Undecidability and Reductions
CSCI 3130 Formal Languages and Automata Theory

Siu On CHAN

Chinese University of Hong Kong

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Undecidability

\[ A_{TM} = \{ \langle M, w \rangle \mid \text{Turing machine } M \text{ accepts input } w \} \]

Turing’s Theorem

The language \( A_{TM} \) is undecidable

Note that a Turing machine \( M \) may take as input its own description \( \langle M \rangle \)
Proof of Turing’s Theorem

Proof by contradiction:

Suppose \( A_{TM} \) is decidable, then some TM \( H \) decides \( A_{TM} \):

\[
\langle M, w \rangle \quad \rightarrow \quad H \quad \rightarrow \quad \text{accept if } M \text{ accepts } w \\
\quad \quad \quad \quad \rightarrow \quad \text{reject if } M \text{ rejects or loops on } w
\]
Proof of Turing’s Theorem

Proof by contradiction:

Suppose $A_{TM}$ is decidable, then some TM $H$ decides $A_{TM}$:

\[
\langle M, w \rangle \rightarrow H \rightarrow \begin{cases} 
\text{accept if } M \text{ accepts } w \\
\text{reject if } M \text{ rejects or loops on } w 
\end{cases}
\]

If $w = \langle M \rangle$,

\[
\langle M, \langle M \rangle \rangle \rightarrow H \rightarrow \begin{cases} 
\text{accept if } M \text{ accepts } \langle M \rangle \\
\text{reject if } M \text{ rejects or loops on } \langle M \rangle 
\end{cases}
\]
Proof of Turing’s theorem

\[ \langle M, \langle M \rangle \rangle \rightarrow H \rightarrow \text{accept if } M \text{ accepts } \langle M \rangle \]

\[ \rightarrow \text{reject if } M \text{ rejects or loops on } \langle M \rangle \]

Let \( H' \) be a TM that does the opposite of \( H \)
accept states in \( H \) becomes reject states in \( H' \), and vice versa

\[ \langle M, \langle M \rangle \rangle \rightarrow H' \rightarrow \text{accept if } M \text{ rejects or loops on } \langle M \rangle \]

\[ \rightarrow \text{reject if } M \text{ accepts } \langle M \rangle \]
Proof of Turing’s theorem

Let $D$ be the following TM:

$\langle M \rangle$ \rightarrow \text{copy} \rightarrow \langle M, \langle M \rangle \rangle \rightarrow H'$

$H'$ accepts if $M$ rejects or loops on $\langle M \rangle$

$H'$ rejects if $M$ accepts $\langle M \rangle$
Proof of Turing’s theorem

What happens when $M = D$?

- $\langle M \rangle$ → accept if $M$ rejects or loops on $\langle M \rangle$
- $\langle M \rangle$ → reject if $M$ accepts $\langle M \rangle$

- $\langle D \rangle$ → accept if $D$ rejects or loops on $\langle D \rangle$
- $\langle D \rangle$ → reject if $D$ accepts $\langle D \rangle$

Contradiction! $D$ cannot exist! $H$ cannot exist!
Proof of Turing’s theorem

What happens when \( M = D \)?

- If \( D \) rejects \( \langle D \rangle \), then \( D \) accepts \( \langle D \rangle \)
- If \( D \) accepts \( \langle D \rangle \), then \( D \) rejects \( \langle D \rangle \)

Contradiction! \( D \) cannot exist! \( H \) cannot exist!
Proof of Turing’s theorem: conclusion

Proof by contradiction

Assume $A_{TM}$ is decidable
Then there are TM $H$, $H'$ and $D$
But $D$ cannot exist!

Conclusion

The language $A_{TM}$ is undecidable
Diagonalization

Write an infinite table for the pairs \((M, w)\)

(Entries in this table are all made up for illustration)
**Diagonalization**

<table>
<thead>
<tr>
<th>all possible Turing machines</th>
<th>inputs $w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>$\langle M_1 \rangle$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$\langle M_2 \rangle$</td>
</tr>
<tr>
<td>$M_3$</td>
<td>$\langle M_3 \rangle$</td>
</tr>
<tr>
<td>$M_4$</td>
<td>$\langle M_4 \rangle$</td>
</tr>
</tbody>
</table>

Only look at those $w$ that describe Turing machines
Diagonalization

<table>
<thead>
<tr>
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<th>$\langle M_1 \rangle$</th>
<th>$\langle M_2 \rangle$</th>
<th>$\langle M_3 \rangle$</th>
<th>$\langle M_4 \rangle$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>acc</td>
<td>loop</td>
<td>rej</td>
<td>rej</td>
<td></td>
</tr>
<tr>
<td>$M_2$</td>
<td>rej</td>
<td>rej</td>
<td>acc</td>
<td>rej</td>
<td></td>
</tr>
<tr>
<td>$M_3$</td>
<td>loop</td>
<td>acc</td>
<td>acc</td>
<td>acc</td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td>rej</td>
<td>acc</td>
<td>rej</td>
<td>rej</td>
<td></td>
</tr>
</tbody>
</table>

If $A_{TM}$ is decidable, then TM $D$ is in the table
Diagonalization

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</tr>
<tr>
<td>$D$</td>
<td>rej</td>
</tr>
</tbody>
</table>
| ...                         | ...        | ...        | ...         | ...

$D$ does the opposite of the diagonal entries

$D$ on $\langle M_i \rangle = $ opposite of $M_i$ on $\langle M_i \rangle$

$\langle D \rangle$ → accept if $D$ rejects or loops on $\langle D \rangle$

$\langle D \rangle$ → reject if $D$ accepts $\langle D \rangle$
Diagonalization

<table>
<thead>
<tr>
<th>Turing machines</th>
<th>$\langle M_1 \rangle$</th>
<th>$\langle M_2 \rangle$</th>
<th>$\langle M_3 \rangle$</th>
<th>$\langle M_4 \rangle$</th>
<th>...</th>
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We run into trouble when we look at $(D, \langle D \rangle)$
Unrecognizable languages

The language $\overline{A}_{TM}$ is recognizable but not decidable

How about languages that are not recognizable?

$$\overline{A}_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that does not accept } w \}$$

$$= \{ \langle M, w \rangle \mid M \text{ rejects or loops on input } w \}$$

Claim

The language $\overline{A}_{TM}$ is not recognizable
Unrecognizable languages

Theorem

If $L$ and $\overline{L}$ are both recognizable, then $L$ is decidable

Proof of Claim from Theorem:

We know $A_{TM}$ is recognizable

if $A_{TM}$ were also, then $A_{TM}$ would be decidable

But Turing’s Theorem says $A_{TM}$ is not decidable
Unrecognizable languages

Theorem

If $L$ and $\overline{L}$ are both recognizable, then $L$ is decidable

Proof idea:

Let $M = \text{TM recognizing } L, M' = \text{TM recognizing } \overline{L}$

The following Turing machine $N$ decides $L$:

On input $w$,

1. Simulate $M$ on input $w$. If $M$ accepts, $N$ accepts.
2. Simulate $M'$ on input $w$. If $M'$ accepts, $N$ rejects.
Unrecognizable languages

Theorem

If $L$ and $\overline{L}$ are both recognizable, then $L$ is decidable

Proof idea:

Let $M = \text{TM recognizing } L$, $M' = \text{TM recognizing } \overline{L}$

The following Turing machine $N$ decides $L$:

On input $w$,

1. Simulate $M$ on input $w$. If $M$ accepts, $N$ accepts.
2. Simulate $M'$ on input $w$. If $M'$ accepts, $N$ rejects.

Problem: If $M$ loops on $w$, we will never go to step 2
Unrecognizable languages

Theorem

If $L$ and $\overline{L}$ are both recognizable, then $L$ is decidable

Proof idea (2nd attempt):

Let $M = \text{TM recognizing } L$, $M' = \text{TM recognizing } \overline{L}$

The following Turing machine $N$ decides $L$:

On input $w$,

For $t = 0, 1, 2, 3, \ldots$

Simulate first $t$ transitions of $M$ on input $w$.

If $M$ accepts, $N$ accepts.

Simulate first $t$ transitions of $M'$ on input $w$.

If $M'$ accepts, $N$ rejects.
Reductions
Another undecidable language

\[
\text{HALT}_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM that halts on input } w \}\]

We’ll show:

\[
\text{HALT}_{\text{TM}} \text{ is an undecidable language}
\]

We will argue that

If \(\text{HALT}_{\text{TM}}\) is decidable, then so is \(A_{\text{TM}}\)

…but by Turing’s theorem, \(A_{\text{TM}}\) is not
Undecidability of halting

If $\text{HALT}_{TM}$ can be decided, so can $A_{TM}$

Suppose $H$ decides $\text{HALT}_{TM}$

$\langle M, w \rangle \rightarrow H$

accept if $M$ halts on $w$
reject if $M$ loops on $w$

We want to construct a TM $S$ that decides $A_{TM}$

$\langle M, w \rangle \rightarrow ?$
accept if $M$ accepts $w$
reject if $M$ rejects or loops on $w$
Undecidability of halting

\[
\text{HALT}_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM that halts on input } w \} \\
\text{A}_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts input } w \}
\]

Suppose \( \text{HALT}_{\text{TM}} \) is decidable
Let \( H \) be a TM that decides \( \text{HALT}_{\text{TM}} \)
The following TM \( S \) decides \( \text{A}_{\text{TM}} \)
On input \( \langle M, w \rangle \):

Run \( H \) on input \( \langle M, w \rangle \)
If \( H \) rejects, reject
If \( H \) accepts, run \( U \) on input \( \langle M, w \rangle \)
  If \( U \) accepts, accept; else reject
Steps for showing that a language $L$ is undecidable:

1. If some TM $R$ decides $L$
2. Using $R$, build another TM $S$ that decides $A_{\text{TM}}$

But $A_{\text{TM}}$ is undecidable, so $R$ cannot exist
Example 1

\[ A'_{TM} = \{ \langle M \rangle \mid M \text{ is a TM that accepts input } \varepsilon \} \]

Is \( A'_{TM} \) decidable? Why?
Example 1

\[ A'_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM that accepts input } \varepsilon \} \]

Is \( A'_{\text{TM}} \) decidable? Why?

Undecidable!

Intuitive reason:
To know whether \( M \) accepts \( \varepsilon \) seems to require simulating \( M \)
But then we need to know whether \( M \) halts

Let’s justify this intuition
Example 1: Figuring out the reduction

Suppose $A'_\text{TM}$ can be decided by a TM $R$

$\langle M' \rangle \xrightarrow{R} \begin{cases} \text{accept if } M' \text{ accepts } \varepsilon \\ \text{reject otherwise} \end{cases}$

We want to build a TM $S'$

$\langle M, w \rangle \xrightarrow{?} \langle M' \rangle \xrightarrow{R} \begin{cases} \text{accept if } M \text{ accepts } w \\ \text{reject otherwise} \end{cases}$

$M'$ should be a Turing machine such that $M'$ on input $\varepsilon = M$ on input $w$
Example 1: Implementing the reduction

\[ \langle M, w \rangle \rightarrow ? \rightarrow \langle M' \rangle \]

\(M'\) should be a Turing machine such that
\(M'\) on input \(\varepsilon = M\) on input \(w\)

Description of the machine \(M'\):
On input \(z\)

1. Simulate \(M\) on input \(w\)
2. If \(M\) accepts \(w\), accept
3. If \(M\) rejects \(w\), reject
Description of $S'$:
On input $\langle M, w \rangle$ where $M$ is a TM

1. Construct the following TM $M'$:

   $M' = a$ TM such that on input $z$,
   Simulate $M$ on input $w$ and accept/reject according to $M$

2. Run $R$ on input $\langle M' \rangle$ and accept/reject according to $R$
Example 1: The formal proof

\[ A_{TM}' = \{ \langle M \rangle \mid M \text{ is a TM that accepts input } \varepsilon \} \]
\[ A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts input } w \} \]

Suppose \( A_{TM}' \) is decidable by a TM \( R \).
Consider the TM \( S \): On input \( \langle M, w \rangle \) where \( M \) is a TM

1. Construct the following TM \( M' \):

\[ M' = \text{a TM such that on input } z, \]
\[ \text{Simulate } M \text{ on input } w \text{ and accept/reject according to } M \]

2. Run \( R \) on input \( \langle M' \rangle \) and accept/reject according to \( R \)

Then \( S \) accepts \( \langle M, w \rangle \) if and only if \( M \) accepts \( w \)
So \( S \) decides \( A_{TM} \), which is impossible
Example 2

\[ A''_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM that accepts some input strings} \} \]

Is \( A''_{\text{TM}} \) decidable? Why?
Example 2

\[ A''_{TM} = \{ \langle M \rangle \mid M \text{ is a TM that accepts some input strings} \} \]

Is \( A''_{TM} \) decidable? Why?

Undecidable!

Intuitive reason:
To know whether \( M \) accepts some strings seems to require simulating \( M \)
But then we need to know whether \( M \) halts

Let’s justify this intuition
Example 2: Figuring out the reduction

Suppose $A''_{TM}$ can be decided by a TM $R$

$\langle M' \rangle \xrightarrow{R} \text{accept if } M' \text{ accepts some strings}$

We want to build a TM $S'$

$\langle M, w \rangle \xrightarrow{?} \langle M' \rangle \xrightarrow{R} \text{accept if } M \text{ accepts } w$

$M'$ should be a Turing machine such that $M'$ accepts some strings if and only if $M$ accepts input $w$
Implementing the reduction

**Task:** Given $\langle M, w \rangle$, construct $M'$ so that
If $M$ accepts $w$, then $M'$ accepts some input
If $M$ does not accept $w$, then $M'$ accepts no inputs

$$M' = \text{a TM such that on input } z,$$

1. Simulate $M$ on input $w$
2. If $M$ accepts, accept
3. Otherwise, reject
Example 2: The formal proof

\[
A''_{TM} = \{ \langle M \rangle \mid M \text{ is a TM that accepts some input} \}
\]
\[
A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts input } w \}
\]

Suppose \( A''_{TM} \) is decidable by a TM \( R \).
Consider the TM \( S \): On input \( \langle M, w \rangle \) where \( M \) is a TM

1. Construct the following TM \( M' \):

\[
M' = \text{a TM such that on input } z,
\]
\[
\text{Simulate } M \text{ on input } w \text{ and accept/reject according to } M
\]

2. Run \( R \) on input \( \langle M' \rangle \) and accept/reject according to \( R \)

Then \( S \) accepts \( \langle M, w \rangle \) if and only if \( M \) accepts \( w \)
So \( S \) decides \( A_{TM} \), which is impossible
Example 3

\[ E_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM that accepts no input} \} \]

Is \( E_{\text{TM}} \) decidable?
Example 3

\[ E_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM that accepts no input} \} \]

Is \( E_{\text{TM}} \) decidable?

Undecidable! We will show:

If \( E_{\text{TM}} \) can be decided by some TM \( R \),

Then \( A''_{\text{TM}} \) can be decided by another TM \( S \)

\[ A''_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM that accepts some input strings} \} \]
Example 3

\[ E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM that accepts no input} \} \]
\[ A''_{TM} = \{ \langle M \rangle \mid M \text{ is a TM that accepts some input} \} \]

Note that \( E_{TM} \) and \( A''_{TM} \) are complement of each other (except ill-formatted strings, which we will ignore)

Suppose \( E_{TM} \) can be decided by some TM \( R \)

Consider the following TM \( S \):

On input \( \langle M \rangle \) where \( M \) is a TM

1. Run \( R \) on input \( \langle M \rangle \)
2. If \( R \) accepts, reject
3. If \( R \) rejects, accept

Then \( S \) decides \( A''_{TM} \), a contradiction
Example 4

\[ EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs such that } L(M_1) = L(M_2) \} \]

Is \( EQ_{TM} \) decidable?
Example 4

\[ \mathsf{EQ}_{\mathsf{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs such that } L(M_1) = L(M_2) \} \]

Is \( \mathsf{EQ}_{\mathsf{TM}} \) decidable?

Undecidable!

We will show that \( \mathsf{EQ}_{\mathsf{TM}} \) can be decided by some TM \( R \)
then \( \mathsf{ET}_{\mathsf{TM}} \) can be decided by another TM \( S \)
Example 4: Setting up the reduction

\[
\begin{align*}
\mathrm{EQ}_{\text{TM}} &= \{ \langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs such that } L(M_1) = L(M_2) \} \\
\mathrm{ETM} &= \{ \langle M \rangle \mid M \text{ is a TM that accepts no input} \}
\end{align*}
\]

Given \( \langle M \rangle \), we need to construct \( \langle M_1, M_2 \rangle \) so that

- If \( M \) accepts no input, then \( M_1 \) and \( M_2 \) accept same set of inputs
- If \( M \) accepts some input, then \( M_1 \) and \( M_2 \) do not accept same set of inputs

Idea: Make \( M_1 = M \)

Make \( M_2 \) accept nothing
Example 4: The formal proof

$$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs such that } L(M_1) = L(M_2) \}$$

$$E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM that accepts no input} \}$$

Suppose $EQ_{TM}$ is decidable and $R$ decides it
Consider the following TM $S$:
On input $\langle M \rangle$ where $M$ is a TM

1. Construct a TM $M_2$ that rejects every input $z$
2. Run $R$ on input $\langle M, M_2 \rangle$ and accept/reject according to $R$

Then $S$ accepts $\langle M \rangle$ if and only if $M$ accepts no input
So $S$ decides $E_{TM}$ which is impossible