1. ONLINE TO PAC

**Theorem 1.1.** If Algorithm $A$ learns $C$ in Online Mistake Bound model with $M$ mistakes. Then some algorithm PAC learns $C$ using

$$m = M + \frac{M + 1}{\varepsilon} \ln \left( \frac{M + 1}{\delta} \right)$$ samples

**Proof.** Can assume $A$ only updates its hypothesis after making a mistake (homework)

PAC Learning Algorithm

Keep feeding to $A$ independent samples from $\text{EX}(c, D)$

Until $A$ correctly classifies $\frac{1}{\varepsilon} \ln \left( \frac{M + 1}{\delta} \right)$ samples in a row

Then output $A$’s current (i.e. last) hypothesis $h$

$A$’s predictions:

$\sqrt{\sqrt{\cdots \sqrt{X}}} \; \sqrt{\sqrt{\cdots \sqrt{X}}} \; \sqrt{\sqrt{\cdots \sqrt{X}}} \; \sqrt{\sqrt{\cdots \sqrt{X}}}$ (repeat $\leq M$ times)

$\leq \frac{1}{\varepsilon} + \ln \left( \frac{M + 1}{\delta} \right)$

$\leq M$ mistakes and $\leq M + 1$ blocks, each with $\leq \frac{1}{\varepsilon} \ln \left( \frac{M + 1}{\delta} \right)$ correct predictions

#samples used $\leq M + \frac{M + 1}{\varepsilon} \ln \left( \frac{M + 1}{\delta} \right)$

We now argue final hypothesis $h_{\text{last}}$ has error $\leq \varepsilon$ with prob. $\geq 1 - \delta$

If $\text{err}_D(h_i, c) \geq \varepsilon$: $\mathbb{P} \left[ h_i \text{ correct } k \overset{\text{def}}{=} \frac{1}{\varepsilon} \ln \left( \frac{M + 1}{\delta} \right) \text{ times} \right] \leq 1 - \varepsilon \leq e^{-\varepsilon k} = \frac{\delta}{M + 1}$

$A$ uses $\leq M + 1$ hypotheses $h_1, \ldots, h_{\text{last}}$, $\mathbb{P}[\text{any of them has error } \geq \varepsilon] \leq (M + 1) \cdot \frac{\delta}{M + 1} = \delta$ \hfill $\square$

If $A$ efficient, so is its PAC version

Implies PAC learning algorithms for
e.g. (sparse) conjuctions/disjunctions, short decision lists, well-seperated LTFs
e.g. disjunctions: Elimination Algorithm makes $\leq n$ mistakes
its PAC version uses $O \left( \frac{n}{\varepsilon} \ln \left( \frac{n}{\delta} \right) \right)$ samples

2. PAC TO ONLINE? No

$X =$ unit interval $= [0, 1]$ $\quad C =$ initial intervals $= \{ [0, b] \mid 0 \leq b \leq 1 \}$

where $[0, b] = \{ x \in \mathbb{R} \mid 0 \leq x \leq b \}$

Can be PAC learned with $(1/\varepsilon) \ln \left( 1/\delta \right)$ samples (same idea as axis-aligned rectangles)

**Claim 2.1.** Any algorithm $A$ for learning closed intervals over $[0, 1]$ in the Online model makes an arbitrarily large number of mistakes

**Proof.** The adversary below forces $A$ to always err

Adversary

Initially $I = [0, 1]$

Repeat

Set $x =$ midpoint of $I$

Feed $x$ to $A$ and gets $A$’s prediction

Label $x$ opposite to $A$’s prediction

If $x$’s correct label is 0, shrinks $I$ by keeping only its left half, else keep only its right half

e.g. 1st round $x^1 = 1/2$, if $A$ predicts $x^1$ as 0, then label $x^1$ as 1, update $I$ as $[1/2, 1]$
All positive samples to the left of all negative samples
Some initial interval correctly classifies all labelled samples so far

$\text{ }$

$X$ above is infinite
How about finite $X$?
Efficient PAC algorithm for $\mathcal{C}$ over finite $X$ implies efficient online algorithm with few mistakes?
Previous example of initial intervals (now over $X = \{1, 2, \ldots, n\}$) has efficient online algorithm

namely Halving algorithm with $\leq \log n$ mistakes
In fact Halving algorithm has very efficient implementation in this case (binary search)
Under reasonable cryptographic assumptions, still no PAC-to-online conversion for finite $X = \{0, 1\}^n$