Looping

Turing machine may not halt

\[
\Sigma = \{0, 1\}
\]

input: \(\varepsilon\)

Inputs can be divided into three types:

- Accept
- Reject
- Infinite loop
We say $M$ halts on input $x$ if there is a sequence of configurations $C_0, C_1, \ldots, C_k$

$C_0$ is starting $C_i$ yields $C_{i+1}$ $C_k$ is accepting or rejecting

A TM $M$ is a decider if it halts on every input

Language $L$ is decidable if it is recognized by a TM that halts on every input
Programming Turing machines: Are two strings equal?

\[ L_1 = \{ w#w \mid w \in \{a, b\}^* \} \]

Description of Turing Machine

1. **Until** you reach 
2. **Read** and remember entry
3. **Write** \(x\)
4. **Move** right past # and past all \(x\)'s
5. **If** this entry is different, **reject**
6. **Write** \(x\)
7. **Move** left past # and to right of first \(x\)
8. **If** you see only \(x\)'s followed by □, **accept**
Programming Turing machines: Are two strings equal?

\[ L_1 = \{ w\#w \mid w \in \{a, b\}^* \} \]
Programming Turing machines: Are two strings equal?

input: aab#aab

configurations:
$q_0$ aab#aab
x $q_{a1}$ ab#aab
xa $q_{a1}$ b#aab
xab $q_{a1}$ #aab
xab# $q_{a2}$ aab
xab $q_2$ #xab
xa $q_3$ b#xab
x $q_3$ ab#xab
$q_3$ xab#ab
x $q_0$ ab#xab

...
Programming Turing machines

\[ L_2 = \{a^i b^j c^k \mid ij = k \text{ and } i, j, k > 0\} \]

High level description of TM:

1. For every a:
2. Cross off the same number of b’s and c’s
3. Uncross the crossed b’s (but not the c’s)
4. Cross off this a
5. If all a’s and c’s are crossed off, accept

\[ \Sigma = \{a, b\} \quad \Gamma = \{a, b, c, \text{a}, \text{b}, \varepsilon, \square\} \]

Example:

1. aabbccccc
2. aabbεεεε
3. aabbεεεε
4. aabbεεεε
5. aabbεεεε
6. aabbεεεε
Programming Turing machines

\[ L_2 = \{ a^i b^j c^k | ij = k \text{ and } i, j, k > 0 \} \]

Low-level description of TM:

Scan input from left to right to check it looks like \( a a^* b b^* c c^* \)
Move the head to the first symbol of the tape
For every \( a \):
  - Cross off the \textbf{same number} of \( b \)’s and \( c \)’s
  - Restore the crossed off \( b \)’s (but not the \( c \)’s)
  - Cross off this \( a \)
If all \( a \)’s and \( c \)’s are crossed off, accept
\[ L_2 = \{ a^i b^j c^k \mid ij = k \text{ and } i, j, k > 0 \} \]

Low-level description of TM:

Scan input from left to right to check it looks like \( a a^* b b^* c c^* \)

Move the head to the first symbol of the tape    How?

For every \( a \):    How?
    Cross off the same number of \( b \)'s and \( c \)'s    How?
    Restore the crossed off \( b \)'s (but not the \( c \)'s)
    Cross off this \( a \)

If all \( a \)'s and \( c \)'s are crossed off, accept
Programming Turing machines

Implementation details:

Move the head to the first symbol of the tape:
Put a special marker on top of the first $a$ $\ åabbcccc$.

Cross off the same number of $b$’s and $c$’s: $\ åabbc$ccc.
Replace $b$ by $b$ $\ åabb$ccc.
Move right until you see a $c$ $\ åabbc$ccc.
Replace $c$ by $c$ $\ åabb$$c$ccc.
Move left just past the last $b$ $\ åabbc$ccc.
If any uncrossed $b$’s are left, repeat $\ åabbc$ccc.

$\Sigma = \{a, b, c\}$ $\Gamma = \{a, b, c, a, b, c, \epsilon, \ å, \ å, \ □\}$
Programming Turing machines: Element distinctness

$L_3 = \{#x_1#x_2 \ldots #x_m \mid x_i \in \{0, 1\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j\}$

Example:  
#01#0011#1 \in L_3

High-level description of TM:

On input $w$
For every pair of blocks $x_i$ and $x_j$ in $w$
  Compare the blocks $x_i$ and $x_j$
  If they are the same, reject
Accept
Programming Turing machines: Element distinctness

\[ L_3 = \{ \#x_1\#x_2\ldots\#x_m \mid x_i \in \{0, 1\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j \} \]

Low-level description:

0. If input is \( \varepsilon \), or has exactly one \#, accept

1. Mark the leftmost \# as \( \hat{\#} \) and move right \( \hat{\#}01\hat{\#}0011\hat{\#}1 \)

2. Mark the next unmarked \# \( \hat{\#}01\hat{\#0011\#}1 \)
Programming Turing machines: Element distinctness

\[ L_3 = \{ \#x_1\#x_2 \ldots \#x_m \mid x_i \in \{0, 1\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j \} \]

3. Compare the two strings to the right of \( \dot{\#} \)
   If they are equal, reject

\[ \dot{\#} \underline{01} \#0011 \#1 \]

4. Move the right \( \dot{\#} \)
   If not possible, move the left \( \dot{\#} \) to the next \( \# \)
   and put the right \( \dot{\#} \) on the next \( \# \)
   If not possible, accept

\[ \dot{\#} 01 \#0011 \#1 \]

5. Repeat Step 3
   \[ \dot{\#} 01 \#0011 \#1 \]
   \[ \#01 \#0011 \#1 \]
   \[ \#01 \#0011 \#1 \]
How to describe Turing Machines

Unlike for DFAs, NFAs, PDAs, we rarely give complete state diagrams of Turing Machines

We usually give a high-level description unless you’re asked for a low-level description or even state diagram

We are interested in algorithms behind the Turing machines
Programming Turing machines: Graph connectivity

$L_4 = \{ \langle G \rangle \mid G \text{ is a connected undirected graph} \}$

How do we feed a graph into a Turing Machine?
How to encode a graph $G$ as a string $\langle G \rangle$?

$(1,2,3,4)((1,4),(2,3),(3,4),(4,2))$

Conventions for describing graphs:
(nodes)(edges)
no node appears twice
edges are pairs (first node, second node)
Programming Turing machines: Graph connectivity

$L_3 = \{ \langle G \rangle \mid G \text{ is a connected undirected graph} \}$

High-level description:

On input $\langle G \rangle$

0. Verify that $\langle G \rangle$ is the description of a graph
   No node/edge repeats; Edge endpoints are nodes

1. Mark the first node of $G$

2. Repeat until no new nodes are marked:
   2.1 For each node, mark it if it is adjacent to an already marked node

3. If all nodes are marked, accept; otherwise reject
Some low-level details:

0. Verify that $\langle G \rangle$ is the description of a graph
No node/edge repeats: Similar to Element distinctness
Edge endpoints are nodes: Also similar to Element distinctness

1. Mark the first node of $G$
Mark the leftmost digit with a dot, e.g. 12 becomes ˙12

2. Repeat until no new nodes are marked:
2.1 For each node, mark it if it is attached to an already marked node
For every dotted node $u$ and every undotted node $v$:
   Underline both $u$ and $v$ from the node list
   Try to match them with an edge from the edge list
   If not found, remove underline from $u$ and/or $v$ and try another pair
Variants of Turing Machines
Multitape Turing machine

Transitions may depend on the contents of all cells under the heads

Different tape heads can move independent
Multitape Turing machine

Multiple tapes are convenient
One tape can serve as temporary storage
How to argue equivalence

Multitape Turing machines are **equivalent** to single-tape Turing machines.

![Diagram showing the equivalence between multitape and single-tape Turing machines.](image)
Simulating multitape Turing machine

\[ \Gamma = \{a, b, \square\} \]

\[ \Gamma = \{a, b, \square, \dot{a}, \dot{b}, \dot{\square}, \#\} \]
Simulating multitape Turing machine

We show how to simulate a multitape Turing machine on a single tape Turing machine.

To be specific, let’s simulate a 3-tape TM

Multitape TM $M$

Single tape TM $S'$
Simulating multitape Turing machine

**Single-tape TM: Initialization**

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
w_1 \ w_2 \ldots \ w_n \\
\vdots \\
\vdots
\end{array}
\rightarrow
\begin{array}{c}
\# \vdots \\
\vdots \\
\# \vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\# \ w_1 \ w_2 \ldots \ w_n \# \ \hat{\ } \ \hat{\ } \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[S: \text{On input } w_1 \ldots w_n: \]

Replace tape contents by \(#w_1 \ w_2 \ldots \ w_n \# \ \hat{\ } \ \hat{\ }\)

Remember that \(M\) is in state \(q_0\)
Simulating multitape Turing machine

Single-tape TM: Simulating multitape TM moves

Suppose Multitape TM $M$ moves like this:

We simulate the move on single-tape TM $S$ like this
Simulating multitape Turing machine

$S$ given input $w_1 \ldots w_n$:
Replace tape contents by $\# \dot{w}_1 \dot{w}_2 \ldots \dot{w}_n \# □ \# □$
Remember (in state) that $M$ is in state $q_0$

$S$ simulates a step of $M$:
Make a pass over tape to find $\dot{x}, \dot{y}, \dot{z}$

\[
\begin{array}{c}
\# x_1 x_2 \ldots \dot{x} \ldots x_i \# y_1 y_2 \ldots \dot{y} \ldots y_j \# z_1 \dot{z}_2 \ldots \dot{z} \ldots z_k
\end{array}
\]

If $M$ at state $q_a$ has transition
\[
\begin{cases}
  x/x' A \\
  y/y' B \\
  z/z' C
\end{cases}
\]
update state/tape accordingly

If $M$ reaches accept (reject) state, $S$ accepts (rejects)
To simulate a model $M$ by another model $N$:

Say how the state and storage of $N$ is used to represent the state and storage of $M$

Say what should be initially done to convert the input of $N$

Say how each transition of $M$ can be implemented by a sequence of transitions of $N$