(1) Suppose you are given a sequence of nonnegative real numbers \( d_{i,j} \) (\( 1 \leq i < j \leq n \)), and you want to know whether there are points \( v_1, \ldots, v_n \) in the \( n \)-dimensional Euclidean space such that their pairwise distances are exactly \( d_{i,j} \) (that is, \( \|v_i - v_j\|_2 = d_{i,j} \) for all \( 1 \leq i < j \leq n \)).

Formulate this problem as the feasibility of a semidefinite program.

(2) (a) Given real numbers \( a_1, \ldots, a_n \) as parameters, consider the following linear program \( P \) (whose variables are \( x_1, \ldots, x_n \) and \( y \)):

\[
\begin{align*}
\text{min} & \quad \sum_{1 \leq i \leq n} x_i \\
\text{s.t.} & \quad a_i - y \leq x_i \quad \forall i \\
& \quad y - a_i \leq x_i \quad \forall i \\
& \quad x_i \geq 0 \quad \forall i
\end{align*}
\]

In an optimal solution, what is \( y \) (in terms of \( a_1, \ldots, a_n \))? (b) Write down the dual to the above linear program (use \( \lambda_i \) as the dual variable for the constraint “\( a_i - y \leq x_i \)” and \( \mu_i \) as the dual variable for the constraint “\( y - a_i \leq x_i \)”).

(c) Let \( D \) denotes your answer to the previous part (so \( D \) is the dual to \( P \)). Suppose one modifies \( D \) as follows to obtain a new linear program \( D' \):

i. replace the constraint “\( \sum_i \mu_i = \sum_i \lambda_i \)” in \( D \) with “\( \sum_i \mu_i = 4 \sum_i \lambda_i \)”

ii. change the objective function to “\( \sum_i (4 \lambda_i - \mu_i) a_i \)”

What is the dual program \( P' \) of \( D' \)? In an optimal solution of \( P' \), what is \( y \) (in terms of \( a_1, \ldots, a_n \))?

Hint: look up “percentile”.

(3) Consider the maximum flow problem from \( s \) to \( t \) on a directed graph, where each edge has capacity one. A fraction \( s-t \) flow solution with value \( k \) is an assignment of each edge \( e \) to a fractional value \( x(e) \) satisfying

\[
\sum_{e \in \delta^{\text{out}}(s)} x(e) = \sum_{e \in \delta^{\text{in}}(t)} x(e) = k,
\]

\[
\sum_{e \in \delta^{\text{in}}(v)} x(e) = \sum_{e \in \delta^{\text{out}}(v)} x(e) \quad \forall v \in V \setminus \{s, t\}
\]

\[
0 \leq x(e) \leq 1 \quad \forall e \in E.
\]
In the above, $\delta_{\text{out}}(v)$ denotes the set of out-going edges from $v$, and $\delta_{\text{in}}(v)$ denotes the set of incoming edges to $v$.

Use the multiplicative weight update algorithm to find an approximation solution to the above linear program by reducing the flow problem to the problem of finding shortest paths between $s$ and $t$. Analyze the convergence rate and the total complexity of your algorithm to compute a flow of value at least $k(1 - \varepsilon)$ for any $\varepsilon > 0$.

(4) Compute all the eigenvalues of the (unnormalized) adjacency matrix of the hypercube graph $H_d$. Also specify the multiplicities of these eigenvalues.

The hypercube $H_d$ has $2^d$ vertices that are identified with binary strings of length $d$. Let $\{0, 1\}^d$ denote the set of such strings. Two different vertices $x, y \in \{0, 1\}^d$ are adjacent if they agree at $d - 1$ positions (and differ at the remaining position).

*Hint:* First guess a nice eigenbasis for the adjacency matrix.

(5) Prove the following strengthening of Alon–Milman inequality by modifying the proof given in class: For any $d$-regular graph $G$, 

$$\phi(G) \leq \sqrt{(2 - \lambda_2)\lambda_2},$$

where $\lambda_2$ is the second smallest eigenvalue of the normalized Laplacian $L$ of $G$.

*Hint:* 

$$\sum_{(i,j) \in E}(y_i + y_j)^2 + \sum_{(i,j) \in E}(y_i - y_j)^2 = 2d \sum_i y_i^2.$$