number of vertices of degree i in $G_{n,p=c/n}$ is $\left(\frac{c^i}{i!}e^{-c} + o(1)\right)n$. Standard methods (eg Lemma 3.10 of [2]) show that this number is concentrated enough that: A.s. for all i such that $\frac{c^i}{i!}n \geq \sqrt{n}$ we have (a) at most $\frac{c^i}{i!}n$ vertices have degree i and (b) at most $\sum_{j \geq i} \frac{c^j}{j!}n$ vertices have degree at least i. Also, it is well-known that the maximum degree in $G_{n,p=c/n}$ is a.s. less than $\log n$ (see eg. Exercise 3.5 of [2]). We will assume that these almost sure properties hold, and show that for every choice of $|S|$, the bound in (a) holds. This establishes our lemma.

Case 1: $|S| \leq n^{2/3}$. Since the maximum degree is less than $\log n$, we have $D \leq |S| \log n$ and so

$$\left(\frac{25|S \cup N(S)|}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{|S|^2} \leq \left(\frac{25|S|(\log n + 1)}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{|S|^2}.$$

That product clearly increases with $|S|$ and so is at most:

$$\left(\frac{25n^{2/3}(\log n + 1)}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{(n^{2/3})^2} = o(1).$$

For the next two cases, we define I to be the largest integer such that $\frac{c^I}{I!}n \geq |S|$, and $i^* \geq I$ to be the largest integer for which $\frac{c^{i^*}}{i^*!}n \geq \sqrt{n}$. It is easily verified that $\sum_{i > i^*} \frac{c^i}{i!}n < 2\frac{c^{i^*+1}}{(i^*+1)!}n < 2\sqrt{n}$, and so fewer than $2\sqrt{n}$ vertices have degree greater than i^*. Since those vertices all have degree at most $\log n$, we have:

$$D < \sum_{i=I}^{i^*} i\frac{c^i}{i!}n + 2\sqrt{n}\log n.$$

Case 2: $|S| > n^{2/3}$ and $I \geq 4c$. Since $I \geq 4c$, it is easily verified that $\frac{c^I}{I!}n + \sum_{i \geq I} i\frac{c^i}{i!}n < 2I\frac{c^I}{I!}n$. Also, $\frac{c^I}{I!}n \geq |S| > n^{2/3} > 2\sqrt{n}\log n$. So our bound on D above, and the fact that we can take $c > 2$, yields $|S| + D < 2I\frac{c^I}{I!}n + 2\sqrt{n}\log n < 3I\frac{c^I}{I!}n < 3I^2\frac{c^{I+1}}{(I+1)!}n < 3I^2|S|$. In the next line, we will use the fact, from the previous sentence, that $|S \cup N(S)| \leq |S| + D$ is at most $3I\frac{c^I}{I!}n$ and at most $3I^2|S|$:

$$\left(\frac{25|S \cup N(S)|}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{|S|^2} < \left(\frac{25 \times 3I^2}{\gamma}\right)^2 \left(\frac{25 \times 3I\frac{c^I}{I!}}{\gamma}\right)^{\frac{1}{2}\gamma k - 2}.$$

This product is easily seen to decrease as $I \geq 4c$ increases, and so it is at most

$$\left(\frac{25 \times 48c^2}{\gamma}\right)^2 \left(\frac{25 \times 12c\frac{c^{4c}}{(4c)!}}{\gamma}\right)^{\frac{1}{2}\gamma k - 2} < \frac{1}{2e^2},$$

for k (and hence c) sufficiently large.

Case 3: $I < 4c$. $D/|S|$ is monotone decreasing as $|S|$ increases, since increasing $|S|$ adds to S^* vertices of degree at most that of all those already in S^*. Therefore, $|S \cup N(S)|/|S|$ is at most the value $(|S|+D)/|S|$ at $I = 4c$ and $|S| = \frac{c^I}{I!}n$. Applying the analysis from Case 2, at that value of $|S|$ we have $|S| + D \le 3I\frac{c^I}{I!}n = 3I|S| = 12c|S|$. Therefore, using the facts that $c < 2k$ and $|S| < \frac{\epsilon_0}{20k}n < \frac{\epsilon_0}{10c}n < \frac{\gamma}{25\times 24c}n$ we have:

$$\left(\frac{25|S \cup N(S)|}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{|S|^2} < \left(\frac{25\times 12c}{\gamma}\right)^2 \left(\frac{25\times 12c|S|}{\gamma n}\right)^{\frac{1}{2}\gamma k-2} < \left(\frac{25\times 24k}{\gamma}\right)^2 \left(\frac{1}{2}\right)^{\frac{1}{2}\gamma k-2} < \frac{1}{2e^2},$$

for k sufficiently large. □

Next we prove our bound on $\lambda(X, (S \cup N(S))\backslash X)$:

A.s. for every pair of disjoint sets $S, X \subset V(K)$ with $|S| \le \frac{\epsilon_0}{20k}n$ and $|X| \ge |S|$ we have:

$$\lambda(X, (S \cup N(S))\backslash X) \le \frac{1}{2}\gamma k|X|. \qquad (9)$$

Proof of (9): We show that there are a.s. no such sets violating (9) in $G_{n,p=c/n}$. We fix $|S| = \sigma n \le \frac{\epsilon_0}{20k}n$, $|X| = xn \ge \sigma n$ and we let ρ be the solution to $\left(\frac{25\rho}{\gamma}\right)^{\frac{1}{2}\gamma k}\frac{1}{\sigma^2} = \frac{1}{2e^2}$. By Lemma 3.1(a), a.s. K is such that for every choice of S we have $|S \cup N(S)| \le \rho n$.

We will bound the expected number of pairs S, X with $|S \cup N(S)| \le \rho n$ and $\lambda(X, (S \cup N(S))\backslash X) > \frac{1}{2}\gamma k|X|$. We first choose S, X, then expose $N(S)$. We assume that $|S \cup N(S)| \le \rho n$ and bound the probability, under that assumption, that $\lambda(X, (S \cup N(S))\backslash X) > \frac{1}{2}\gamma k|X|$. This yields a bound of at most:

$$\begin{aligned}
&\binom{n}{\sigma n}\binom{n}{xn}\binom{(\rho n)(xn)}{\frac{1}{2}\gamma kxn}\left(\frac{c}{n}\right)^{\frac{1}{2}\gamma kxn} \\
<\ &\left(\frac{e}{\sigma}\right)^{\sigma n}\left(\frac{e}{x}\right)^{xn}\left(\frac{ec\rho}{\frac{1}{2}\gamma k}\right)^{\frac{1}{2}\gamma kxn} \\
<\ &\left(\frac{e}{\sigma}\right)^{2xn}\left(\frac{25\rho}{\gamma}\right)^{\frac{1}{2}\gamma kxn} \qquad \text{since } \sigma \le x \text{ and } c < 2k \\
=\ &\left(\frac{1}{2}\right)^{xn} \qquad \text{since } \left(\frac{25\rho}{\gamma}\right)^{\frac{1}{2}\gamma k} = \frac{\sigma^2}{2e^2}.
\end{aligned}$$

The sum of $\left(\frac{1}{2}\right)^{xn}$ over all $i = |X|, j = |S|$ with $|X| \ge \max(|S|, \sqrt{\log n})$ is at most

$$\sum_{i\ge\sqrt{\log n}}\sum_{j\le i}\left(\frac{1}{2}\right)^i = \sum_{i\ge\sqrt{\log n}} i\left(\frac{1}{2}\right)^i = o(1).$$

Therefore, a.s. there are no sets S, X violating (9) with $|X| \geq \sqrt{\log n}$ and with $|S \cup N(S)| \leq \rho n$. By Lemma 3.1(a), this implies that a.s. there are no S, X violating (9) with $|X| \geq \sqrt{\log n}$.

For the case where $|S| \leq |X| < \sqrt{\log n}$, let H be the subgraph induced by X, S and the endpoints in $S \cup N(S)$ of more than $\frac{1}{2}\gamma k|X|$ edges from X. It is straightforward to show that H has more edges than vertices: Indeed, if ℓ is the number of vertices of H in $N(S)$, then $|E(H)| > \ell + \frac{1}{2}\gamma k|X| \geq \ell + 2|X|$ and $|V(H)| = \ell + |S| + |X| \leq \ell + 2|X|$. But H has at most $|S| + |X| + \frac{1}{2}\gamma k|X| = O(\sqrt{\log n})$ vertices. So by Lemma 1.5, a.s. no such H exists. □

Our final bound is that a.s. for every set $S \subset V(K)$ with $|S| \leq \frac{\epsilon_0}{20k}n$ we have:

$$\lambda(S, N(S)) \leq |N(S)| + \frac{k}{4}|S|. \qquad (7)$$

Proof[2] of (7): Again, we prove that (7) a.s. holds for every such S in $G_{n,p=c/n}$. Consider any set S of σn vertices. Let ν be the solution to $(25\nu)^{k/4}/\sigma = \frac{1}{2e}$. By Lemma 3.1(b), a.s. $|N(S)| \leq \nu n$.

We expose the edges from S to $N(S)$ as follows: First, for every $v \notin S$, we test the presence of an edge from v to each of the vertices in S, one-at-a-time, and stop testing as soon as we discover the first edge. This determines the vertices of $N(S)$. Next, for each $u \in N(S)$, we test the presence of an edge from u to each vertex in S for which the test was not carried out during the first step. The total number of edge-tests carried out in the second step is less than $|S| \times |N(S)|$. Note that S violates (7) iff more than $\frac{k}{4}|S|$ edges are exposed during the second step. So the expected number of sets S of size σn that violate (7) and for which $|N(S)| \leq \nu n$ is at most:

$$\begin{aligned}
& \binom{n}{\sigma n} \times \binom{\sigma n \times \nu n}{\frac{k}{4}\sigma n} \left(\frac{c}{n}\right)^{\frac{k}{4}\sigma n} \\
< \; & \left(\frac{e}{\sigma}\right)^{\sigma n} \left(\frac{ec\sigma\nu n^2}{\frac{k}{4}\sigma n^2}\right)^{\frac{k}{4}\sigma n} \\
< \; & \left(\frac{e}{\sigma}\right)^{\sigma n} (25\nu)^{\frac{k}{4}\sigma n} \qquad \text{since } c < 2k \\
= \; & \left(\frac{1}{2}\right)^{\sigma n}.
\end{aligned}$$

The sum of $\left(\frac{1}{2}\right)^{|S|}$ over all $|S| \geq \log n$ is $o(1)$. For the case where $|S| < \log n$, we use the well known fact that a.s. the maximum degree in $G_{n,p=c/n}$ is less than $\log n$ (see eg. Exercise

[2] We are grateful to an anonymous referee for suggesting this approach, which is simpler and more elegant than our original proof.

3.5 in [2]). Thus, replacing νn above by $|S|\log n$ we obtain a smaller bound of

$$\left(\frac{en}{|S|}\right)^{|S|}\left(\frac{25|S|\log n}{n}\right)^{\frac{k}{4}|S|}<\frac{1}{n^2}.$$

Multiplying this by the $\log n$ choices for $|S| < \log n$ yields that the expected number of violating sets is $o(1)$, thus establishing that a.s. (7) holds for all such S. $\square$

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