CEG5010: The Symmetric Table Addition Method

Introduction

- Lookup tables can be used to implement any function
- The Symmetric Table Addition Method (STAM) can be used to approximate any differentiable function
- In STAM, symmetry and leading zeros are introduced to reduce amount of memory needed (Stine and Schulte 1998)

Taylor Series

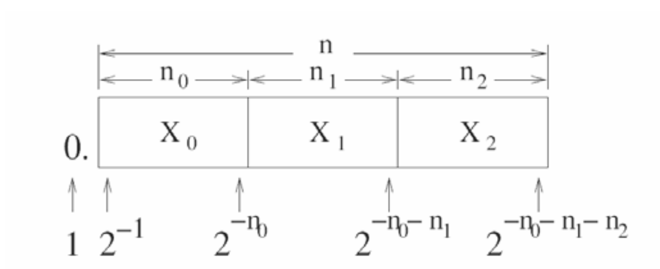
$$
\begin{aligned}
f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots \\
&\quad + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n \\
&= \sum_{i=0}^{n} \frac{f^{(i)}(a)(x-a)^i}{i!} + R_n
\end{aligned}
$$

where

$$
\begin{aligned}
R_n &= \int_a^x f^{(n+1)}(u)\frac{(x-u)^n}{n!}\mathrm{d}u \\
&= \frac{f^{(n+a)}(\xi)(x-a)^{n+1}}{(n+1)!} \quad \text{for} \quad a < \xi < x
\end{aligned}
$$

Bipartite Table Method

- Operand x divided into 3 parts $x=x_0+x_1+x_2$



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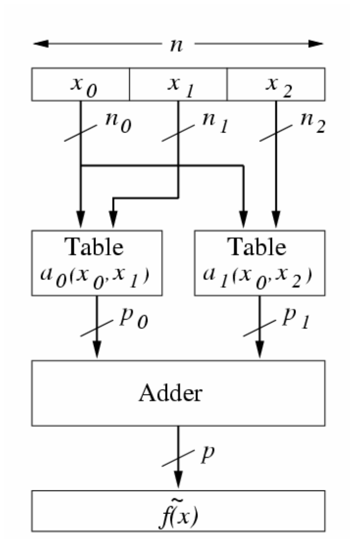
Bipartite Method (BM)



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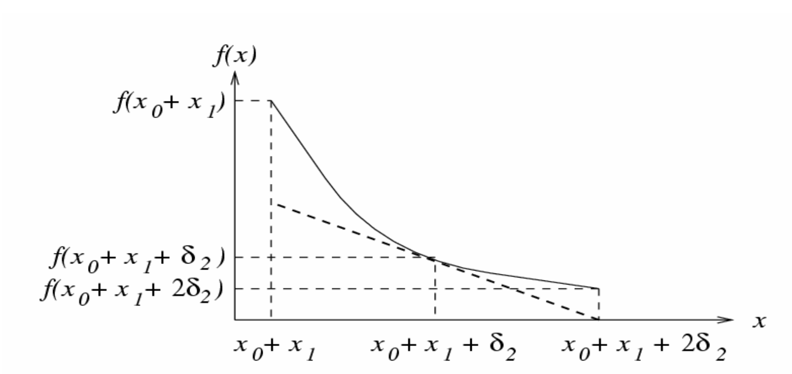
BM coefficients

- Computed by taking first 2 terms of Taylor series around $x_0+x_1+\delta_2$ (δ_2 is half the range of x_2)



BM coefficients

$$
\begin{aligned}
f(x) &= f(x_0 + x_1 + x_2) \\
&\approx f(x_0 + x_1 + \delta_2) + f'(x_0 + x_1 + \delta_2)(x_2 - \delta_2) \\
&\approx f(x_0 + x_1 + \delta_2) + f'(x_0 + \delta_1 + \delta_2)(x_2 - \delta_2) \\
&= a_0(x_0, x_1) + a_1(x_0, x_2)
\end{aligned}
$$

$$
\begin{aligned}
\delta_1 &= (2^{-n_0} - 2^{-n_0-n_1})/2 \\
&= 2^{-n_0-1} - 2^{-n_0-n_1-1} \\
\delta_2 &= (2^{-n_0-n_1} - 2^{-n_0-n_1-n_2})/2 \\
&= 2^{-n_0-n_1-1} - 2^{-n_0-n_1-n_2-1}
\end{aligned}
$$

BM coefficients

- Note that $a_1(x_0,x_2)$ cannot depend on x_1 and is replaced by δ_1, a value which is half the range of x_1

$$
\begin{aligned}
f(x) &= f(x_0 + x_1 + x_2) \\
&\approx f(x_0 + x_1 + \delta_2) + f'(x_0 + x_1 + \delta_2)(x_2 - \delta_2) \\
&\approx f(x_0 + x_1 + \delta_2) + f'(x_0 + \delta_1 + \delta_2)(x_2 - \delta_2) \\
&= a_0(x_0, x_1) + a_1(x_0, x_2)
\end{aligned}
$$

Symmetry in $a_1(x_0,x_2)$

- There is a bound on a_1

$$\begin{aligned} |a_1(x_0, x_2)| &= |f'(x_0 + \delta_1 + \delta_2)(x_2 - \delta_2)| \\ &< |f'(\xi_1)|2^{-n_0-n_1-1} \end{aligned}$$

- Leading to the number of leading zeros to be approximately

$$n_0 + n_1 + 1 + \log_2(|f(\xi_0)/f'(\xi_1)|)$$

(the leading zeros do not need to be stored)

Symmetry in $a_1(x_0,x_2)$

- By symmetry, $a_1(x_1,x_2)$ can be reduced in size from $2^{n_0+n_2} \times p_1$ bits to $2^{n_0+n_2-1} \times (p_1 -1)$ bits

		$a_1(x_0, x_2)$	
x	$x_2 2^7$	decimal	binary
0.500000	000	+0.0166016	0.0000010001
0.507812	001	+0.0107422	0.0000001011
0.515625	010	+0.0068359	0.0000000111
0.523438	011	+0.0029297	0.0000000011
0.531250	100	-0.0029297	1.1111111101
0.539062	101	-0.0068359	1.1111111001
0.546875	110	-0.0107422	1.1111110101
0.554688	111	-0.0166016	1.1111101111

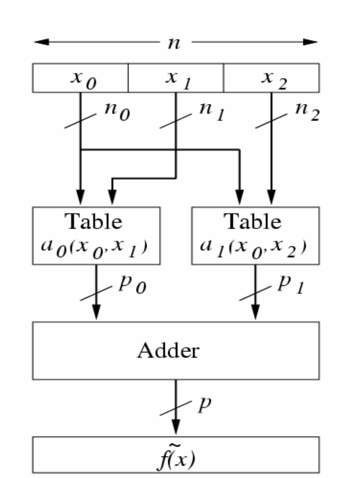
Table 1. Table Entries of $a_1(x_0, x_2)$.

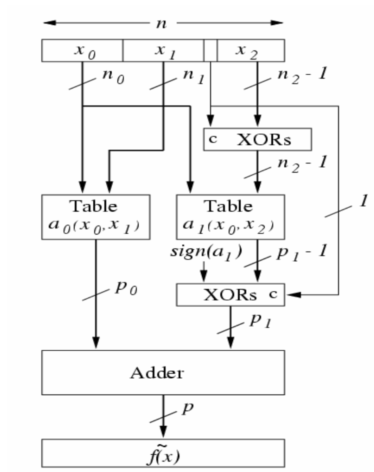
f(x)=cos(x)
x_0=0.5
x_1=0
n_0=2
n_1=2
n_2=3
p=7
g=2

Symmetry

- Table is reduced by using
  - $2\delta_2$-x_2 is the one's complement of x_2
  - $a_1(x_0, 2\delta_2-x_2)$ is the one's complement of $a_1(x_0, x_2)$

Symmetric Bipartite Table Method (SBTM)





4 Error sources

- ε_0: approximating f(x) with only the first 2 terms of the Taylor series

$$\begin{aligned} \epsilon_0 &= \left|\sum_{i=2}^{\infty} \frac{f^{(i)}(x_0 + x_1 + \delta_2)(x_2 - \delta_2)^i}{i!}\right| \\ &\approx \frac{|f''(x_0 + x_1 + \delta_2)|(x_2 - \delta_2)^2}{2!} \\ &< |f''(\xi_2)|2^{-2n_0 - 2n_1 - 2} \end{aligned}$$

ξ_i is the point on [0,1) at which $|f^i(x)|$ is max

4 Error sources

- ε_1: replacing the 2nd term of the Taylor series with $a_1(x_0,x_2)$

$$\begin{aligned} \epsilon_1 &\approx |f''(x_0 + \delta_1 + \delta_2)(x_1 - \delta_1)(x_2 - \delta_2)| \\ &< |f''(\xi_2)|2^{-2n_0 - n_1 - 2} \end{aligned}$$

4 Error sources

- ε_2: rounding $a_0(x_0,x_1)$ and $a_1(x_0,x_2)$ to p_f+g fractional bits

$$\begin{aligned}\epsilon_2 &\leq 2 \times 2^{-p_f-g-1} \\ &= 2^{-p_f-g}\end{aligned}$$

4 Error sources

- ε_3: rounding to the final result

$$\epsilon_3 \leq 2^{-p_f-1} - 2^{-p_f-g-1}$$

Error analysis

- Assume p_f fraction bits and p_f+g coefficient bits, need to ensure that the sum of these errors is smaller than 1 unit in the last place (ulp)

$$\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq 2^{-p_f}$$

- This leads to the constraint

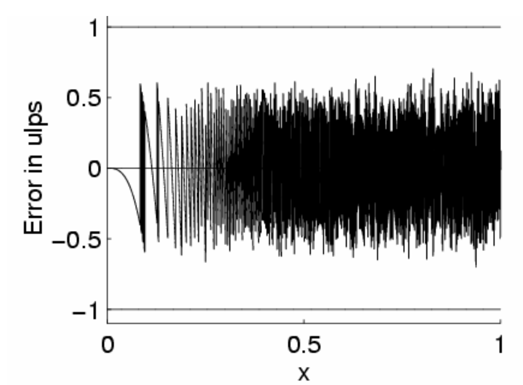
$$|f''(\xi_2)|2^{-2n_0-n_1-2}(1+2^{-n_1}) + 2^{-p_f-g-1} \leq 2^{-p_f-1}$$

- Satisfied by

$$\begin{aligned} g &= 2 \\ 2n_0 + n_1 &\leq p_f + log_2(f''(\xi_2)) \end{aligned}$$

Example

- Approximate sin(x) to p_f=12 bits on [0,1)
- n_0=4, n_1=4, n_2=4
- $2\times4 + 4 \geq 12 + \lceil \log_2(1/\sqrt{2}) \rceil$



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Examples

Table 2. Ranges and Derivative Values for Selected Functions.

$f(x)$	Input	Output	$\mid f'(\xi_1) \mid$	$\mid f''(\xi_2) \mid$
$1/x$	[1, 2)	(0.5, 1]	1	2
$\sqrt{x}$	[1, 2)	$[1, \sqrt{2})$	1/2	1/4
$\sin(x)$	[0, 1)	$[0, 1/\sqrt{2})$	1	$1/\sqrt{2}$
2^x	[0, 1)	[1, 2)	$2 \ln 2$	$2(\ln(2))^2$
$\log_2(x)$	[1, 2)	[0, 1)	$\log_2(e)$	$\log_2(e)$
$\ln(x)$	[1, 2)	$[0, \ln(2))$	1	1
$1/\sqrt{x}$	[1, 2)	$(1/\sqrt{2},1]$	1/2	3/4

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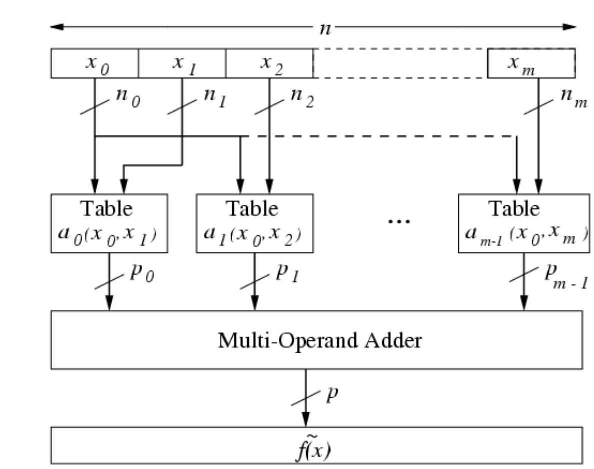
Symmetric Table Addition Method

- Generalization of STAM with multiple tables



STAM approximation

$$
\begin{aligned}
\widetilde{f(x)} &= f(x_0 + x_1 + \sum_2^m \delta_i) + \\
&\quad f'(x_0 + \delta_1 + \sum_2^m \delta_i)(\sum_2^m x_i - \sum_2^m \delta_i) \\
&= a_0(x_0, x_1) + \sum_2^m a_{i-1}(x_0, x_i)
\end{aligned}
$$

$$
a_{i-1}(x_0, x_i) = f'(x_0 + \delta_1 + \sum_2^m \delta_k)(x_i - \delta_i) \quad 2 \le i \le m
$$

STAM error constraints

$$
\begin{aligned}
2n_0 + n_1 &\le p_f + log_2(|f''(\xi_2)|) \\
g &\le 2 + log_2(m-1)
\end{aligned}
$$

Partitions and table sizes (16 bit)

Table 3. Bit Partitions and Table Sizes for 16-Bit Operands.

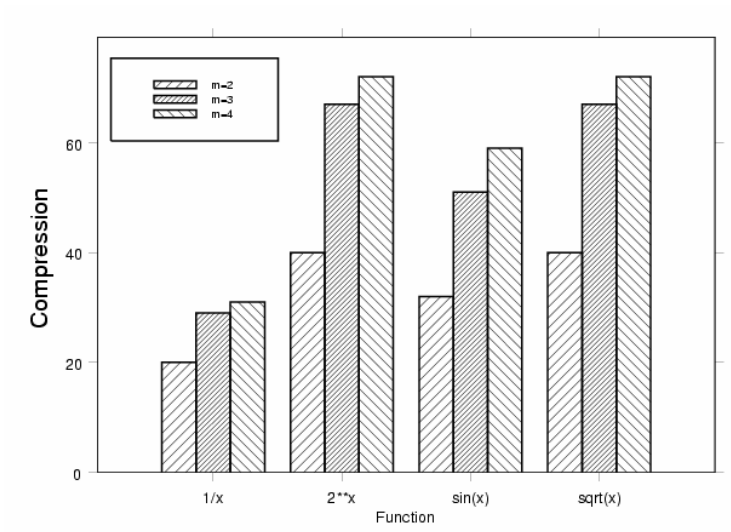
$f(x)$	Tables	Bit Partitions	Table Sizes	Total Memory
$1/x$	1	15	$2^{15} \cdot 15$	491,520
$1/x$	2	$6, 4, 5$	$2^{10}(17+7)$	24,576
$1/x$	3	$7, 2, 3, 3$	$2^{9}(18+9+6)$	16,896
$1/x$	4	$7, 2, 2, 2, 2$	$2^{9} \cdot 19 + 2^{8}(10+8+6)$	15,872
$\sqrt{x}$	1	15	$2^{15} \cdot 15$	491,520
$\sqrt{x}$	2	$4, 5, 6$	$2^{9}(17+7)$	12,288
$\sqrt{x}$	3	$5, 3, 3, 4$	$2^{8} \cdot 18 + 2^{7} \cdot 9 + 2^{8} \cdot 6$	7,296
$\sqrt{x}$	4	$5, 3, 2, 2, 3$	$2^{8} \cdot 19 + 2^{6}(10+8) + 2^{7} \cdot 6$	6,784
$\sin(x)$	1	16	$2^{16} \cdot 16$	1,048,576
$\sin(x)$	2	$6, 4, 6$	$2^{10} \cdot 18 + 2^{11} \cdot 7$	32,768
$\sin(x)$	3	$7, 2, 3, 4$	$2^{9}(19+9) + 2^{10} \cdot 6$	20,480
$\sin(x)$	4	$7, 2, 2, 2, 3$	$2^{9} \cdot 20 + 2^{8}(10+8) + 2^{9} \cdot 6$	17,920
2^x	1	16	$2^{16} \cdot 15$	983,040
2^x	2	$5, 5, 6$	$2^{10}(17+7)$	24,576
2^x	3	$6, 3, 3, 4$	$2^{9} \cdot 18 + 2^{8} \cdot 9 + 2^{9} \cdot 6$	14,592
2^x	4	$6, 3, 2, 2, 3$	$2^{9} \cdot 19 + 2^{7}(10+8) + 2^{8} \cdot 6$	13,568

Partitions and table sizes (24 bit)

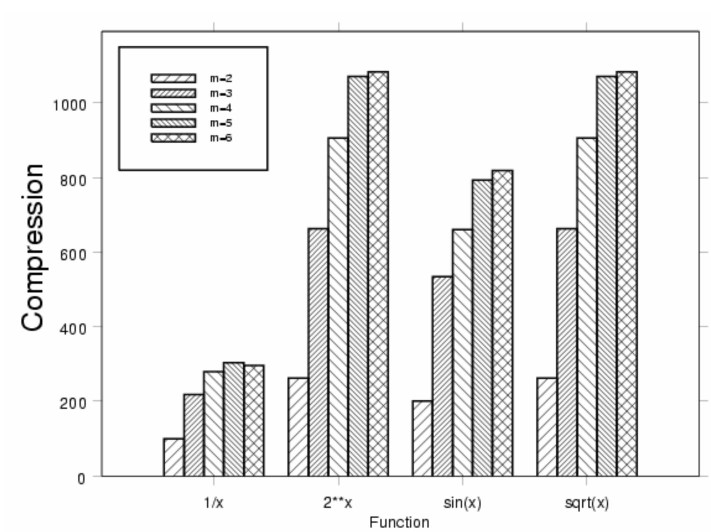
Table 4. Bit Partitions and Table Sizes for 24-Bit Operands.

$f(x)$	Tables	Bit Partitions	Table Sizes	Total Memory
$1/x$	1	23	$2^{23} \cdot 23$	192,937,984
$1/x$	2	$9, 7, 7$	$2^{16} \cdot 25 + 2^{15} \cdot 9$	1,933,312
$1/x$	3	$11, 3, 4, 5$	$2^{14}(26+12) + 2^{15} \cdot 8$	884,736
$1/x$	4	$11, 3, 3, 3, 3$	$2^{14} \cdot 27 + 2^{13}(13+10+7)$	688,128
$1/x$	5	$11, 3, 2, 2, 2, 3$	$2^{14} \cdot 27 + 2^{12}(13+11+9) + 2^{13} \cdot 7$	634,880
$1/x$	6	$11, 3, 1, 2, 2, 2, 2$	$2^{14} \cdot 28 + 2^{11} \cdot 14 + 2^{12}(13+11+9+7)$	651,264
$\sqrt{x}$	1	23	$2^{23} \cdot 23$	192,937,984
$\sqrt{x}$	2	$7, 7, 9$	$2^{14} \cdot 25 + 2^{15} \cdot 10$	737,280
$\sqrt{x}$	3	$8, 5, 5, 5$	$2^{13} \cdot 26 + 2^{12}(12+7)$	290,816
$\sqrt{x}$	4	$9, 3, 3, 4, 4$	$2^{12} \cdot 27 + 2^{11} \cdot 14 + 2^{12}(11+7)$	212,992
$\sqrt{x}$	5	$9, 3, 2, 3, 3, 3$	$2^{12} \cdot 27 + 2^{10} \cdot 14 + 2^{11}(12+9+6)$	180,224
$\sqrt{x}$	6	$9, 3, 2, 2, 2, 2, 3$	$2^{12} \cdot 28 + 2^{10}(15+13+11+9) + 2^{11} \cdot 7$	178,176
$\sin(x)$	1	24	$2^{24} \cdot 24$	402,653,184
$\sin(x)$	2	$8, 8, 8$	$2^{16} \cdot 26 + 2^{15} \cdot 9$	1,998,848
$\sin(x)$	3	$10, 4, 5, 5$	$2^{14} \cdot (27+12+7)$	753,664
$\sin(x)$	4	$10, 4, 3, 3, 4$	$2^{14} \cdot 28 + 2^{12}(13+10) + 2^{13} \cdot 7$	610,304
$\sin(x)$	5	$11, 2, 2, 3, 3, 3$	$2^{13} \cdot 28 + 2^{12} \cdot 14 + 2^{13}(12+9+6)$	507,904
$\sin(x)$	6	$11, 2, 2, 2, 2, 2, 3$	$2^{13} \cdot 29 + 2^{12}(15+13+11+9) + 2^{13} \cdot 7$	491,520
2^x	1	24	$2^{24} \cdot 23$	385,875,968
2^x	2	$8, 7, 9$	$2^{15} \cdot 25 + 2^{16} \cdot 10$	1,474,560
2^x	3	$9, 5, 5, 5$	$2^{14} \cdot 26 + 2^{13}(12+7)$	581,632
2^x	4	$10, 3, 3, 4, 4$	$2^{13} \cdot 27 + 2^{12} \cdot 14 + 2^{13}(11+7)$	425,984
2^x	5	$10, 3, 2, 3, 3, 3$	$2^{13} \cdot 27 + 2^{11} \cdot 14 + 2^{12}(12+9+6)$	360,448
2^x	6	$10, 3, 2, 2, 2, 2, 3$	$2^{13} \cdot 28 + 2^{11}(15+13+11+9) + 2^{12} \cdot 7$	356,352

Compression Ratio (16 bit)



Compression ratio (24 bit)



Conclusion

- The STAM method uses the first two terms of the Taylor series to construct an approximation to an arbitrary differentiable function
- Considerable area reduction over simple table lookup
- High speed (only adders and multipliers required)

References

- J. E. Stine and M. J. Schulte, ``The Symmetric Table Addition Method for Accurate Function Approximation,'' in Journal of VLSI Signal Processing, vol. 21, no. 2, pp. 167-177, June, 1999.
- http://www.cse.lehigh.edu/~caar/STAM.html (includes a program for computing the tables)