Path Decompositions of Multigraphs

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Abstract

Let G be a loopless finite multigraph. For each vertex x of G, denote its degree and multiplicity by $d(x)$ and $\mu(x)$ respectively. Define

$$\phi(x) = \begin{cases} \text{the least even integer} \geq \mu(x) & \text{if } d(x) \text{ is even} \\ \text{the least odd integer} \geq \mu(x) & \text{if } d(x) \text{ is odd.} \end{cases}$$

In this paper, it is shown that every multigraph G admits a *faithful path decomposition* — a partition $\mathcal{P}$ of the edges of G into simple paths such that every vertex x of G is an end of exactly $\phi(x)$ paths in $\mathcal{P}$. This result generalizes Lovász's path decomposition theorem, Li's perfect path double cover theorem (conjectured by Bondy), and a result of Fan concerning path covers of weighted graphs. It also implies an upper bound on the number of paths in a minimum path decomposition of a multigraph, which motivates a generalization of Gallai's path decomposition conjecture.

1 Introduction

In this paper we consider loopless finite multigraphs. Given a multigraph G, a **path decomposition (PD)** of G is a partition of the edges of G into simple paths. Path decompositions have received considerable attention for simple graphs. A well known conjecture of Gallai (cf. [6]) states that every connected simple graph on n vertices admits a PD with at most $\lceil n/2 \rceil$ paths. In 1966, Lovász [6] proved that every simple odd graph[1] G contains a PD $\mathcal{P}$ such that every vertex of G is an end of exactly one path in $\mathcal{P}$, which confirmed Gallai's conjecture for simple odd graphs. In 1980, Donald [3] showed that every simple graph on n vertices admits a PD with

[1]An *odd graph* is a graph where each vertex is incident with an odd number of edges.

at most $\lfloor 3n/4 \rfloor$ paths. The concept of a path decomposition is also closely related to a **path cover** — a collection of simple paths of G that covers every edge at least once. In 1988, Bondy [1] introduced the notion of a **perfect path double cover** of a simple graph G (a path cover $\mathcal{P}$ such that each edge of G is covered by exactly two paths in $\mathcal{P}$ and each vertex of G is an end of exactly two paths in $\mathcal{P}$), and conjectured that every simple graph with no isolated vertex has such a path cover. His conjecture was settled by Li [5] in 1989. Recently, Fan [4] has studied path covers of weighted graphs where each edge e has weight $w(e) \in \{0, 1, 2\}$.

The main concerns in the above work were the cardinality of a path decomposition, the number of paths in a path decomposition (or a path cover) that contain a vertex as ends, and the number of paths in a path cover that cover an edge. These aspects are interconnected and, indeed, they can be studied in a unified way by considering path decompositions of multigraphs.

Let $G = (V, E)$ be a multigraph. For any two vertices x and y of G, $\rho_G(x, y)$ denotes the number of (multiple) edges joining x and y. For each vertex x of G, $N_G(x)$ represents the set of vertices adjacent to x, $\mu_G(x) = \max\{\rho_G(x, y) : y \in N_G(x)\}$ (note that if $S = \emptyset$ then $\max S = 0$) specifies the **multiplicity** of x, and $d_G(x)$ stands for the **degree** (the total number of edges incident with x) of x. Then $\mu(G) = \max\{\mu_G(x) : x \in V\}$ is the **multiplicity** of G. A vertex x is **isolated** if $d_G(x) = 0$, **even** if $d_G(x)$ is even, and **odd** if $d_G(x)$ is odd. Notice that for any PD $\mathcal{P}$ of G, every odd vertex of G is an end of an odd number of paths in $\mathcal{P}$, and every even vertex of G is an end of an even number of paths in $\mathcal{P}$. For a collection $\mathcal{P}$ of simple paths of G, let $\xi_{\mathcal{P}}(x)$ denote the number of paths in $\mathcal{P}$ that contain vertex x as an end. For any integer k, let $\lceil k \rceil_e$ be the least even integer $\geq k$ and $\lceil k \rceil_o$ the least odd integer $\geq k$. Define a function $\phi_G : V \to N$ according to the degree and multiplicity of a vertex $x \in V$ as follows:

$$\phi_G(x) = \begin{cases} \lceil \mu_G(x) \rceil_e & \text{if } x \text{ is an even vertex} \\ \lceil \mu_G(x) \rceil_o & \text{otherwise.} \end{cases}$$

A PD $\mathcal{P}$ of G is a **faithful path decomposition (FPD)** if every vertex x of G is an end of exactly $\phi_G(x)$ paths in $\mathcal{P}$. In this paper we will prove the following theorem and discuss its implications.

Theorem 1 *Every multigraph admits a faithful path decomposition.*

This theorem generalizes Lovász's path decomposition theorem [6], and Li's perfect path double cover theorem [5]. It can also be transformed into a theorem on path covers of weighted graphs, which implies a result of Fan [4]. Moreover, the theorem gives an upper bound on the cardinality of a **minimum path decomposition** (a path decomposition with the minimum number of paths) of a multigraph; the bound is sharp for certain multigraphs and nearly sharp for a large family of multigraphs. This, subsequently, motivates a generalization of Gallai's path decomposition conjecture.

2 Extending a path decomposition

Let $G = (V, E)$ be a multigraph. Let e be an arbitrary edge of G and H be the multigraph obtained from G by deleting edge e. First we consider a technique, introduced by Lovász [6], for extending a certain PD $\mathcal{Q}$ of H to a PD $\mathcal{P}$ of G without increasing the number of paths.

Let u and v be the two ends of e, and set $v_0 = u$. Suppose that there is a sequence $(v_0, v_1, \ldots, v_k)$ of distinct vertices, where $v_1, \ldots, v_k \in N_H(v)$, such that

1. for each v_i, $0 \le i < k$, there is a path $Q_i \in \mathcal{Q}$ that starts at v_i and hits v_{i+1} just before it goes through v; and
2. for v_k, there is a path $Q_k \in \mathcal{Q}$ that starts at v_k but does not go through v.

Then $(Q_0, \ldots, Q_k)$ forms a **path-fan from** u **to** v. For each $0 \le i < k$, let

$$P_i = (Q_i \setminus \{v_{i+1}v\}) \cup \{v_iv\};$$

and let

$$P_k = Q_k \cup \{v_kv\}.$$

Then each P_i, $0 \le i < k$, starts at v_{i+1} and hits v_i just before it goes through v; and P_k starts at v and immediatedly goes through v_k. Therefore

$$\mathcal{P} = (\mathcal{Q} \setminus \{Q_0, \ldots, Q_k\}) \cup \{P_0, \ldots, P_k\}$$

is a PD of G satisfying

$$\xi_{\mathcal{P}}(u) = \xi_{\mathcal{Q}}(u) - 1,\ \xi_{\mathcal{P}}(v) = \xi_{\mathcal{Q}}(v) + 1,$$

and $\xi_{\mathcal{P}}(x) = \xi_{\mathcal{Q}}(x)$ for every vertex $x \in V \setminus \{u, v\}$. We say that $\mathcal{P}$ is a $(-1, +1)$-**extension of** $\mathcal{Q}$ **from** u **to** v, and that $\mathcal{Q}$ is $(-1, +1)$-**extendible from** u **to** v. Figure 1 illustrates a path-fan and its $(-1, +1)$-extension.

The above technique will be the main tool in proving Theorem 1. Therefore we first give a sufficient condition for a PD $\mathcal{Q}$ of H to be $(-1, +1)$-extendible from u to v.

Lemma 2 *Let $G = (V, E)$ be a multigraph. Let e, with ends u and v, be an arbitrary edge of G and $H = G - e$. For any path decomposition $\mathcal{Q}$ of H, if $\xi_{\mathcal{Q}}(x) \ge \rho_G(x, v)$ for every vertex $x \in N_G(v)$ then $\mathcal{Q}$ is $(-1, +1)$-extendible from u to v, i.e., there is a PD $\mathcal{P}$ of G that satisfies $\xi_{\mathcal{P}}(u) = \xi_{\mathcal{Q}}(u) - 1, \xi_{\mathcal{P}}(v) = \xi_{\mathcal{Q}}(v) + 1$ and $\xi_{\mathcal{P}}(x) = \xi_{\mathcal{Q}}(x)$ for every vertex $x \in V \setminus \{u, v\}$.*

Proof. We use a method introduced by Li [5]. Construct a digraph $D = (V_D, A_D)$ with $V_D = N_G(v) \cup \{z\}$ (where $z \notin N_G(v)$ is a special vertex) whose arcs are defined

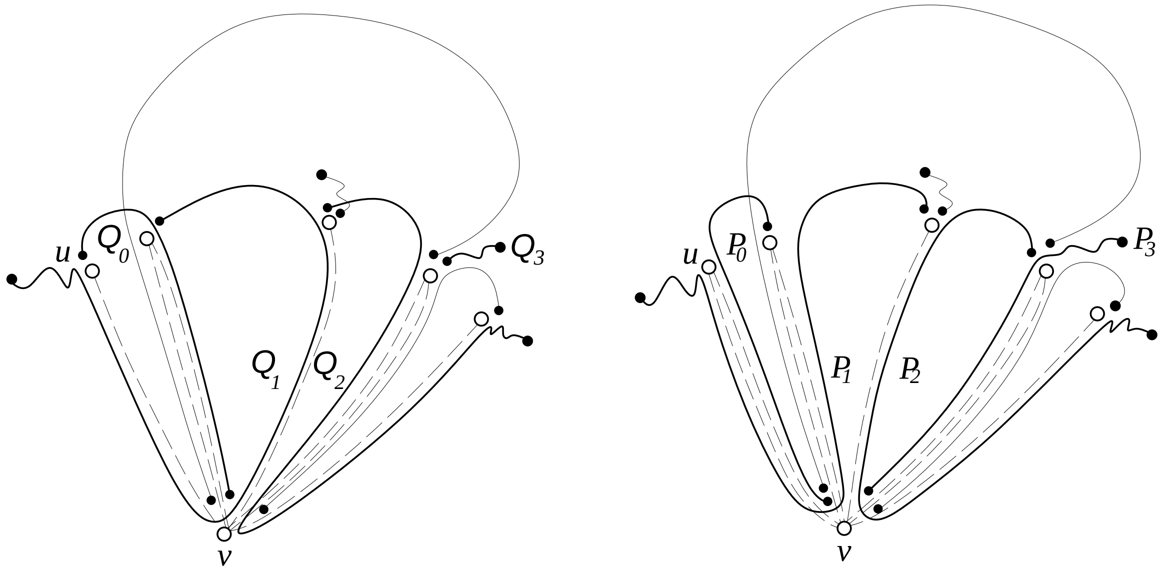

Figure 1: A path-fan (Q_0, Q_1, Q_2, Q_3) from u to v and its $(-1, +1)$-extension to (P_0, P_1, P_2, P_3)

as follows: For each vertex $x \in N_G(v)$ and each path $Q \in \mathcal{Q}$ which starts at vertex x, if Q does not go through v, then put arc (x, z) in A_D. Otherwise there is a vertex $y \in N_G(v)$ that Q hits just before Q goes through v; and we put arc (x, y) in A_D. Note that D may contain multiarcs or loops.

First, we show that there is a directed (u, z)-path in D. By the construction of D, we see that the out-degree $d_D^+(x) = \xi_{\mathcal{Q}}(x)$ and the in-degree $d_D^-(x) \leq \rho_H(x, v)$ for every vertex $x \in N_G(v)$. Therefore

$$d_D^+(x) - d_D^-(x) \geq \xi_{\mathcal{Q}}(x) - \rho_H(x, v).$$

Notice $\rho_G(u, v) = \rho_H(u, v) + 1$ and $\rho_G(x, v) = \rho_H(x, v)$ for every $x \in N_G(v) \setminus \{u\}$. By the assumption that $\xi_{\mathcal{Q}}(x) \geq \rho_G(x, v)$ for every $x \in N_G(v)$, we have

$$d_D^+(u) - d_D^-(u) \geq 1$$

and

$$d_D^+(x) - d_D^-(x) \geq 0 \text{ for every } x \in N_G(v) \setminus \{u\}.$$

Hence for any longest trail T in D starting at vertex u, no vertex in $N_G(v)$ can be the other end of T. Thus D contains a (u, z)-trail and hence a directed (u, z)-path $v_0 v_1 \ldots v_k v_{k+1}$, where $v_0 = u$ and $v_{k+1} = z$. Now for $0 \leq i \leq k$, let Q_i be the path in $\mathcal{Q}$ corresponding to arc (v_i, v_{i+1}) of D. Then $(Q_0, \ldots, Q_k)$ is a path-fan from u to v, and thus $\mathcal{Q}$ is $(-1, +1)$-extendible from u to v. ∎

3 Faithful path decompositions

We now prove our main theorem: *every multigraph admits a FPD.*

Proof of Theorem 1. We use induction on the number of edges in a multigraph. Clearly, $\emptyset$ is a FPD for any edgeless multigraph. Assume that any multigraph with less than $k \geq 1$ edges admits a FPD, and let $G = (V, E)$ be a multigraph with k edges. We consider two cases depending on the multiplicity of G.

Case 1. $\mu(G) = 1$. Without loss of generality, we may assume that G is connected (otherwise we can consider each connected component of G independently). Since $\mu(G) = 1$, G is a simple graph. Thus for any vertex x, $\phi_G(x) = 1$ if $d_G(x)$ is odd and $\phi_G(x) = 2$ if $d_G(x)$ is even. If G contains a vertex v with $d_G(v) = 1$, then let v' be the vertex adjacent to v and $H = G - v$. By the induction hypothesis, H admits a FPD $\mathcal{P}$. We construct a FPD of G as follows: If $d_H(v') = 0$ or $d_H(v')$ is odd then we add a path consisting of edge vv' to $\mathcal{P}$; otherwise ($d_H(v')$ is even) there is a path P in $\mathcal{P}$ that contains v' as an end and we extend P to include edge $v'v$. If G contains a vertex u with $d_G(u) = 2$ then let a and b be the two vertices adjacent to u and $G' = G - u + ab$. By the induction hypothesis, G' admits a FPD $\mathcal{Q}$. Let $Q = Q_a ab Q_b$ be the path in $\mathcal{Q}$ that contains edge ab. Then a FPD of G is obtained from $\mathcal{Q}$ by replacing Q with two paths $Q_a au$ and ubQ_b.

It remains to consider the case where the minimum degree of G is at least three. In this case, G contains a cycle $C = v_1 v_2 ... v_t v_1$, where $t \geq 3$, and the graph H_0 obtained from G by deleting all edges in C contains no isolated vertex. Applying the induction hypothesis to each connected component of H_0, we obtain a FPD $\mathcal{P}_0$ of H_0. Since the parity of the degree of each vertex x in H_0 is the same as that in G, we have

$$\xi_{\mathcal{P}_0}(x) = \phi_{H_0}(x) = \phi_G(x) = \begin{cases} 1 & \text{if } d_G(x) \text{ is odd} \\ 2 & \text{if } d_G(x) \text{ is even.} \end{cases} \tag{1}$$

For convenience, we use v_0 as a synonym for v_t and let $H_i = H_{i-1} + v_i v_{i-1}$ for each $1 \leq i \leq t$. Then $G = H_t$. We construct a PD $\mathcal{P}_t$ of H_t as follows: For each i, $1 \leq i \leq t$, in increasing order, we successively construct a PD $\mathcal{P}_i$ of H_i from the existing PD $\mathcal{P}_{i-1}$ of H_{i-1} by a $(-1, +1)$-extension from v_i to v_{i-1}. The existence of such a PD $\mathcal{P}_i$ follows from Lemma 2, because for each i, $1 \leq i \leq t$, $\xi_{\mathcal{P}_{i-1}}(x) \geq 1$ for every vertex $x \in V \setminus \{v_{i-1}\}$. In fact, we have

$$\xi_{\mathcal{P}_t}(x) = \xi_{\mathcal{P}_{t-1}}(x) = \cdots = \xi_{\mathcal{P}_0}(x)$$

for every vertex $x \in V \setminus \{v_1, \ldots, v_t\}$. Furthermore,

$$\xi_{\mathcal{P}_t}(v_i) = \cdots = \xi_{\mathcal{P}_{i+1}}(v_i) = \xi_{\mathcal{P}_i}(v_i) + 1 = (\xi_{\mathcal{P}_{i-1}}(v_i) - 1) + 1 = \xi_{\mathcal{P}_{i-1}}(v_i) = \cdots = \xi_{\mathcal{P}_0}(v_i)$$

for every i satisfying $1 \leq i \leq t - 1$, and

$$\xi_{\mathcal{P}_t}(v_t) = \xi_{\mathcal{P}_{t-1}}(v_t) - 1 = \cdots = \xi_{\mathcal{P}_1}(v_t) - 1 = (\xi_{\mathcal{P}_0}(v_t) + 1) - 1 = \xi_{\mathcal{P}_0}(v_t).$$

Combining with (1), we see that $\mathcal{P}_t$ is a FPD of G.

Case 2. $\mu(G) > 1$. In this case, G contains two distinct edges e and e' that share the same ends, say u and v. Construct a multigraph H from G by deleting edges e and e'. Then $\rho_G(u,v) = \rho_H(u,v) + 2$ and $\mu_H(x) \le \mu_G(x) \le \mu_H(x) + 2$ for each $x \in \{u,v\}$. Since the parity of $d_H(x)$ is the same as that of $d_G(x)$, we can deduce from the definition of ϕ that either $\phi_G(x) = \phi_H(x)$ or $\phi_G(x) = \phi_H(x)+2$. Therefore, by the induction hypothesis that $\phi_H(x) = \xi_{\mathcal{Q}}(x)$, we have either $\phi_G(x) = \xi_{\mathcal{Q}}(x)$ or $\phi_G(x) = \xi_{\mathcal{Q}}(x) + 2$.

Consider multigraph $G' = H + e'$. If $\phi_G(u) = \xi_{\mathcal{Q}}(u)+2$ then let $\mathcal{P}' = \mathcal{Q} \cup \{\{e'\}\}$. Otherwise, since

$$\xi_{\mathcal{Q}}(y) = \phi_G(y) \ge \rho_{G'}(y,v) \text{ for each } y \in N_{G'}(v),$$

there is a $(-1,+1)$-extension $\mathcal{P}'$ of $\mathcal{Q}$ from u to v by Lemma 2. In both cases, we have

$$\begin{cases} \xi_{\mathcal{P}'}(u) = \phi_G(u) - 1 \\ \xi_{\mathcal{P}'}(v) = \xi_{\mathcal{Q}}(v) + 1. \end{cases}$$

This gives

$$\xi_{\mathcal{P}'}(v) = \begin{cases} \phi_G(v) - 1 & \text{if } \phi_G(v) = \xi_{\mathcal{Q}}(v) + 2 \\ \phi_G(v) + 1 & \text{if } \phi_G(v) = \xi_{\mathcal{Q}}(v). \end{cases}$$

Now consider multigraph $G = G' + e$. If $\phi_G(v) = \xi_{\mathcal{Q}}(v)+2$, then let $\mathcal{P} = \mathcal{P}' \cup \{\{e\}\}$. Otherwise it is readily checked that

$$\xi_{\mathcal{P}'}(z) \ge \rho_G(z,u) \text{ for each } z \in N_G(u).$$

Therefore there is a $(-1,+1)$-extension $\mathcal{P}$ of $\mathcal{P}'$ from v to u by Lemma 2. It is easy to see that $\mathcal{P}$ is a FPD of G. This completes the proof. ∎

Remark The above proof also implies a polynomial time algorithm for constructing a FPD in a multigraph.

4 Consequences

Theorem 1 can be used to deduce various results on path decompositions and path covers, which include Lovász's path decomposition theorem [6], Li's perfect path double cover theorem [5] (conjectured by Bondy [1]), a result of Fan [4] about path covers of weighted graphs, and a generalization of Gallai's path decomposition conjecture (cf. [6]).

For an arbitrary vertex x of a simple graph G, recall that $\phi_G(x) = 1$ if x is an odd vertex and $\phi_G(x) = 2$ if x is a nonisolated even vertex. The following result is an immediate consequence of Theorem 1:

Corollary 3 *Every simple graph admits a path decomposition $\mathcal{P}$ such that every odd vertex is an end of exactly one path in $\mathcal{P}$, and every nonisolated even vertex is an end of exactly two paths in $\mathcal{P}$.*

Recall that for any PD $\mathcal{P}$ of G, every odd vertex of G is an end of an odd number of paths in $\mathcal{P}$, and every even vertex of G is an end of an even number of paths in $\mathcal{P}$. We easily deduce from Corollary 3 the following two complementary results for simple graphs:

Corollary 4 (Lovász [6]) *A simple graph G admits a path decomposition $\mathcal{P}$ such that each vertex of G is an end of exactly one path in $\mathcal{P}$ if and only if G is an odd graph.*

Corollary 5 *A simple graph admits a path decomposition $\mathcal{P}$ such that each vertex of G is an end of exactly two paths in $\mathcal{P}$ if and only if G is an even graph with no isolated vertex.*

In the case of path covers, Theorem 1 can be used to obtain a result on *path k-covers* of a simple graph G (a path cover $\mathcal{P}$ that covers every edge of G exactly k times). Construct a multigraph H from G by replacing each edge e with k edges joining the two ends of e. Let x be an arbitrary nonisolated vertex of H. If k is even, then $\phi_H(x) = k$; otherwise $\phi_H(x) = k$ if x is an odd vertex and $\phi_H(x) = k+1$ if x is an even vertex. By Theorem 1, we see that H admits a FPD $\mathcal{P}$ in which $\xi_{\mathcal{P}}(x) = \phi_H(x)$ for every vertex x of H. Therefore the image of $\mathcal{P}$ in G yields a path k-cover of G which possesses the property stated in the following result:

Corollary 6 *For any integer $k \geq 1$, every simple graph admits a path k-cover $\mathcal{P}$ such that if k is even then every nonisolated vertex is an end of exactly k paths in $\mathcal{P}$; otherwise every odd vertex is an end of exactly k paths and every nonisolated even vertex is an end of exactly $k+1$ paths in $\mathcal{P}$.*

The case $k = 2$ of the above result yields the following theorem of Li:

Corollary 7 (Li [5]) *Any simple graph without isolated vertex admits a perfect path double cover.*

There is also a natural correspondence between a path decomposition of a multigraph and a *faithful path cover* of a *weighted graph*, where a *weighted graph* F is a simple graph in which each edge e has a nonnegative integer $w(e)$ associated with it, and a *faithful path cover* of F is a collection $\mathcal{P}$ of simple paths of F that covers each edge e of F exactly $w(e)$ times. For each vertex x of F, let

$$\psi_F(x) = \begin{cases} \lceil \max\{w(xv) : v \in N_F(x)\} \rceil_e & \text{if } \sum\limits_{v \in N_F(x)} w(xv) \text{ is even} \\ \lceil \max\{w(xv) : v \in N_F(x)\} \rceil_o & \text{if } \sum\limits_{v \in N_F(x)} w(xv) \text{ is odd.} \end{cases}$$

Define a *ψ-faithful path cover* of F to be a faithful path cover $\mathcal{P}$ such that each vertex x of F is an end of exactly $\psi_F(x)$ paths in $\mathcal{P}$. Now if we construct a multigraph H from F by replacing each edge e of F with $w(e)$ edges joining the two ends of e, then for any vertex x of F, $d_H(x) = \sum_{v\in N_F(x)} w(xv)$, $\mu_H(x) = \max\{w(xv) : v \in N_F(x)\}$, and $\phi_H(x) = \psi_F(x)$. Hence a FPD of H can be transformed into a ψ-faithful path cover of F and, in the language of path covers of weighted graphs, Theorem 1 is equivalent to the following result:

Corollary 8 *Every weighted graph admits a ψ-faithful path cover.*

Applying Corollary 8 to weighted graphs where edge weights are numbers in $\{0, 1, 2\}$, we obtain the following result of Fan [4], which is also a generalization of Corollary 7:

Corollary 9 *Let F be a weighted graph where $w(e) \in \{0, 1, 2\}$ for every edge e and $\sum_{v\in N_F(x)} w(xv)$ is even for every vertex x. Then F admits a faithful path cover $\mathcal{P}$ such that every nonisolated vertex is an end of exactly two paths in $\mathcal{P}$.*

We now consider the size of a path decomposition of a multigraph G. The *path decomposition number* of G, denoted by $\pi(G)$, is the number of paths in a minimum path decomposition of G. By Theorem 1, G admits a PD $\mathcal{P}$ such that each odd vertex x is an end of exactly $\lceil \mu_G(x) \rceil_o$ paths in $\mathcal{P}$, and each even vertex y is an end of exactly $\lceil \mu_G(y) \rceil_e$ paths in $\mathcal{P}$. Therefore we have the following upper bound for $\pi(G)$:

Corollary 10 *Let $G = (V, E)$ be a multigraph, and let V_o and V_e be the sets of odd and even vertices respectively. Then*

$$\pi(G) \le \frac{1}{2}\sum_{x\in V} \phi_G(x) = \frac{1}{2}(\sum_{x\in V_o} \lceil \mu_G(x) \rceil_o + \sum_{y\in V_e} \lceil \mu_G(y) \rceil_e)$$

It can be verified that μK_n, the complete multigraph constructed from the complete graph K_n by substituting each edge e with μ edges joining the two ends of e, attains the upper bound of Corollary 10 if at least one of n and μ is even.[2] Furthermore, it is interesting to note that the upper bound in Corollary 10 is nearly sharp for a large family of multigraphs. In fact, for any given $\mu \ge 1$, there are infinitely many "arbitrary" multigraphs of multiplicity μ whose path decomposition numbers differ from the upper bound in Corollary 10 by at most $(\mu + 1)/2$. To see this, let $\mu_1 \le \mu_2 \le \ldots \le \mu_{n-1} = \mu$, be an arbitrary sequence of nonnegative integers. Let H be an arbitrary even multigraph on $n - 1$ vertices $x_1, \ldots, x_{n-1}$ in which $\mu_H(x_i) \le \mu_i$ for $1 \le i \le n - 1$. Construct a multigraph G from H by adding a new vertex x_n and μ_i edges between x_n and x_i for each $1 \le i \le n - 1$. Then $\mu_G(x_i) = \mu_i$ and

[2] See the appendix for a proof of the path decomposition number of μK_n.

$\phi_G(x_i) = \mu_i$ for $1 \leq i \leq n-1$, since x_i is an even vertex in G if and only if μ_i is even. In multigraph G, there are $\sum_{i=1}^{n-1} \mu_i$ edges incident with vertex x_n. Therefore any path decomposition of G contains at least $\frac{1}{2}\sum_{i=1}^{n-1} \mu_i$ paths, since each path can contain at most two edges incident with x_n. Combining with Corollary 10, we have

$$\frac{1}{2}\sum_{i=1}^{n-1} \mu_i \leq \pi(G) \leq \frac{1}{2}\sum_{i=1}^{n} \phi_G(x_i) = \frac{\phi_G(x_n)}{2} + \frac{1}{2}\sum_{i=1}^{n-1} \mu_i.$$

Notice that $\mu(G) = \mu$ and $\phi_G(x_n) \leq \mu + 1$. It is clear that G possesses the claimed property.

In spite of the above discussion, the bound in Corollary 10 still leaves room for improvement. The author proposes the following conjecture that gives a better bound for connected multigraphs.

Conjecture 1 *Let $G = (V, E)$ be a connected multigraph. Then*

$$\pi(G) \leq \lceil \frac{1}{2} \sum_{x \in V} \mu_G(x) \rceil.$$

It can be readily checked that the above conjecture is true for any complete multigraph μK_n. The conjecture is also valid (by Corollary 10) for multigraphs in which every even vertex has an even multiplicity and every odd vertex has an odd multiplicity.

We can replace $\mu_G(x)$ in the above conjecture by the multiplicity $\mu(G)$ of G to get a weaker conjecture: $\pi(G) \leq \lceil n \cdot \mu(G)/2 \rceil$ *for any connected multigraph G on n vertices.* In this weak form, the connectedness is not required if $\mu(G)$ is even. Clearly, Conjecture 1 (its weak form as well) implies Gallai's conjecture; however, it is not clear whether the conjecture (even its weak form) is equivalent to Gallai's conjecture. Notice that for a disconnected simple graph G, we can only obtain $\pi(G) \leq \lceil 2n/3 \rceil$ from Gallai's conjecture (cf. [3]). So we can not deduce the bound in the conjecture or in its weak form from Gallai's conjecture by means of decomposing a multigraph into simple graphs, since some simple graphs in the decomposition may have to be disconnected. As a final remark, we point out that it can be shown that the truth of Conjecture 1 for multigraphs of multiplicities ≥ 4 can be deduced from the truth of the conjecture for multigraphs of multiplicities 2 and 3. Therefore, multigraphs of small multiplicities deserve special attention.

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Appendix. The path decomposition number of μK_n

Theorem A *For any two integers $\mu \geq 1$ and $n \geq 2$, $\pi(\mu K_n) = \lceil \mu n/2 \rceil$.*

Proof. Since μK_n contains $\mu n(n-1)/2$ edges and the longest simple path in μK_n contains $n-1$ edges, we clearly have $\pi(\mu K_n) \geq \lceil \mu n/2 \rceil$. We need to show $\pi(\mu K_n) \leq \lceil \mu n/2 \rceil$.

Consider K_n first. If n is even, then each vertex of K_n is odd. It follows from Corollary 4 that K_n can be decomposed into $n/2$ paths. Otherwise n is odd. Then the following well known construction (cf. [2] p.229 (5.1.6)) yields an edge partition of K_n into k spanning cycles, where $k = (n-1)/2$. Label the vertices of K_n by $0, 1, \ldots, 2k$ and arrange the vertices $1, 2, \ldots, 2k$ in a circle clockwise with 0 at the centre. Now $C_1 = (0, 1, 2, 2k, 3, 2k-1, 4, \ldots, k+2, k+1, 0)$ forms a spanning cycle of K_n. Then the clockwise rotations of C_1 yield an edge partition of K_n into k spanning cycles $C_1, C_2, \ldots, C_k$. To obtain a path decomposition of K_n, we delete the edge $\{i, i+1\}$ from each C_i, where $1 \leq i \leq k$, to form path P_i. Note that these deleted edges form a path P in K_n. Then $\{P_1, P_2, \ldots, P_k, P\}$ is a path decomposition of K_n with $k+1 = \lceil n/2 \rceil$ paths. Therefore $\pi(K_n) \leq \lceil n/2 \rceil$.

For $2K_n$, we easily deduce from Theorem 1 that it admits a PD with n paths. So $\pi(2K_n) \leq n$. Notice that for an even μ, μK_n can be expressed as an edge disjoint union of $\mu/2$ copies of $2K_n$, and that for an odd μ, μK_n can be expressed as an edge disjoint union of a copy of K_n and $(\mu-1)/2$ copies of $2K_n$. Combining the above results, we can easily deduce that $\pi(\mu K_n) \leq \lceil \mu n/2 \rceil$. This completes the proof. ∎