Turing Machines and Their Variants

CSCI 3130 Formal Languages and Automata Theory

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Fall 2018

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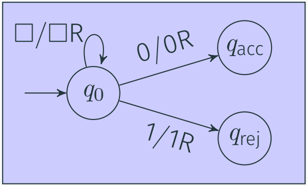
Looping

Turing machine may not halt

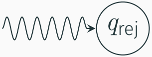


$\Sigma = \{0, 1\}$

input: ε

Inputs can be divided into three types:

Accept



Reject

Infinite loop

Halting

We say M halts on input x if there is a sequence of configurations

$$C_0, C_1, \ldots, C_k$$

C_0 is starting $\quad$ C_i yields C_{i+1} $\quad$ C_k is accepting or rejecting

A TM M is a decider if it halts on every input

Language L is decidable if it is recognized by a TM that halts on every input

Programming Turing machines: Are two strings equal?

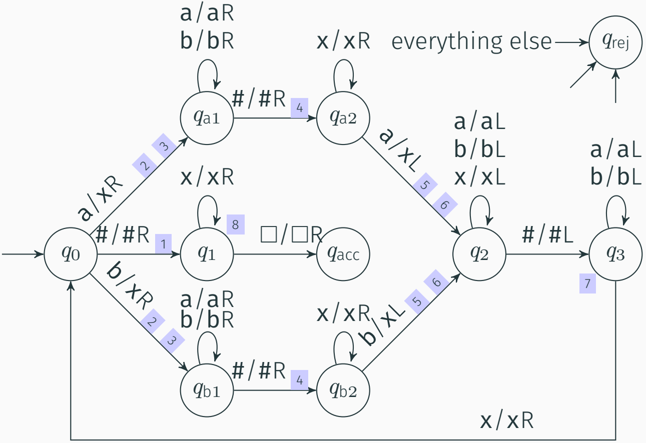
$$L_1 = \{w\#w \mid w \in \{\mathsf{a}, \mathsf{b}\}^*\}$$

Description of Turing Machine

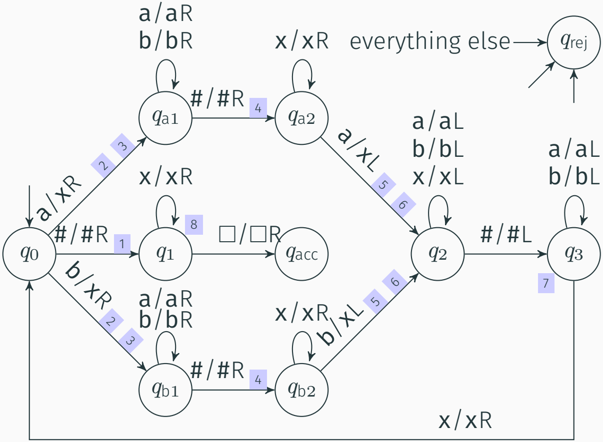
Until you reach #	
Read and remember entry	$\mathtt{x\underline{b}baa\#xbbaa}$
Write x	$\mathtt{x\underline{x}baa\#xbbaa}$
Move right past # and past all x's	$\mathtt{xxbaa\#x\underline{b}baa}$
If this entry is different, reject	
Write x	$\mathtt{xxbaa\#x\underline{x}baa}$
Move left past # and to right of first x	$\mathtt{xx\underline{b}aa\#xxbaa}$
If you see only x's followed by $\square$, accept	

Programming Turing machines: Are two strings equal?

$$L_1 = \{w\#w \mid w \in \{\mathsf{a},\mathsf{b}\}^*\}$$



Programming Turing machines: Are two strings equal?



input:

aab#aab

configurations:

q_0 aab#aab

x q_{a1} ab#aab

xa q_{a1} b#aab

xab q_{a1} #aab

xab# q_{a2} aab

xab q_2 #xab

xa q_3 b#xab

x q_3 ab#xab

q_3 xab#xab

x q_0 ab#xab

⋮

Programming Turing machines

$$L_2 = \{\mathsf{a}^i\mathsf{b}^j\mathsf{c}^k \mid ij = k \text{ and } i,j,k > 0\}$$

High level description of TM:

1. For every a:
2. Cross off the same number of b's and c's
3. Uncross the crossed b's (but not the c's)
4. Cross off this a
5. If all a's and c's are crossed off, accept

Example:

1 aabbcccc
2 aa~~bbcc~~cc
3 aabb~~cc~~cc
4 ~~a~~abb~~cc~~cc
5 ~~a~~abb~~cc~~cc
2 ~~a~~a~~bbcccc~~
3 ~~aabbcccc~~

$$\Sigma = \{\mathsf{a}, \mathsf{b}\} \qquad \Gamma = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \cancel{\mathsf{a}}, \cancel{\mathsf{b}}, \cancel{\mathsf{c}}, \square\}$$

$$L_2 = \{\mathsf{a}^i\mathsf{b}^j\mathsf{c}^k \mid ij = k \text{ and } i, j, k > 0\}$$

Low-level description of TM:

Scan input from left to right to check it looks like $\mathsf{aa}^*\mathsf{bb}^*\mathsf{cc}^*$

Move the head to the first symbol of the tape

For every a:

- Cross off the same number of b's and c's
- Restore the crossed off b's (but not the c's)
- Cross off this a

If all a's and c's are crossed off, accept

Programming Turing machines

$$L_2 = \{\mathsf{a}^i\mathsf{b}^j\mathsf{c}^k \mid ij = k \text{ and } i, j, k > 0\}$$

Low-level description of TM:

Scan input from left to right to check it looks like $\mathsf{aa}^*\mathsf{bb}^*\mathsf{cc}^*$

Move the head to the first symbol of the tape How?

For every a:

- Cross off the same number of b's and c's How?
- Restore the crossed off b's (but not the c's)
- Cross off this a

If all a's and c's are crossed off, accept

Implementation details:

Move the head to the first symbol of the tape:

Put a special marker on top of the first a	ȧabbcccc
Cross off the same number of b's and c's:	ȧabbcccc
Replace b by ~~b~~	ȧa~~b~~bcccc
Move right until you see a c	ȧa~~b~~bcccc
Replace c by ~~c~~	ȧa~~b~~b~~c~~ccc
Move left just past the last ~~b~~	ȧa~~bb~~~~c~~ccc
If any uncrossed b's are left, repeat	ȧa~~bb~~~~c~~ccc
	ȧa~~bb~~~~cc~~cc

$$\Sigma = \{a, b, c\} \qquad \Gamma = \{a, b, c, \cancel{a}, \cancel{b}, \cancel{c}, \dot{a}, \cancel{\dot{a}}, \Box\}$$

Programming Turing machines: Element distinctness

$L_3 = \{\texttt{\#}x_1\texttt{\#}x_2 \ldots \texttt{\#}x_m \mid x_i \in \{\texttt{0}, \texttt{1}\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j\}$

Example: $\texttt{\#01\#0011\#1} \in L_3$

High-level description of TM:

On input w

For every pair of blocks x_i and x_j in w

  Compare the blocks x_i and x_j

  If they are the same, reject

Accept

Programming Turing machines: Element distinctness

$L_3 = \{\texttt{\#}x_1\texttt{\#}x_2 \ldots \texttt{\#}x_m \mid x_i \in \{\texttt{0}, \texttt{1}\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j\}$

Low-level desrciption:

0. If input is ε, or has exactly one #, accept
1. Mark the leftmost # as $\dot{\texttt{\#}}$ and move right $\quad \dot{\texttt{\#}}\texttt{01\#0011\#1}$
2. Mark the next unmarked # $\quad \dot{\texttt{\#}}\texttt{01}\dot{\texttt{\#}}\texttt{0011\#1}$

Programming Turing machines: Element distinctness

$L_3 = \{\#x_1\#x_2 \ldots \#x_m \mid x_i \in \{0, 1\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j\}$

3. Compare the two strings to the right of $\dot{\#}$ — $\dot{\#}\underline{01}\dot{\#}\underline{0011}\#1$
   If they are equal, reject
4. Move the right $\dot{\#}$ — $\dot{\#}01\#0011\dot{\#}1$
   If not possible, move the left $\dot{\#}$ to the next $\#$
   and put the right $\dot{\#}$ on the next $\#$
   If not possible, accept
5. Repeat Step 3 — $\dot{\#}\underline{01}\#0011\dot{\#}\underline{1}$

$\#01\dot{\#}0011\dot{\#}1$

$\#01\dot{\#}\underline{0011}\dot{\#}\underline{1}$

Unlike for DFAs, NFAs, PDAs, we rarely give complete state diagrams of Turing Machines

We usually give a high-level description

unless you're asked for a low-level description or even state diagram

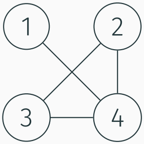
We are interested in algorithms behind the Turing machines

Programming Turing machines: Graph connectivity

$L_4 = \{\langle G \rangle \mid G \text{ is a connected undirected graph}\}$

How do we feed a graph into a Turing Machine?

How to encode a graph G as a string $\langle G \rangle$?



`(1,2,3,4)((1,4),(2,3),(3,4),(4,2))`

Conventions for describing graphs:

`(`nodes`)(`edges`)`
no node appears twice
edges are pairs `(`first node`,` second node`)`

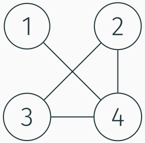
Programming Turing machines: Graph connectivity

$L_3 = \{\langle G \rangle \mid G \text{ is a connected undirected graph}\}$

High-level description:

On input $\langle G \rangle$

0. Verify that $\langle G \rangle$ is the description of a graph
   No node/edge repeats; Edge endpoints are nodes
1. Mark the first node of G
2. Repeat until no new nodes are marked:
   2.1 For each node, mark it if it is adjacent to an already marked node
3. If all nodes are marked, accept; otherwise reject



Programming Turing machines: Graph connectivity

Some low-level details:

0. Verify that $\langle G \rangle$ is the description of a graph

No node/edge repeats: Similar to Element distinctness

Edge endpoints are nodes: Also similar to Element distinctness

1. Mark the first node of G

Mark the leftmost digit with a dot, e.g. 12 becomes $\dot{1}2$

2. Repeat until no new nodes are marked:

2.1 For each node, mark it if it is attached to an already marked node

For every dotted node u and every undotted node v:

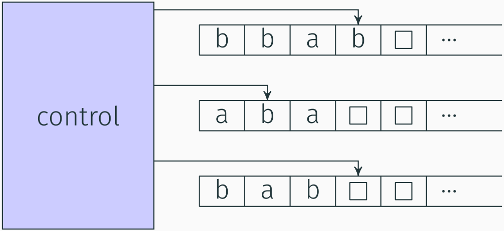
Underline both u and v from the node list

Try to match them with an edge from the edge list

If not found, remove underline from u and/or v and try another

Variants of Turing machines

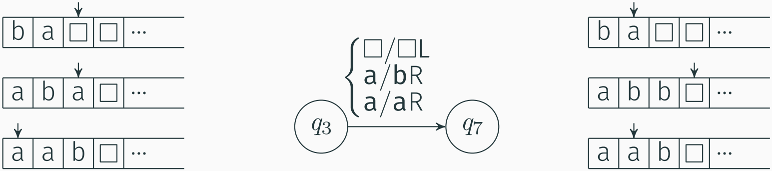
Multitape Turing machine



Transitions may depend on the contents of all cells under the heads

Different tape heads can move independent

Multitape Turing machine

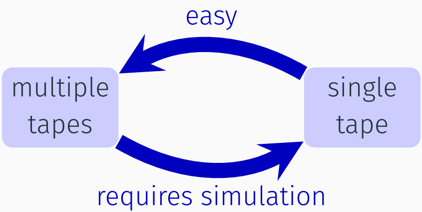


Multiple tapes are convenient

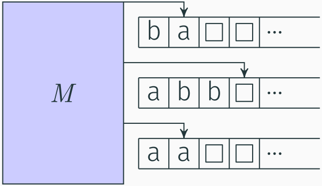
One tape can serve as temporary storage

How to argue equivalence

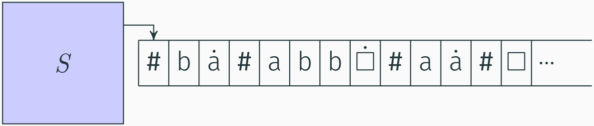
Multitape Turing machines are equivalent to singlne-tape Turing machines



Simulating multitape Turing machine



$\Gamma = \{\mathrm{a}, \mathrm{b}, \square\}$



$\Gamma = \{\mathrm{a}, \mathrm{b}, \square, \dot{\mathrm{a}}, \dot{\mathrm{b}}, \dot{\square}, \#\}$

Simulating multitape Turing machine

We show how to simulate a multitape Turing machine on a single tape Turing machine

To be specific, let's simulate a 3-tape TM

Multitape TM M

x_1	$\cdots$	x_r	$\cdots$	x_i	$\square$	

y_1	$\cdots$	$\cdots$	y_s	$\cdots$	y_j	$\square$	

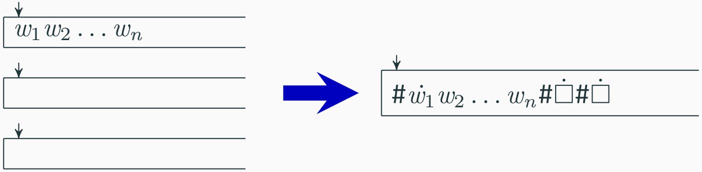
z_1	$\cdots$	z_t	$\cdots$	z_k	$\square$	

Single tape TM S

$\#x_1x_2\ldots\dot{x}_r\ldots x_i\#y_1y_2\ldots\dot{y}_s\ldots y_j\#z_1z_2\ldots\dot{z}_t\ldots z_k$

Simulating multitape Turing machine

Single-tape TM: Initialization



S: On input $w_1 \ldots w_n$:

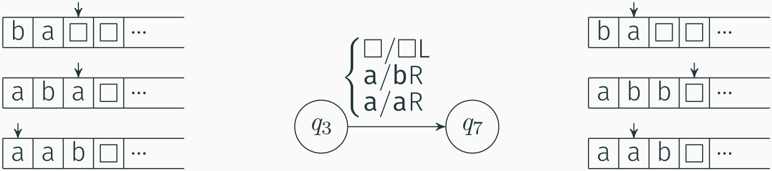
Replace tape contents by $\#\,\dot{w}_1\,w_2\ldots w_n\#\dot{\square}\#\dot{\square}$

Remember that M is in state q_0

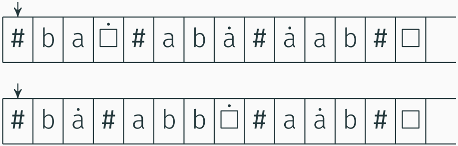
Simulating multitape Turing machine

Single-tape TM: Simulating multitape TM moves

Suppose Multitape TM M moves like this:



We simulate the move on single-tape TM S like this



Simulating multitape Turing machine

S given input $w_1 \dots w_n$:

Replace tape contents by $\#\dot{w}_1 w_2 \dots w_n \# \dot{\square} \# \dot{\square}$

Remember (in state) that M is in state q_0

S simulates a step of M:

Make a pass over tape to find $\dot{x}$, $\dot{y}$, $\dot{z}$

$$\downarrow$$
$$\boxed{\# x_1 x_2 \dots \dot{x} \dots x_i \# y_1 y_2 \dots \dot{y} \dots y_j \# z_1 z_2 \dots \dot{z} \dots z_k}$$

If M at state q_a has transition

$$q_a \xrightarrow{\begin{cases} x/x'\text{A} \\ y/y'\text{B} \\ z/z'\text{C} \end{cases}} q_b$$

update state/tape accordingly

If M reaches accept (reject) state, S accepts (rejects)

Simulation

To simulate a model M by another model N:

Say how the state and storage of N is used to represent the state and storage of M

Say what should be initially done to convert the input of N

Say how each transition of M can be implemented by a sequence of transitions of N