Notes 10: Cheeger–Alon–Milman inequality

1. Conductance and expansion

Last lecture we saw that a graph is connected if and only if the second smallest eigenvalue \(\lambda_2\) of its Laplacian \(L_G\) is strictly larger than the smallest eigenvalue \(\lambda_1\) (which is zero). Today we will show a robust version of this result: a graph is “well-connected” if and only if \(\lambda_2\) is much bigger than \(\lambda_1\).

One way to measure how well a graph \(G\) is connected is expansion.

**Definition 1.1.** Given a graph \(G\) with edge weights \(w : E \to \mathbb{R}_+\), the degree of vertex \(i\) is \(\deg(i) = \sum_{j \sim i} w_{ij}\) and the total degree of a vertex subset \(S \subseteq V\) is \(\deg(S) = \sum_{i \in S} \deg(i)\).

The conductance of a vertex subset \(S \subseteq V\) is

\[
\varphi(S) = w(S, \overline{S})/\deg(S),
\]

where \(w(S, \overline{S}) = \sum_{i \in S, j \notin S, i \sim j} w_{ij}\) is the total edge weight across the cut from \(S\) to \(\overline{S}\).

The expansion of a graph is

\[
\varphi(G) = \min_{S \subset V, S \neq \emptyset} \varphi(S).
\]

The condition \(\deg(S) \leq \deg(V)/2\) in expansion is equivalent to \(\deg(S) \leq \deg(\overline{S})\).

The conductance of a subset or the expansion a graph is always between 0 and 1. A graph is disconnected if and only if \(\varphi(G) = 0\).

2. Normalized Laplacians

We are going to compare graph expansion to Laplacian eigenvalues. We will assume the graph has no isolated vertices (of degree 0).

Recall that \(L_G = \sum_{i \sim j} w_{ij}(1_i - 1_j)(1_i - 1_j)^\top = D - A\), where \(D\) is the diagonal matrix with \(D_{ii} = \deg(i)\) and \(A\) is the adjacency matrix. (We will drop subscript \(G\) and write \(L = L_G\).)

\(D\) has eigenvalues in the range \([0, \Delta]\) and \(A\) has eigenvalues in the range \([-\Delta, \Delta]\), where \(\Delta = \max_{i \in V} \deg(i)\) is the maximum degree. Hence \(L\) has eigenvalues in the range \([0, 2\Delta]\) (the zero lowerbound is due to positive semidefiniteness of \(L\)).

We want to remove the dependence on (maximum) degree and normalize the Laplacian, so that its eigenvalues are between \([0, 2]\).

How should we normalize the Laplacian? A naive attempt is \(L/\Delta\), but computing \(\Delta\) is nonlinear (being the maximum over vertices).

A better way to normalize is to “divide” \(L\) by a positive definite matrix, and use the following fact (proof omitted).

**Claim 2.1.** If \(n\)-by-\(n\) real symmetric matrices \(X\) and \(Y\) satisfy \(X \succ 0\) and \(Y \succ 0\), then \(Y^{-1/2}XY^{-1/2} \succ 0\).

**Definition 2.2.** The normalized adjacency matrix is \(A = D^{-1/2}AD^{-1/2}\).

The normalized Laplacian is \(L = D^{-1/2}LD^{-1/2} = D^{-1/2}(D - A)D^{-1/2} = I - A\).

**Proposition 2.3.** Eigenvalues of \(A\) are between \(-1\) and 1. Eigenvalues of \(L\) are between 0 and 2.

**Proof.** \(D - A = \sum_{i \sim j} w_{ij}(1_i - 1_j)(1_i - 1_j)^\top \succ 0\). Therefore \(I - A = D^{-1/2}(D - A)D^{-1/2} \succ 0\) by the above claim. This means all eigenvalues of \(A\) are at most 1.

Similarly \(D + A = \sum_{i \sim j} w_{ij}(1_i + 1_j)(1_i + 1_j)^\top \succ 0\). Therefore \(I + A = D^{-1/2}(D + A)D^{-1/2} \succ 0\) by the above claim. This means all eigenvalues of \(A\) are at least \(-1\).

Eigenvalue bounds for \(L\) follows from eigenvalue bounds for \(A = I - L\). \(\square\)

In fact 0 is always an eigenvalue of \(L\), with eigenvector \(v_1 = D^{1/2}1\), because

\[
Lv_1 = D^{-1/2}LD^{-1/2}D^{1/2}1 = D^{-1/2}L1 = 0.
\]

One can show that \(L\) and \(L\) have the same zero eigenspace.
3. Cheeger–Alon–Milman Inequality

Let \(0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2\) be the eigenvalues of the normalized Laplacian matrix \(L\) of a graph \(G\).

In the previous lecture, we showed that \(\lambda_2 = 0 (= \lambda_1)\) if and only if the graph is connected.
We now quantify well-connectedness of a graph via \(\lambda_2\) (the gap between the two smallest eigenvalues).

**Theorem 3.1** (Cheeger–Alon–Milman).

\[
\frac{\lambda_2}{2} \leq \varphi(G) \leq \sqrt{2\lambda_2}.
\]

We first prove the easy direction (left inequality).
By Courant–Fischer, taking \(v_1 = D^{1/2}\mathbf{1}\) to be an eigenvector of \(L\) with eigenvalue 0,
\[
\lambda_2 = \min_{x \perp v_1} \frac{x^\top Lx}{x^\top x} = \min_{x \perp v_1} \frac{x^\top D^{-1/2}LD^{-1/2}x}{x^\top x} = \min_{y \perp v_1} \frac{y^\top Ly}{y^\top Dy},
\]
where \(y = D^{-1/2}x\).
The condition \(D^{1/2}y \perp v_1\) means \(0 = (D^{1/2}y)^\top D^{1/2}\mathbf{1} = y^\top D\mathbf{1} = \sum_{i \in V} \deg(i)y_i\).
Also, the denominator in the Rayleigh quotient is \(y^\top Dy = \sum_{i \in V} \deg(i)y_i^2\).
In summary,
\[
\lambda_2 = \min_{\sum_{i \in V} \deg(i)y_i = 0} \frac{\sum_{i,j} w_{ij}(y_i - y_j)^2}{\sum_{i \in V} \deg(i)y_i^2}.
\]
To upperbound \(\lambda_2\), we construct a vector \(y\) from a vertex subset \(S\) of small conductance.
A natural choice is \(y = \mathbf{1}_S\), the indicator function for \(S\), i.e. \(y_i = 1\) if \(i \in S\) and \(y_i = 0\) if \(i \notin S\).
Then the numerator \(\sum_{i,j} w_{ij}(y_i - y_j)^2 = w(S, \overline{S})\) and denominator \(\sum_{i \in V} \deg(i)y_i^2 = \deg(S)\), so the quotient gives us exactly \(\varphi(S)\).
But this \(y\) fails to satisfy the orthogonality constraint, because \(0 \not= \sum_{i \in V} \deg(i)y_i = \deg(S)\).
Instead we pick constants \(a\) and \(b\) and set \(y_i = a\) if \(i \in S\) and \(y_i = b\) if \(i \notin S\), and require \(0 = \sum_{i \in V} \deg(i)y_i = \deg(S)a + \deg(\overline{S})b\).
Solving gives \(a = 1/\deg(S)\) and \(b = -1/\deg(\overline{S})\).
For this \(y\),
\[
\lambda_2 \leq \frac{\sum_{i,j} w_{ij}(y_i - y_j)^2}{\sum_{i \in V} \deg(i)y_i^2} = \frac{w(S, \overline{S}) \left(\frac{1}{\deg(S)} + \frac{1}{\deg(\overline{S})}\right)^2}{\deg(S) \frac{1}{\deg(S)^2} + \deg(\overline{S}) \frac{1}{\deg(\overline{S})^2}}
\]
\[
= \frac{w(S, \overline{S}) \left(\frac{1}{\deg(S)} + \frac{1}{\deg(\overline{S})}\right)^2}{\deg(S) + \deg(\overline{S})}
\]
\[
= w(S, \overline{S}) \left(\frac{1}{\deg(S)} + \frac{1}{\deg(\overline{S})}\right) = w(S, \overline{S}) \frac{\deg(S) + \deg(\overline{S})}{\deg(S) \deg(\overline{S})} = w(S, \overline{S}) \frac{\deg(V)}{\deg(S) \deg(\overline{S})}
\]
\[
\leq 2\varphi(S) \quad (\text{using } \deg(S) \leq \deg(V)/2)
\]
We will prove the hard direction (right inequality) of Cheeger–Alon–Milman in the next lecture.