
Collaborating on homework and consulting references is encouraged, but you must write your own solutions in your own words, and list your collaborators and your references. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Please answer clearly and concisely. Explain your answers.

(1) (a) An ellipsoid $\mathcal{E}_A$ centered at the origin is the image of the unit ball under the linear map A, that is,

$$\mathcal{E}_A \stackrel{\text{def}}{=} \{Ax \mid x \in \mathbb{R}^n, \|x\| \leqslant 1\}$$

where A is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. If A is invertible, show that $\mathcal{E}_A$ is precisely the set of vectors $y \in \mathbb{R}^n$ such that $y^T(AA^T)^{-1}y \leqslant 1$.

(b) If A and B are invertible linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, show that $\mathcal{E}_A \subseteq \mathcal{E}_B$ if and only if $AA^T \preccurlyeq BB^T$.

This shows that the semidefinite ordering $\preccurlyeq$ (known as Loewner ordering) corresponds to inclusion relationship of certain ellipsoids. In fact part (b) still holds without assuming A and B to be invertible, but you do not need to consider that general case.

(2) Let $\mathcal{S} \subseteq \{1, \ldots, n\}$ be a family of subsets over a universe of size n. The following program finds the maximum entropy probability distribution p supported on $\mathcal{S}$ subject to marginal probability constraints:

$$\begin{aligned} \max \sum_{S \in \mathcal{S}} p_S \log \frac{1}{p_S} & \\ \sum_{S \in \mathcal{S}} p_S &= 1 \\ \text{for } 1 \leqslant i \leqslant n \qquad \sum_{S \in \mathcal{S},\ S \ni i} p_S &= b_i \\ \forall S \in \mathcal{S} \quad p_S &\geqslant 0 \end{aligned}$$

By considering the optimality condition of the Lagrangian, show that any maximizer satisfying $p_S > 0$ for all $S \in \mathcal{S}$ must be of the form

$$p_S = \frac{\prod_{i \in S} e^{\lambda_i}}{\sum_{T \in \mathcal{S}} \prod_{i \in T} e^{\lambda_i}}$$

for some real numbers λ_i.

(3) (a) Given real numbers $a_1, \ldots, a_n$ as parameters, consider the following linear program P (whose variables are $x_1, \ldots, x_n$ and y):

$$\begin{aligned} \min \sum_{1 \leqslant i \leqslant n} x_i & \\ a_i - y \leqslant x_i & \qquad \forall i \\ y - a_i \leqslant x_i & \qquad \forall i \\ x_i \geqslant 0 & \qquad \forall i \end{aligned}$$

In an optimal solution, what is y (in terms of $a_1, \ldots, a_n$)?

(b) Write down the dual to the above linear program (use λ_i as the dual variable for the constraint "$a_i - y \leqslant x_i$" and μ_i as the dual variable for the constraint "$y - a_i \leqslant x_i$").

(c) Let D denotes your answer to the previous part (so D is the dual to P). Suppose one modifies D' as follows to obtain a new linear program D':

i. replace the constraint "$\sum_i \mu_i = \sum_i \lambda_i$" in D with "$\sum_i \mu_i = 4 \sum_i \lambda_i$"

ii. change the objective function to "$\sum_i (4\lambda_i - \mu_i) a_i$"

What is the dual program P' of D'? In an optimal solution of P', what is y (in terms of $a_1, \ldots, a_n$)?

Hint: look up "percentile".

(4) Suppose you are given a function $f : \{0,1\}^n \to \{0,1\}$ that is the OR of some k input bits (and independent of the other input bits), where k is much smaller than n. You do not know which k bits f depends on. You are asked to predict the output of f on a sequence of input strings with the following algorithm: initially set all weights w_i to be 1 for $1 \leqslant i \leqslant n$, and on input string $x = x_1 \ldots x_n$ predict 1 when $w_1 x_1 + \cdots + w_n x_n \geqslant n$, and predict 0 otherwise. You will then learn whether your prediction agrees with $f(x)$. Every time your prediction is wrong, halve or double weights appropriately.

How do you halve or double your weights? Show that you will make $O(k \log n)$ mistakes.

(5) Prove the following strengthening of Alon–Milman inequality by modifying the proof given in class: For any d-regular graph G,

$$\phi(G) \leqslant \sqrt{(2 - \lambda_2)\lambda_2},$$

where λ_2 is the second smallest eigenvalue of the normalized Laplacian $\mathcal{L}$ of G.

Hint: $\sum_{(i,j)\in E} (y_i + y_j)^2 + \sum_{(i,j)\in E} (y_i - y_j)^2 = 2d \sum_i y_i^2$.