# Game Chromatic Index of *k*-Degenerate Graphs

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**Abstract:** We consider the following edge coloring game on a graph $G$. Given $t$ distinct colors, two players Alice and Bob, with Alice moving first, alternately select an uncolored edge $e$ of $G$ and assign it a color different from the colors of edges adjacent to $e$. Bob wins if, at any stage of the game, there is an uncolored edge adjacent to colored edges in all $t$ colors; otherwise Alice wins. Note that when Alice wins, all edges of $G$ are properly colored. The *game chromatic index* of a graph $G$ is the minimum number of colors for which Alice has a winning strategy. In this paper, we study the

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edge coloring game on $k$-degenerate graphs. We prove that the game chromatic index of a $k$-degenerate graph is at most $\Delta + 3k - 1$, where $\Delta$ is the maximum vertex degree of the graph. We also show that the game chromatic index of a forest of maximum degree 3 is at most 4 when the forest contains an odd number of edges.  J Graph Theory 36: 144–155, 2001

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## 1. INTRODUCTION

Consider the following two person coloring game on a graph $G = (V, E)$. Given $t$ distinct colors, two players Alice and Bob, with Alice moving first, alternately select an uncolored vertex $v$ of $G$ and assign it a color different from the colors of vertices adjacent to $v$. Bob wins if, at any stage of the game, there is an uncolored vertex adjacent to colored vertices in all $t$ colors; otherwise Alice wins. Note that all vertices of $G$ are properly colored when Alice wins, and that Alice always wins if $t \geq |V|$.

Bodlaender [1] studied the computational complexity pertaining to the above coloring game, and defined the *game chromatic number* $\chi_g(G)$ of a graph $G$ to be the smallest number of colors for which Alice has a winning strategy. Recently, there has been a growing interest in the coloring game [2–4, 6–9]. A notable development involves the game chromatic number of a planar graph, which was conjectured to have a constant bound by Bodlaender [1], proved to be at most 33 by Kierstead and Trotter [7], improved to 30 by Dinski and Zhu [2], reduced to 19 by Zhu [9], and further reduced to 18 by Kierstead [6].

In this article, we consider a variation of the coloring game where Alice and Bob color edges instead of vertices. In this edge coloring game, Bob wins if, at any stage of the game, there is an uncolored edge adjacent to colored edges in all $t$ colors; otherwise Alice wins. We define the *game chromatic index* $\chi'_g(G)$ of a graph $G$ to be the smallest number of colors for which Alice has a winning strategy for the edge coloring game on $G$. Note that for any graph $G$ of maximum degree $\Delta$, we have $\Delta \leq \chi'_g(G) \leq 2\Delta - 1$ since no edge is adjacent to more than $2\Delta - 2$ edges. This motivates us to consider graphs whose chromatic game indices are bounded above by $\Delta + C$ for some constant $C$.

To facilitate the study of game chromatic index, we consider the following *edge ordering game* on a graph $G$: Alice and Bob, with Alice moving first, alternately select an unselected edge and put it at the end of the linear order formed by the edges selected earlier. The outcome of a game is a linear order on the edge set $E$ of $G$. For any edge $e$ of $G$, any adjacent edge of $e$ that precedes $e$ in the linear order is called a *preceding neighbor* of $e$. In this edge ordering game, Alice's goal is to minimize the maximum number of preceding neighbors of edges, and Bob's goal is to maximize it. It turns out that we can use this edge ordering game to obtain an upper bound of game chromatic index. Suppose that Alice has a strategy for the edge ordering game on $G$ which guarantees that, regardless of how Bob plays, each edge of $G$ has at most $t$ preceding neighbors in the resulting edge order of $E$. Then Alice can use this strategy for the edge coloring game to guarantee her win with $t+1$ colors: she simply uses her strategy for the edge ordering game on $G$ to select the next edge to be colored. This implies $\chi'_g(G) \leq t+1$.

We now define $k$-degenerate graphs. Let $L$ be a linear order on the vertex set $V$ of a graph $G$. For a vertex $v \in V$, the *back degree of $v$ relative to $L$* is the number of adjacent vertices of $v$ that precede $v$ in $L$, i.e., $|\{u \in V : uv \in E \text{ and } L(u) < L(v)\}|$. The *back degree of $L$* is the maximum back degree of vertices relative to $L$. A graph $G$ is *$k$-degenerate* if there is a linear order $L$ on $V$ whose back degree is at most $k$. Note that $G$ is $k$-degenerate iff every induced subgraph of $G$ has a vertex of degree at most $k$. Many interesting families of graphs are $k$-degenerate for some constant $k$. For example, graphs embeddable on some fixed surface, graphs avoiding a certain minor, and graphs of bounded arboricity. In particular, planar graphs are 5-degenerate, outerplanar graphs are 2-degenerate, partial $k$-trees are $k$-degenerate, and graphs of arboricity $i$ is $(2i-1)$-degenerate.

In this article, we study game chromatic indices of $k$-degenerate graphs. In Section 2, we use the edge ordering game to prove that the game chromatic index of a $k$-degenerate graph is at most $\Delta+3k-1$. In Section 3, we show that the game chromatic index of a forest of maximum degree 3 is at most 4 when the forest contains an odd number of edges. We summarize in Table 1 our upper bounds for game chromatic indices of some $k$-degenerate graphs.

TABLE 1. Upper bound of $\chi'_g(G)$ on some $k$-degenerate graphs

| Graph $G$ | Upper bound of $\chi'_g(G)$ |
|---|---|
| Arboricity $i$ | $\Delta+6i-4$ |
| Partial $k$-tree | $\Delta+3k-1$ |
| Planar | $\Delta+14$ |
| Outerplanar | $\Delta+5$ |
| Forest | $\Delta+2$ |

## 2. EDGE ORDERING GAME ON *k*-DEGENERATE GRAPHS

In this section we will present a strategy for Alice to play the edge ordering game on a $k$-degenerate graph $G$ such that, no matter how Bob plays the game, each edge $e$ has at most $\Delta+3k-2$ preceding neighbors in the resulting linear order of edges. This implies that the game chromatic index of a $k$-degenerate graph is at most $\Delta+3k-1$.

We will use a digraph in describing Alice's strategy, and thus first fix some definitions and notation for digraphs. For an edge $e = uv$ in a digraph $G$, $u$ is the *tail* of $e$ and $v$ the *head* of $e$. For a vertex $x$ of $G$, edges with $x$ as head and tail, respectively, are *in-edges* and *out-edges* of $x$, and the number of out-edges of $x$ is the *out-degree* $d^+(x)$ of $x$. A vertex with no out-edges is a *sink*, and $G$ is *acyclic* if it contains no directed cycle. In an acyclic digraph, the *level* of an edge $e$ is the length of a longest directed path from the head of $e$ to a sink.

We now begin our description of Alice's strategy. Let $G = (V, E)$ be a $k$-degenerate graph and $L$ be a linear order on $V$ with maximum back degree at most $k$. Alice will regard $G$ as a digraph by orienting every edge $uv$ of $G$ from $u$ to $v$ whenever $L(u) > L(v)$. Under this orientation, $G$ is an acyclic digraph where every vertex has out-degree at most $k$.

During a game, Alice maintains a subgraph $G_a$, called *active subgraph*, of $G$ that contains all sinks. Edges and vertices in $G_a$ are *active edges* and *active vertices*, respectively. At any stage of a game, an edge not in $G_a$ is an *inactive edge*, and an edge that has been selected by either player before that stage is a *selected edge*.

We also need the following notion of an extension of a directed $(u, v)$-path $P$ in $G$. Let $Q$ be a directed path in $G$ from $v$ to an active vertex such that all internal vertices of $Q$ are inactive. Since $G$ is acyclic, the concatenation $PQ$ is a directed path, and is called an *extension* of $P$.

In her first move, Alice selects an in-edge $e$ of a sink and set $G_a$ to be the graph consisting of edge $e$ and all sinks of $G$. Then each time after Bob has selected an edge $uv$, Alice uses the following three-step procedure to select an unselected edge in her move.

**Step 1.** Set path $P = uv$ initially and then repeat the following until the last vertex of $P$ has no inactive out-edges: Pick up an inactive out-edge $e$ of the last vertex of $P$ and replace path $P$ by an extension of the concatenation $Pe$ of $P$ and $e$. (Note that path $P$ maintains a path that grows until it cannot grow further through an inactive edge.)

**Step 2.** If the last vertex of $P$ has an unselected active out-edge, select such an edge; otherwise arbitrarily select an unselected edge of minimum level.

**Step 3.** Add $P$ and the newly selected edge to the active subgraph $G_a$.

Note that in Step 1, whenever the head of edge $e$ is not an active vertex, $Pe$ is always extended to a longer path as the active subgraph $G_a$ contains all sinks. Furthermore, since $G$ is acyclic, Step 1 always terminates and produces a directed path $P$ whose last vertex has no inactive out-edges. See Fig. 1 for an example of an execution of Step 1. Also note that Alice always tries to select an unselected active edge before considering an unselected edge of minimum level.

For a vertex $x$, let $a_i(x)$, $a_o(x)$, and $s_o(x)$, respectively, be the numbers of active in-edges, active out-edges, and selected out-edges of $x$. Then we observe the following two facts from Alice's strategy.

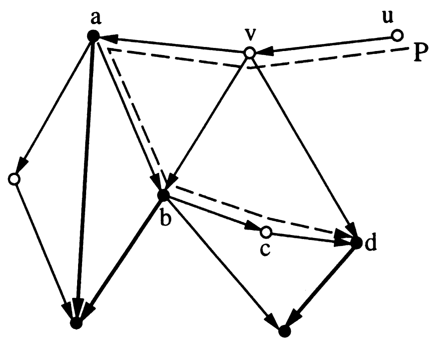


FIGURE 1. An execution of Step 1. Black vertices are active vertices, thick edges are active edges, and the dashed line segments indicate path $P$.

**Lemma 2.1.** *Each time when Alice finishes a move, every selected edge is an active edge.*

***Proof.*** The edges selected by both Alice and Bob in the round just finished are added to the active subgraph in Step 3 of Alice's strategy.

**Lemma 2.2.** *Each time when Alice finishes a move, every vertex $x$ satisfies*

$$a_o(x) \geq \min\{d^+(x), a_i(x)\}$$

and

$$s_o(x) \geq \min\{d^+(x), a_i(x) - d^+(x)\}.$$

***Proof.*** The two inequalities clearly hold after Alice's first move, and Lemma 2.1 implies that $a_o(x)$ is never decreased. We need only show that each time an inactive in-edge of $x$ becomes active, an inactive out-edge of $x$, if it exists, also turns active; and after all out-edges of $x$ become active, each time an inactive in-edge of $x$ becomes active, an unselected active out-edge of $x$, if it exists, will be selected by Alice.

The only time an inactive in-edge of $x$ becomes active is either when the directed path $P$ constructed in Step 1 passes through or terminates at $x$, or when the inactive in-edge is of minimum level and selected by Alice in Step 3. In the former case, if $P$ passes through $x$, then an inactive out-edge of $x$ indeed turns active; otherwise, $x$ has no inactive out-edges ($a_o(x) = d^+(x)$) and indeed Alice selects an unselected active out-edge of $x$, if it exists. In the latter case, because of the minimality of the level of the inactive in-edge, all out-edges of $x$ are selected edges, and hence also active edges by Lemma 2.1. Therefore, the two inequalities are not violated after each of Alice's moves. ∎

We are now ready to obtain an upper bound on the game chromatic index of a $k$-degenerate graph.

**Theorem 2.3.** *For any $k$-degenerate graph $G$, $\chi'_g(G) \le \Delta + 3k - 1$.*

***Proof.*** Alice uses her edge ordering strategy given in this section to select the next edge to be colored. Then we need only show that during the game, any unselected edge of $G$ is adjacent to at most $\Delta + 3k - 2$ selected edges, and hence is adjacent to at most $\Delta + 3k - 2$ colored edges when it is to be colored. Let $e = xy$ be an unselected edge right after Alice finishes a move.

Since $y$ has degree at most $\Delta$, it has at most $\Delta - 1$ active in-edges and out-edges; and since $x$ has out-degree at most $k$, it has at most $k - 1$ active out-edges other than $xy$. Because $xy$ is an unselected edge, $x$ has at most $k - 1$ selected out-edges. By Lemma 2.2, $x$ has at most $k - 1 + d^+(x) \le 2k - 1$ active in-edges. Therefore, the total number of active edges adjacent to $e$ is at most $\Delta + 3k - 3$, and at most $\Delta + 3k - 2$ after Bob's move. By Lemma 2.1, at most $\Delta + 3k - 2$ selected edges are adjacent to $e$ at any stage of the game. Therefore the game chromatic index of $G$ is at most $\Delta + 3k - 1$. ■

## 3. FORESTS OF MAXIMUM DEGREE THREE

We have deduced in the previous section that the game chromatic index of a forest is at most $\Delta + 2$. On the other hand, it is easy to construct a tree whose game chromatic index is at least $\Delta + 1$. In an attempt to characterize forests of game chromatic index at most $\Delta + 1$, we investigate the game chromatic index of forests of maximum degree 3 in this section. We will show that if a forest of maximum degree 3 has an odd number of edges, then Alice has a winning strategy with four colors.

Let $F$ be a forest of maximum degree 3 where some edges are properly colored by colors from a set of four distinct colors. We obtain a family of *independent subtrees* of $F$ by cutting each colored edge of $F$ in the middle. Thus, each colored edge becomes two half-edges with the same color but in two different independent subtrees, and each uncolored edge belongs to one independent subtree only. During a game, when an uncolored edge $e$ is to be colored, only the independent subtree $T$ containing $e$ need to be considered; and once $e$ is colored, $T$ is broken into two smaller independent subtrees by cutting $e$ into two half-edges. Therefore, we can consider each independent subtree independently during a game.

**Remark.** Using the notion of independent subtrees, we have a simpler winning strategy for Alice that uses $\Delta + 2$ colors on a forest. During a game, Alice need only make sure that every independent subtree contains at most $\Delta$ colored edges after her move. Then after Bob's move, at most one independent subtree can contain $\Delta + 1$ colored edges, and Alice can easily break it into two independent subtrees each containing no more than $\Delta$ colored edges.

We call an uncolored edge of $F$ a *safe edge* if it is adjacent to at least three colored edges with two of them having the same color. Note that a safe edge can always be colored properly at any later stage using one of the four colors. A *legal color* for an edge $e$ is any color that has not been used for any edges adjacent to $e$.

For clarity, the following notation is used in all figures of this section: a thick edge denotes a colored edge, a dashed line indicates a path with zero or more edges, boldface letters **a**, **b**, **c**, and **d** represent the four distinct colors, and "Alice[**x**]" with an arrow pointing to an edge means that Alice's move is to color the edge with color **x**. Boldface letters **x** and **y** denote two different colors from the color set.

**Theorem 3.1.** *If $F$ is a forest of maximum degree 3 with an odd number of edges, then* $\chi'_g(F) \leq 4$.

***Proof.*** To prove that Alice has a winning strategy with four colors, we present a strategy for Alice guaranteeing that each time after her move but before Bob's next move, each independent subtree $T$ of $F$ satisfies the following *color invariants*:

1. No uncolored edge in $T$ is adjacent to one uncolored edge and three distinctively colored edges.
2. If $T$ contains four or five colored edges, then it has at least one safe edge.
3. If $T$ contains more than five colored edges, then it is one of the independent subtrees in Fig. 2.

Note that such a strategy will ensure that both Alice and Bob can always make a legal move until all edges of $F$ are colored properly, and thus is a winning strategy for Alice.

Clearly, the color invariants are satisfied within the first two moves. Suppose that the color invariants are satisfied after Alice finishes a move. Because of the color invariants, Bob can always pick up an uncolored edge of $F$ (otherwise the game is finished) and use a legal color to color it. This edge belongs to a unique

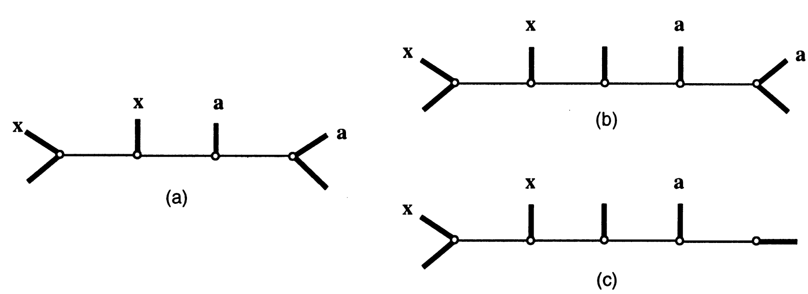


FIGURE 2. Three special independent subtrees. Color **x** can equal color **a** and colors for unmarked thick edges are arbitrary. Note that Alice will not produce the independent subtree in (c) but Bob may.

independent subtree before Bob's move. After Bob's move, this independent subtree is broken into two independent subtrees while all other independent subtrees remain unchanged. It is a routine matter to check that at most one of these two new independent subtrees may violate the color invariants.

We now describe a strategy for Alice to color an uncolored edge of $F$ so that each independent subtree satisfies the color invariants after her move. If no independent subtree violates the color invariants, Alice arbitrarily selects an independent subtree that contains an odd number of uncolored edges (such an independent subtree exists as $F$ always has an odd number of uncolored edges each time before Alice's move). Otherwise, she selects the only independent subtree $T$ that violates the color invariants. Note that Alice will never select an independent subtree in Fig. 2(b) or (c), and thus $T$ has at most six colored edges.

**Case 1.** *$T$ contains at most four colored edges.* In this case, $T$ may only violate color invariants (1) and (2). If $T$ has two adjacent colored edges, then there is an uncolored edge $e$ adjacent to these two colored edges. Note that $e$ is the only edge in $T$ that may violate color invariant (1). Alice tries a legal color, say color **c**, for $e$ to see if the resulting independent subtrees satisfy the color invariants. If yes, Alice colors $e$ with color **c** in her move; otherwise, one of the two resulting independent subtrees violates color invariant (1) and the two possible configurations of $T$ are shown in Figure 3(a) and (b). For Fig. 3(a), Alice makes her move as indicated in the diagram. For Figure 3(b), if $e$ does not violate color invariant (1), Alice colors edge $g$ with color **a**; otherwise one of the two leftmost colored edges is in color **d**, and she colors edge $f$ with color **d**.

If $T$ does not have adjacent colored edges, then it may only violate color invariant (2). Alice picks an edge $e$ of $T$ whose removal disconnects $T$ into two

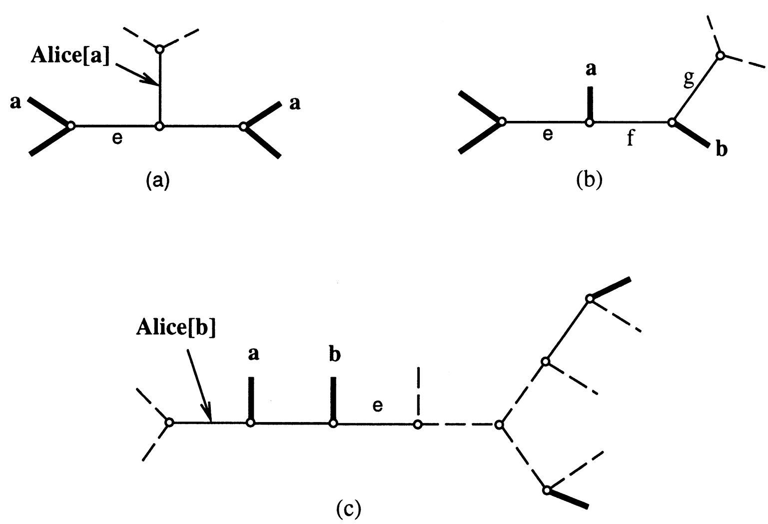


FIGURE 3. Case 1: $T$ has at most four colored edges (one colored edge in ($c$) may not appear).

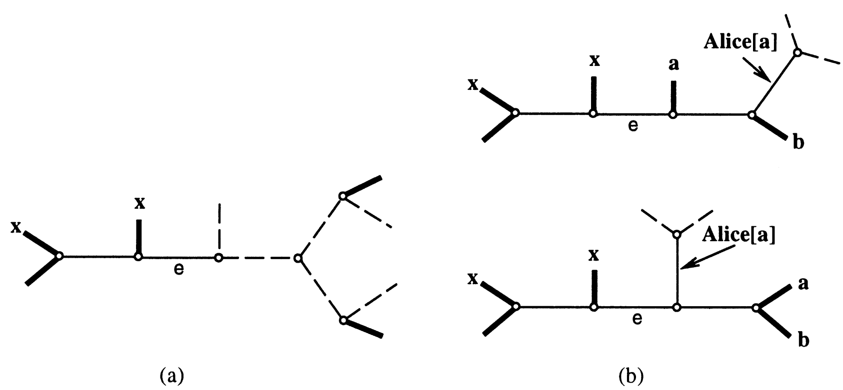


FIGURE 4. Case 2: $T$ contains five colored edges.

independent subtrees each containing at most two colored edges. Again, Alice tries a legal color, say color **c**, for $e$ to see if the resulting independent subtrees satisfy the color invariants. If yes, Alice colors $e$ with color **c** in her move; otherwise, color invariant (1) is violated by the resulting independent subtrees (other color invariants will not be violated) and the structure of $T$ and Alice's move are shown in Fig. 3(c).

**Case 2.** *$T$ contains five colored edges.* By color invariant (2), $T$ contains a safe edge and its structure is shown in Fig. 4(a). Note that if $T$ violates color invariant (1) then $e$ is the only edge that violates it. Alice tries a legal color for $e$. If it does not cause any violation to color invariants, Alice colors edge $e$ with the legal color. Otherwise, color invariant (1) is violated, and the structure of $T$ and Alice's move are given in Fig. 4(b) Note that the independent subtree in Fig. 2(a) is produced after Alice's move.

**Case 3.** *$T$ contains six colored edges.* By color invariant (2), $T$ contains a safe edge ($T$ can be the independent subtree in Fig. 2(a)), and Fig. 5(a) shows the structure of $T$. Note that $T$ may violate color invariants (1) and (3), but $e$ is the only edge that may violate color invariant (1). Again, Alice tries a legal color for $e$. If it does not cause a violation to color invariants, Alice colors edge $e$ with the legal color; otherwise, at least one of $T_1$ and $T_2$ violates color invariant (1).

**Case 3.1.** *$T_1$ violates color invariant* (1). The structure of $T_1$ is shown in Fig. 5(b). If $e$ is also adjacent to an edge of color **b**, then the structure of $T$ is shown in Fig. 6(a). Otherwise **b** is a legal color for $e$. In the former case, if edge $f$ is not adjacent to an edge of color **a**, Alice colors edge $f$ with color **a**; otherwise we have the structure of $T$ shown in Fig. 6(b) and Alice's move is indicated in the figure (it produces an independent subtree in Fig. 2(b)). In the latter case, if coloring $e$ with color **b** does not cause $T_2$ to violate color invariant (1), Alice colors $e$ with **b**; otherwise, the structure of $T$ is shown in Fig. 6(c) and Alice's move is indicated in the figure. (Again, it produces an independent subtree in Fig. 2(b).) Note that $e$ does not violate color invariant (1) after Alice's move.

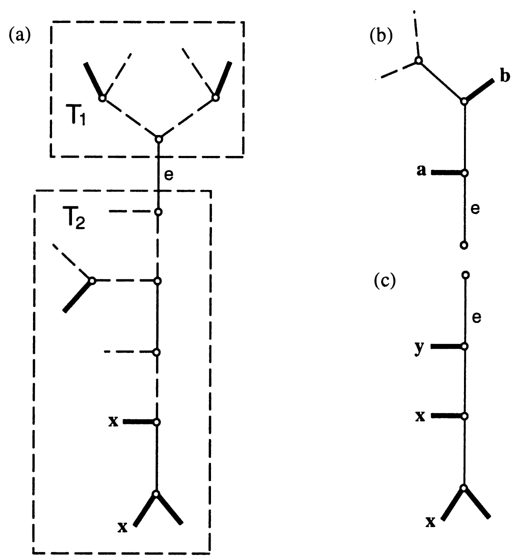


FIGURE 5. Case 3: (a) Structure of $T$. (b) Structure of $T_1$ when it violates color invariant (1). (c) Structure of $T_2$ when it violates color invariant (1).

**Case 3.2.** *$T_2$ violates color invariant* (1). The structure of $T_2$ is shown in Fig. 5(c). If $e$ is not adjacent to an edge of color **x**, then **x** is a legal color for $e$; otherwise $T$ has the structure shown in Fig. 7(a). In the former case, if coloring $e$ with color **x** causes $T_1$ to violate color invariant (1), we have the situation discussed in Case 3.1; otherwise Alice colors $e$ with color **x**. In the latter case, if $e$ is adjacent to three colored edges then we have $T$ as shown in Fig. 7(b), and Alice colors edge $f$ with the color of the topmost colored edge if it is different

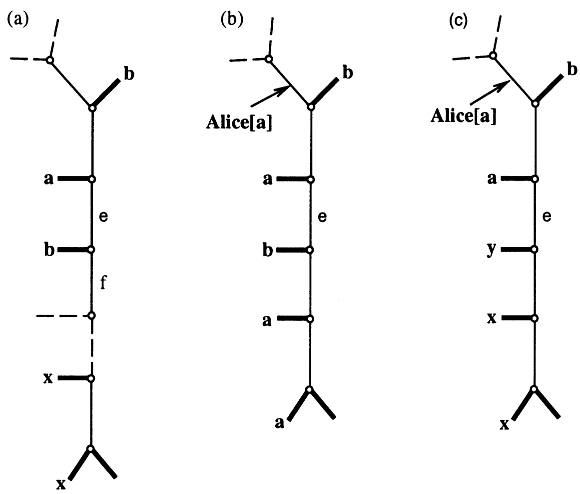


FIGURE 6. Case 3.1: $T_1$ violates color invariant (1).

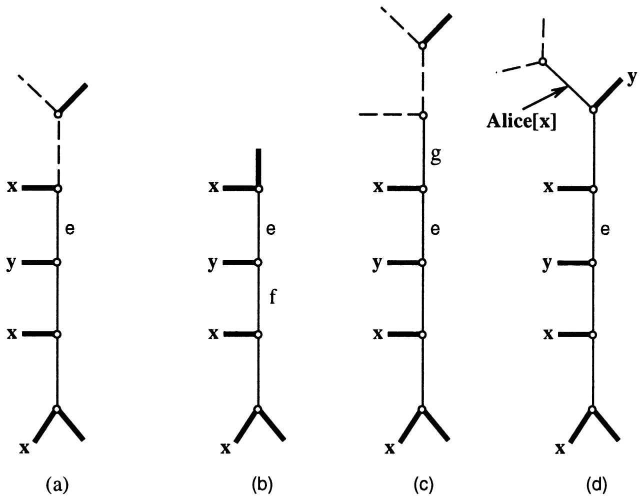


FIGURE 7. Case 3.2: $T_2$ violates color invariant (1).

from **y**, else she colors $f$ with a third color **z**. Otherwise, we have the situation in Fig. 7(c). If edge $g$ is not adjacent to an edge of color **y**, Alice colors edge $g$ with color **y**, else we have the situation in Fig. 7(d) as $T$ has an odd number of uncolored edges, and an independent subtree in Fig. 2(b) is produced after Alice's move.

It is a routine matter to check that in all cases, independent subtrees after Alice's move satisfy the color invariants. This completes the proof. ■

**Corollary 3.2.** *If $T$ is a tree of maximum degree 3 in which all internal vertices have degree 3, then* $\chi'_g(T) \leq 4$.

***Proof.*** Every vertex of $T$ is of odd degree, and thus $T$ contains an even number of vertices, implying an odd number of edges. ■

We note that our strategy for Alice fails to work if a forest of maximum degree 3 has an even number of edge. This is because an independent subtree $T$ in Fig. 2(b) or (c) may be produced during a game. Notice that there are four uncolored edges in $T$. When the number of uncolored edges in the forest is even, Alice will eventually be forced to color an uncolored edge in $T$ first (recall that Alice moves first). If $\mathbf{x} = \mathbf{a}$ and all unmarked thick edges are in color **b**, then it is easy to check that Alice loses once she colors an uncolored edge in $T$. On the other hand, if Bob colors an uncolored edge in $T$ first, no obstruction is produced to prevent Alice from winning. Thus $T$ does not cause a problem for Alice when the forest has an odd number of edges, since she can always avoid touching such a subtree until Bob colors an uncolored edge in $T$.

**Remark.** Recently, He et al. [5] have proved, by a similar method, that the game chromatic index of every forest of maximum degree 3 is at most 4.

## ACKNOWLEDGMENT

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