Lecture Outline 9
Topics in Graph Algorithms (CSCI5320-16S)

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Keywords: Parameterized intractability, FPT-reduction, W[1]-complete, and W[2]-complete.

1. **Parameterized intractability**: Similar to the theory of NP-completeness, the theory of parameterized intractability is built on complete classes formed by reductions from seed problems.

2. Short Turing Machine Acceptance: W[1]-complete but FPT for fixed alphabet.

   Given a nondeterministic Turing Machine M, string x and integer k (in unary), determine whether M accepts x in $\leq k$ steps.

   The problem is the seed problem for the basic intractable class W[1]-complete, analogous to that Turing Machine Acceptance is the seed problem for NPC.

3. **FPT-reduction**: An FPT-reduction (a.k.a. parameterized many-one reduction) from a parameterized problem Π to another parameterized problem Π' is a function that maps instance $(I, k) \in \Pi$ to $(I', k') \in \Pi'$ such that

   (a) (I, k) is an yes-instance iff (I', k') is,

   (b) the function is computable in time $f(k)|I|^{O(1)}$, and

   (c) $k' \leq g(k)$ for some computable function $g(k)$.

   Many polynomial reductions for NPC are not FPT-reductions because of (c), and most FPT-reductions in practice are polynomial reductions satisfying (c).

   Fact: $\Pi' \in \text{FPT} \Longrightarrow \Pi \in \text{FPT}$ and therefore $\Pi \notin \text{FPT} \Longrightarrow \Pi' \notin \text{FPT}$. Furthermore, FPT-reductions are transitive.

4. **Basic class W[1]**: A parameterized problem $\Pi \in W[1]$ if there is an FPT-reduction from Π to Short Turing Machine Acceptance, *W[1]-hard* if every problem in W[1] admits an FPT-reduction to Π, and *W[1]-complete* if Π is W[1]-hard and $\Pi \in W[1]$.

5. **Basic W[1]-complete problems**: Short Turing Machine Acceptance, Weighted q-CNF Satisfiability for each fixed $q \geq 2$. k-Clique, Independent k-Set and Set Packing.

6. Weighted q-CNF Satisfiability: Does a CNF formula with each clause having no more than q literals admit a satisfying truth assignment with weight k?

   Set Packing: Does a collection C of sets contain k mutually disjoint sets?

7. $\Pi \in W[1]$ if there is an FPT-reduction from Π to Weighted 2-CNF Satisfiability. For instance, Independent k-Set $\in W[1]$ as its instance can be expressed as a weight-k truth assignment for the 2-CNF expression $\bigwedge_{v_i v_j \in E} (\overline{x_i} \vee \overline{x_j})$.

8. Induced Tree is W[1]-hard (Cai)

   Does G contain k vertices V' such that $G[V']$ is a tree?

   FPT-reduction from Independent k-Set. Construct (G', k') from (G, k) of Independent k-Set by adding a $(k-1)$-star $K_{1,k-1}$ and connecting the center of the star to every vertex of G, and setting $k' = 2k$.

9. Set Packing is W[1]-hard

   FPT-reduction from Independent k-Set. For each vertex v of G, let E_v be the set of edges incident with v. The collection of sets is $\{E_v : v \in V\}$.

10. Maximum k-Vertex Cover is W[1]-hard (Cai 2000) (a.k.a. Partial Vertex Cover)

    Does G contain k vertices that cover at least l edges?

    FPT-reduction from Independent k-Set. Attach $\Delta - d(v)$ leaves to each vertex v, where Δ is the maximum degree of G. Set $l = k\Delta$.

11. Maximum k-Vertex Multicomponent Cut is W[1]-hard (Cai 2006)

    Does G contain k vertices whose removal results in at least l components?

    FPT-reduction from k-Clique. Subdivide each edge of $G = (V, E)$. Add a $(k+1)$-clique K and add all possible edges between K and V. Set $l = \binom{k}{2} + 1$.

    Open Problem Maximum k-Edge Multicomponent Cut: Does G contain k edges whose removal results in at least l components?

12. W[1]-hardness of Vertex Colouring on split+kv graphs (Cai 2003)

    We give an FPT-reduction from Independent k-Set. Let $G = (V, E)$ be an arbitrary instance of Independent k-Set. For convenience, we assume $V = \{v_1, v_2, \ldots, v_n\}$. Construct from G a split+kv graph $G' = (V', E')$ as follows (see Figure 1 for an example):

    (a) Set $V' = V \cup E \cup V_k$, where V_k is a set of k new vertices disjoint from $V \cup E$.
    (b) Connect every pair of vertices in V to form a complete graph on V, and connect every pair of vertices in V_k to form a complete graph on V_k.
    (c) For each vertex $v_i v_j \in E$, connect it with every vertex but v_i and v_j in V.
    (d) Connect every vertex in V_k with every vertex in E.

    The construction clearly takes polynomial time. We now show that G contains an independent set of size k iff G' is n-colourable.

    Suppose that G contains an independent set I of size k. Then $V - I$ is a vertex cover in G of size $n - k$, and we can construct a vertex n-colouring of G' as follows:

    (a) Colour vertex $v_i \in V$ by colour i.

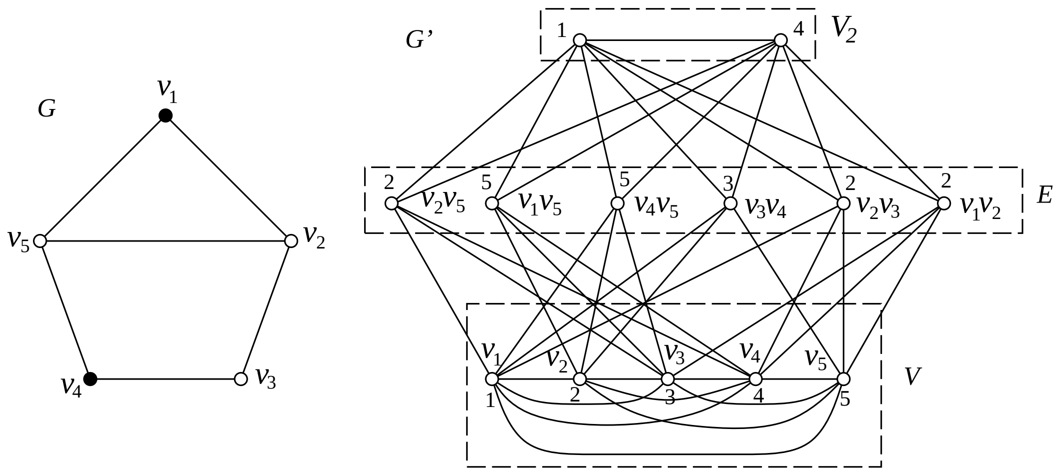

Figure 1: The construction of G', where $k = 2$ and $I = \{v_1, v_4\}$ is an independent set in G.

(b) Arbitrarily colour the k vertices in V_k by the k colours used for vertices in the independent set I of V.

(c) For each vertex $v_iv_j \in E$, since $V - I$ is a vertex cover of G, at least one of v_i and v_j is in $V - I$. This implies that at least one of the colours used for vertices v_i and v_j (i.e., colours i or j) is not used for vertices in V_k. Therefore colour vertex v_iv_j by colour i if $v_i \in V - I$ and colour j otherwise.

Conversely, suppose that f is an n-colouring of G'. Without loss of generality, we may assume that $f(v_i) = i$ for each vertex $v_i \in V$. Then for each vertex $v_iv_j \in E$ of G' , $f(v_iv_j)$ equals either i or j since v_iv_j is adjacent to every vertex but v_i and v_j in V. Let Z_k be the set of k colours used for V_k. Then $S = \{1, 2, \ldots, n\} - Z_k$ equals the set of colours used for the vertices in E since every vertex in V_k is adjacent to every vertex in E. From this, we deduce that $\{v_i : i \in S\}$ share at least one vertex with each vertex in E. Therefore $\{v_i : i \in S\}$ is a vertex cover of size $n - k$ in G, and hence $V - \{v_i : i \in S\}$ is an independent set of size k in G. ∎

13. **W-hierarchy**: FPT $\subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq W[t] \subseteq \cdots$.

    For $t \geq 2$, a parameterized problem $\Pi \in W[t]$ is there is an FPT-reduction from Π to Weighted t-Normalized Satisfiability.

    A Boolean expression is t-normalized if it is of the form $\bigwedge\bigvee\bigwedge\cdots$ with $t-1$ alternations of $\bigwedge$ and $\bigvee$. A 2-normalized expression is the same as a CNF expression. The Weighted t-Normalized Satisfiability problem asks whether a Boolean expression in t-normalized form has a satisfying truth assignment with weight k.

14. **Basic W[2]-complete problems**: Weighted CNF Satisfiability, Dominating k-Set, Hitting Set, Set Cover.

15. Hitting Set is W[2]-hard.

    Reduction from Dominating k-Set: The collection of sets is $\{N[v] : v \in V\}$.

16. W[2]-hardness of C_4-FREE CONTRACTION (Cai and Guo 2013)

First we note that we can destroy an induced 4-cycle C in two ways: either contract an edge of C or an edge in a 2-path between two non-consecutive vertices of C to create a chord for C.

We give an FPT-reduction from DOMINATING SET. For this purpose we construct, for a graph G with $V(G) = \{v_1, \cdots, v_n\}$, a graph G' in polynomial time as follows:

(a) Create an independent set $X = \{x_1, \cdots, x_n\}$ and a clique $Y = \{y_1, \cdots, y_n\}$.

(b) For each vertex v_i of G, add edge x_iy_i.

(c) For each edge v_iv_j of G, add edges x_iy_j and x_jy_i.

(d) Add a new vertex u and connect it with every vertex in clique Y.

(e) For each vertex x_i, add a 2-path Q_i between x_i and u.

It is easy to verify the following properties of G', which are useful to establish a connection between dominating sets in G and edge contactions in G' to make it C_4-free:

(a) $G'[X \cup Y \cup \{u\}]$ is a split graph, and hence C_l-free for all $l \geq 4$.

(b) For every edge x_iy_j in G', (x_i, y_j, u) and Q_i form an induced 4-cycle.

(c) G' contains no induced cycle larger than 4.

(d) Contracting edge y_iu destroys all induced 4-cycles containing y_i.

(e) Contracting edges in Q_i destroy induced 4-cycles containing x_i only.

By Property (c), we need only show that G has a dominating set with $\leq k$ vertices iff G' contains $\leq k$ edges whose contraction yields a C_4-free graph.

Suppose that S is a dominating set of G with $\leq k$ vertices. We contract edges $\{uy_i : v_i \in S\}$ in G' into u^* to obtain a graph G^*. Since S is a dominating set of G, u^* is adjacent to all vertices in X. Therefore for every vertex x_i, every induced 4-cycle through x_i contains chord x_iu^* in G^*, implying that the largest induced cycle in G^* has size 3 and hence G^* is C_4-free.

Conversely, suppose that G' contains at most k edges E' whose contraction results in a C_4-free graph. Consider an arbitrary vertex x_i. If E' contains any edge in Q_i, we can use Properties (d) and (e) to replace it with edge y_iu to obtain another solution. Therefore we can do so for every vertex x_i to obtain a solution E^* from E' such that $|E^*| \leq |E'|$ and, for every vertex x_i, E^* contains an edge incident with some vertex $y_{f(i)} \in N_{G'}(x_i)$. Let $Y' = \{y_{f(i)} : 1 \leq i \leq n\}$. Then $|Y'| \leq |E'| \leq k$, $Y' \subseteq Y$, and every x_i is adjacent to some vertex in Y'. Therefore Y' dominates all vertices in X, and hence $\{v_i : y_i \in Y'\}$ is a dominating set of G with $\leq k$ vertices. ∎