Notes 17: Effective resistance

As in the last lecture, let \( H = (V,E) \) be a connected, undirected graph (representing an electrical network) with positive edge weights \( w : E \to \mathbb{R}_+ \).

The goal of this lecture is to develop tools for fast algorithms to approximately solve Laplace equations.

1. Effective resistance

Given any nodes \( a \) and \( b \), we can treat the whole electrical network \( H \) as a single resistor between \( a \) and \( b \). What is the resistance of this resistor?

If we inject one unit of external current at \( a \) and remove one unit of current at \( b \), we can measure the resulting potential difference \( v(a) - v(b) \). Ohm’s law tells us to expect

\[
v(a) - v(b) = i(a,b) R_{\text{eff}}(a,b).
\]

Thus, we define the effective resistance \( R_{\text{eff}}(a,b) \) between \( a \) and \( b \) so that this equation holds.

This corresponds to the external current vector \( u = 1_a - 1_b \). The above discussion implies the voltage vector due to \( u \) is \( v = L^+ u \). The potential difference \( v(a) - v(b) \), and hence \( R_{\text{eff}}(a,b) \), is \((1_a - 1_b)^\top L^+ (1_a - 1_b)\).

Since \( L \) is positive semidefinite, so is \( L^+ \), and therefore it has a square-root \( L^{+/2} \). In terms of spectral decomposition using nonnegative eigenvalues \( \lambda_\ell \) and eigenvectors \( \psi_\ell \),

\[
L = \sum_\ell \lambda_\ell \psi_\ell \psi_\ell^\top \quad \implies \quad L^{+/2} = \sum_\ell \frac{1}{\sqrt{\lambda_\ell}} \psi_\ell \psi_\ell^\top.
\]

Therefore

\[
R_{\text{eff}}(a,b) = (1_a - 1_b)^\top L^+ (1_a - 1_b) = (1_a - 1_b)^\top (L^{+/2})^\top L^{+/2} (1_a - 1_b) = \|L^{+/2} 1_a - L^{+/2} 1_b\|^2.
\]

In other words, if we represent every node \( a \) as the vector \( L^{+/2} 1_a \), then \( R_{\text{eff}}(a,b) \) is the squared Euclidean distance between the corresponding vectors \( L^{+/2} 1_a \) and \( L^{+/2} 1_b \). This map \( a \mapsto L^{+/2} 1_a \) is sometimes called the effective resistance embedding.

2. Equivalent networks, Gaussian elimination

We just considered what happens when two nodes are under external influence — the rest of the network can be represented as a single resistor. We now do the same when a subset \( B \subseteq V \) of nodes are under external influence.

We call \( B \) the set of boundary nodes and \( I = V \setminus B \) the set of internal nodes. You may imagine that we can attach electrodes of batteries to nodes in \( B \) but not in \( I \). So we can set voltages of nodes in \( V \), while voltages of nodes in \( I \) are determined by electrical flow of the batteries.

When \( B = V \), the Laplacian operator \( L \) maps voltage vector \( v \in \mathbb{R}^B \) to vector of external currents \( u \in \mathbb{R}^B \). Now for a general subset \( B \subseteq V \), we want to find a matrix \( L_B \) such that

\[
u_B = L_B v_B.
\]

Turns out \( L_B \) is a Laplacian matrix (easy exercise), and is obtained by applying Gaussian elimination to remove the internal nodes.

To be concrete, we take \( V = \{1, \ldots, n\} \), \( B = \{2, \ldots, n\} \), and we eliminate the internal node 1 using Gaussian elimination. Given any voltage vector \( v_B \in \mathbb{R}^B \), we want to find \( v \in \mathbb{R}^V \) such that \( v(b) = v_B(b) \) for every \( b \in B \), and

\[
0 = u(1) = \sum_{b \sim 1} i(1,b) = \sum_{b \sim 1} w(1,b)(v(1) - v(b)).
\]

Rearranging,

\[
v(1) = \frac{1}{d(1)} \sum_{b \sim 1} w(1,b)v(b).
\]
This means \( v(1) \) is a weighted average of voltages of its neighbors \( b \). It also means when solving the Laplace equation \( u = Lv \), we will substitute \( v(1) \) as the right-hand-side whenever \( v(1) \) appears. The term \( v(1) \) only appears in the equation for \( u(a) \) when \( a \) is a neighbor of 1, and the equation is

\[
u(a) = d(a)v(a) - \sum_{b \sim a} w(a,b)v(b) .
\]

After substituting \( v(1) \), the equation for \( u(a) \) becomes

\[
u(a) = d(a)v(a) - \sum_{b \sim a, b \neq 1} w(a,b)v(b) - \frac{w(1,a)}{d(1)} \sum_{b \sim 1, b \neq a} w(1,b)v(b) .
\]

One of the term in the last sum is in fact node \( a \), so the equation should be rewritten as

\[
u(a) = d(a)v(a) - \sum_{b \sim a, b \neq 1} w(a,b)v(b) - \frac{w(1,a)^2}{d(1)} v(a) - \sum_{b \sim a, b \neq 1} w(a,b)v(b) - \frac{w(1,a)}{d(1)} \sum_{b \sim 1, b \neq a} w(1,b)v(b) .
\]

This is exactly the result of applying Gaussian elimination to eliminate the variable \( v(1) \) using the equation \( u(1) = 0 \).

### 3. Distance

A distance \( d \) (also known as a metric) is any real-valued function on pair of vertices such that

- (Nonnegativity) \( d(a,b) \geq 0 \) for any vertices \( a \) and \( b \)
- (Identity of indiscernibles) \( d(a,b) = 0 \) if and only if \( a = b \)
- (Symmetry) \( d(a,b) = d(b,a) \) for any \( a \) and \( b \)
- (Triangle inequality/subadditivity) \( d(a,c) \leq d(a,b) + d(b,c) \) for any \( a, b \) and \( c \)

We now argue that effective resistance \( R_{\text{eff}} \) is a distance. The first three properties easily follow from §1 of this notes. It remains to prove the last property (triangle inequality).

We need the following simple observation: Given a unit electrical flow from \( a \) to \( b \), the corresponding voltage vector \( v \in \mathbb{R}^V \) satisfies \( v(a) \geq v(c) \geq v(b) \) for any node \( c \).

This observation holds because the voltage of any internal node \( c \) is a weighted average of its neighbors. To formally prove it, one can first consider the equivalent network with boundary \( B = \{ a, b, c \} \). The voltage of \( c \) in this equivalent network, after \( v(a) \) and \( v(b) \) are fixed, will be a weighted average of \( v(a) \) and \( v(b) \), and hence between them.

**Proposition 3.1.** \( R_{\text{eff}}(a,c) \leq R_{\text{eff}}(a,b) + R_{\text{eff}}(b,c) \).

**Proof.** Let \( u_{a,b} = 1_a - 1_b \) be the external current for the unit current flow from \( a \) to \( b \). Similarly, \( u_{b,c} = 1_b - 1_c \) and \( u_{a,c} = 1_a - 1_c \). Note that

\[
u_{a,c} = u_{a,b} + u_{b,c} .
\]

Let \( v_{a,b} = L^+ u_{a,b} \) be the voltage vector for \( u_{a,b} \). Likewise \( v_{b,c} = L^+ u_{b,c} \) and \( v_{a,c} = L^+ u_{a,c} \). By linearity,

\[
v_{a,c} = v_{a,b} + v_{b,c} ,
\]

and

\[
R_{\text{eff}}(a,c) = v_{a,c}(a) - v_{a,c}(c) = v_{a,b}(a) - v_{a,b}(c) + v_{b,c}(a) - v_{b,c}(c) .
\]

By above observation, the first two terms

\[
v_{a,b}(a) - v_{a,b}(c) \leq v_{a,b}(a) - v_{a,b}(b) = R_{\text{eff}}(a,b)
\]

and similarly \( v_{b,c}(a) - v_{b,c}(c) \leq v_{b,c}(b) - v_{b,c}(c) = R_{\text{eff}}(b,c) \). \( \square \)
4. Equivalent electrical power

Effective resistance between \( a \) and \( b \) in a network is defined as the resistance of the equivalent resistor. Turns out the network and its equivalent resistor share more common properties than just the same resistance: they also dissipate the same power per unit flow.

**Proof.** The power dissipated per unit of \( a-b \) flow in the equivalent resistor is exactly \( R_{\text{eff}}(a,b) \), due to Joule’s law \( P = I^2 R \).

The power dissipated in the network per unit of \( a-b \) flow is \( i^\top W^{-1} i \), where \( W \) is the diagonal matrix of edge weights, and \( i \) is the unit electrical flow from \( a \) to \( b \). Since \( i \) is induced by some voltage \( v \in \mathbb{R}^V \) and \( i = WBv \), the power dissipated is

\[
i^\top W^{-1} i = (WBv)^\top W^{-1} (WBv) = v^\top B^\top WBv = v^\top Lv .
\]

And since

\[
R_{\text{eff}}(a,b) = (1_a - 1_b)^\top L^+ (1_a - 1_b) = (Lv)^\top L^+ (Lv) = v^\top Lv,
\]

the network dissipates the same power as the equivalent resistor.

In the last equation, the first equality relating effective resistance and \( L^+ \) is proved to §1 of this notes; the second equality is due to \( Lv = 1_a - 1_b \) (that is, \( v \) is the voltage vector so that one unit of current flows from \( a \) to \( b \)); the last equality is \( LL^+L = L \).

5. Connectivity

Given an unweighted, undirected graph \( G \), effective resistance is loosely related to its edge-connectivity (minimum number of edges to remove to disconnect the graph).

For example, if there are at least \( k \) edge-disjoint paths from \( a \) to \( b \), each of length at most \( \ell \), then \( R_{\text{eff}}(a,b) \leq \ell/k \). To see this, we may increase the resistance of all edges outside of these \( k \) paths to infinity. The effective resistance between \( a \) and \( b \) in this new network is at most \( \ell/k \) by direct calculation. The effective resistance in the new network upperbounds that in the original network, because effective resistance is the power dissipated by the unit electrical flow from \( a \) to \( b \), and the power can only increase with resistance by Rayleigh’s monotonicity principle.