Notes 17: Effective resistance

As in the last lecture, let $H = (V, E)$ be a connected, undirected graph (representing an electrical network) with positive edge weights $w : E \to \mathbb{R}_+$. The goal of this lecture is to develop tools for fast algorithms to approximately solve Laplace equations.

1. Effective resistance

Given any nodes $a$ and $b$, we can treat the whole electrical network $H$ as a single resistor between $a$ and $b$. What is the resistance of this resistor?

If we inject one unit of external current at $a$ and remove one unit of current at $b$, we can measure the resulting potential difference $v(a) - v(b)$. Ohm’s law tells us to expect

$$v(a) - v(b) = i(a, b) R_{\text{eff}}(a, b).$$

Thus, we define the effective resistance $R_{\text{eff}}(a, b)$ between $a$ and $b$ so that this equation holds.

This corresponds to the external current vector $u = \mathbb{1}_a - \mathbb{1}_b$. The above discussion implies the voltage vector due to $u$ is $v = L^+u$. The potential difference $v(a) - v(b)$, and hence $R_{\text{eff}}(a, b)$, is $(\mathbb{1}_a - \mathbb{1}_b)^T L^+(\mathbb{1}_a - \mathbb{1}_b)$.

Since $L$ is positive semidefinite, so is $L^+$, and therefore it has a square-root $L^{+/2}$. In terms of spectral decomposition using nonnegative eigenvalues $\lambda_\ell$ and eigenvectors $\psi_\ell$,

$$L = \sum_\ell \lambda_\ell \psi_\ell \psi_\ell^\top \implies L^{+/2} = \sum_{\ell : \lambda_\ell > 0} \frac{1}{\sqrt{\lambda_\ell}} \psi_\ell \psi_\ell^\top.$$

Therefore

$$R_{\text{eff}}(a, b) = (\mathbb{1}_a - \mathbb{1}_b)^T L^+(\mathbb{1}_a - \mathbb{1}_b) = (\mathbb{1}_a - \mathbb{1}_b)^T (L^{+/2})^T L^{+/2} (\mathbb{1}_a - \mathbb{1}_b) = \|L^{+/2} \mathbb{1}_a - L^{+/2} \mathbb{1}_b\|^2.$$

In other words, if we represent every node $a$ as the vector $L^{+/2} \mathbb{1}_a$, then $R_{\text{eff}}(a, b)$ is the squared Euclidean distance between the corresponding vectors $L^{+/2} \mathbb{1}_a$ and $L^{+/2} \mathbb{1}_b$. This map $a \mapsto L^{+/2} \mathbb{1}_a$ is sometimes called the effective resistance embedding.

2. Equivalent networks, Gaussian elimination

We just considered what happens when two nodes are under external influence — the rest of the network can be represented as a single resistor. We now do the same when a subset $B \subseteq V$ of nodes are under external influence.

We call $B$ the set of boundary nodes and $I = V \setminus B$ the set of internal nodes. You may imagine that we can attach electrodes of batteries to nodes in $B$ but not in $I$. So we can set voltages of nodes in $V$, while voltages of nodes in $I$ are determined by electrical flow of the batteries.

When $B = V$, the Laplacian operator $L$ maps voltage vector $v \in \mathbb{R}^B$ to vector of external currents $u \in \mathbb{R}^B$. Now for a general subset $B \subseteq V$, we want to find a matrix $L_B$ such that

$$u_B = L_B v_B.$$

Turns out $L_B$ is a Laplacian matrix (easy exercise), and is obtained by applying Gaussian elimination to remove the internal nodes.

To be concrete, we take $V = \{1, \ldots, n\}$, $B = \{2, \ldots, n\}$, and we eliminate the internal node 1 using Gaussian elimination. Given any voltage vector $v_B \in \mathbb{R}^B$, we want to find $v \in \mathbb{R}^V$ such that $v(b) = v_B(b)$ for every $b \in B$, and

$$0 = u(1) = \sum_{b \sim 1} i(1, b) = \sum_{b \sim 1} w(1, b)(v(1) - v(b)).$$

Rearranging,

$$v(1) = \frac{1}{d(1)} \sum_{b \sim 1} w(1, b)v(b).$$
This means \( v(1) \) is a weighted average of voltages of its neighbors \( b \). It also means when solving the Laplace equation \( u = Lv \), we will substitute \( v(1) \) as the right-hand-side whenever \( v(1) \) appears. The term \( v(1) \) only appears in the equation for \( u(a) \) when \( a \) is a neighbor of 1, and the equation is

\[
u(a) = d(a)v(a) - \sum_{b \sim a} w(a, b)v(b).
\]

After substituting \( v(1) \), the equation for \( u(a) \) becomes

\[
u(a) = d(a)v(a) - \sum_{b \sim a, b \neq 1} w(a, b)v(b) - \frac{w(1, a)}{d(1)} \sum_{b \sim 1, b \neq a} w(1, b)v(b).
\]

One of the term in the last sum is in fact node \( a \), so the equation should be rewritten as

\[
u(a) = d(a)v(a) - \sum_{b \sim a, b \neq 1} w(a, b)v(b) - \frac{w(1, a)^2}{d(1)}v(a)
\]

\[
= \left(d(a) - \frac{w(1, a)^2}{d(1)}\right)v(a) - \sum_{b \sim a, b \neq 1} w(a, b)v(b) - \frac{w(1, a)}{d(1)} \sum_{b \sim 1, b \neq a} w(1, b)v(b).
\]

This is exactly the result of applying Gaussian elimination to eliminate the variable \( v(1) \) using the equation \( u(1) = 0 \).

3. Distance

A distance \( d \) (also known as a metric) is any real-valued function on pair of vertices such that

- (Nonnegativity) \( d(a, b) \geq 0 \) for any vertices \( a \) and \( b \)
- (Identity of indiscernibles) \( d(a, b) = 0 \) if and only if \( a = b \)
- (Symmetry) \( d(a, b) = d(b, a) \) for any \( a \) and \( b \)
- (Triangle inequality/subadditivity) \( d(a, c) \leq d(a, b) + d(b, c) \) for any \( a, b \) and \( c \)

We now argue that effective resistance \( R_{\text{eff}} \) is a distance. The first three properties easily follow from §1 of this notes. It remains to prove the last property (triangle inequality).

We need the following simple observation: Given a unit electrical flow from \( a \) to \( b \), the corresponding voltage vector \( v \in \mathbb{R}^V \) satisfies \( v(a) \geq v(c) \geq v(b) \) for any node \( c \).

This observation holds because the voltage of any internal node \( c \) is a weighted average of its neighbors. To formally prove it, one can first consider the equivalent network with boundary \( B = \{a, b, c\} \). The voltage of \( c \) in this equivalent network, after \( v(a) \) and \( v(b) \) are fixed, will be a weighted average of \( v(a) \) and \( v(b) \), and hence between them.

**Proposition 3.1.** \( R_{\text{eff}}(a, c) \leq R_{\text{eff}}(a, b) + R_{\text{eff}}(b, c) \).

**Proof.** Let \( u_{a,b} = 1_a - 1_b \) be the external current for the unit current flow from \( a \) to \( b \). Similarly, \( u_{b,c} = 1_b - 1_c \) and \( u_{a,c} = 1_a - 1_c \). Note that

\[
u_{a,c} = u_{a,b} + u_{b,c}.
\]

Let \( v_{a,b} = L^+u_{a,b} \) be the voltage vector for \( u_{a,b} \). Likewise \( v_{b,c} = L^+u_{b,c} \) and \( v_{a,c} = L^+u_{a,c} \). By linearity,

\[
v_{a,c} = v_{a,b} + v_{b,c},
\]

and

\[
R_{\text{eff}}(a, c) = v_{a,c}(a) - v_{a,c}(c) = v_{a,b}(a) - v_{a,b}(c) + v_{b,c}(a) - v_{b,c}(c).
\]

By above observation, the first two terms

\[
v_{a,b}(a) - v_{a,b}(c) \leq v_{a,b}(a) - v_{a,b}(b) = R_{\text{eff}}(a, b)
\]

and similarly \( v_{b,c}(a) - v_{b,c}(c) \leq v_{b,c}(b) - v_{b,c}(c) = R_{\text{eff}}(b, c) \). 

\(\square\)
4. Equivalent electrical power

Effective resistance between $a$ and $b$ in a network is defined as the resistance of the equivalent resistor. Turns out the network and its equivalent resistor share more common properties than just the same resistance: they also dissipate the same power per unit flow.

Proof 1. The power dissipated per unit of $a$-$b$ flow in the equivalent resistor is exactly $R_{\text{eff}}(a, b)$, due to Joule’s law $P = I^2R$.

The power dissipated in the network per unit of $a$-$b$ flow is $i^\top W^{-1}i$, where $W$ is the diagonal matrix of edge weights, and $i$ is the unit electrical flow from $a$ to $b$. Since $i$ is induced by some voltage $v \in \mathbb{R}^V$ and $i = WBv$, the power dissipated is

$$i^\top W^{-1}i = (WBv)^\top W^{-1}(WBv) = v^\top B^\top WBv = v^\top Lv.$$  

And since

$$R_{\text{eff}}(a, b) = (1_a - 1_b)^\top L^+(1_a - 1_b) = (Lv)^\top L^+(Lv) = v^\top Lv,$$

the network dissipates the same power as the equivalent resistor.

In the last equation, the first equality relating effective resistance and $L^+$ is proved to §1 of this notes; the second equality is due to $Lv = 1_a - 1_b$ (that is, $v$ is the voltage vector so that one unit of current flows from $a$ to $b$); the last equality is $LL^+ = L$. □

Below is a slightly more direct proof that requires more familiarity with pseudo-inverse.

Proof 2. Using the definition of effective resistance and the fact that the unit electrical flow $i$ is induced by voltage vector $v \in \mathbb{R}^V$ with $v(a) - v(b) = R_{\text{eff}}(a, b)$,

$$R_{\text{eff}}(a, b) = (1_a - 1_b)^\top L^+(1_a - 1_b) = (Lv)^\top L^+(Lv) = v^\top Lv.$$

Using $1_a - 1_b = B^\top i$ and $L^+ = (B^\top WB)^+ = B^+ W^{-1}(B^+)^\top$, the above can be rewritten as

$$(i^\top B)B^+ W^{-1}(B^+)^\top (B^\top i) = (\Pi i)^\top W^{-1}(\Pi i) = i^\top W^{-1}i,$$

which is the power dissipated by the network. Here $\Pi = BB^+$ is the symmetric projection to the range of $B$ (that $i$ belongs to). □

5. Connectivity

Given an unweighted, undirected graph $G$, effective resistance is loosely related to its edge-connectivity (minimum number of edges to remove to disconnect the graph).

For example, if there are at least $k$ edge-disjoint paths from $a$ to $b$, each of length at most $\ell$, then $R_{\text{eff}}(a, b) \leq \ell/k$. To see this, we may increase the resistance of all edges outside of these $k$ paths to infinity. The effective resistance between $a$ and $b$ in this new network is at most $\ell/k$ by direct calculation. The effective resistance in the new network upperbounds that in the original network, because effective resistance is the power dissipated by the unit electrical flow from $a$ to $b$, and the power can only increase with resistance by Rayleigh’s monotonicity principle.