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Price differentiation and control in the Kelly mechanism

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ABSTRACT

The design and implementation of resource allocation and pricing for computing and network resources are crucial for system and user performance. Among various designing objectives, we target on maximizing the social welfare, i.e., the summation of all user utilities. The challenge comes from the fact that users are autonomous and their utilities are unknown to the system designer. Under the Kelly mechanism, users bid and proportionally share resources. When user population is large and "price-taking" can be assumed, the Kelly mechanism maximizes the social welfare; however, under oligopolistic competitions, this mechanism might induce an efficiency loss up to 25% of the welfare optimum.

We generalize the Kelly mechanism by designing a price differentiation and show that the efficiency gap can be closed. In particular, we analyze the resource competition game under the generalized mechanism and show that any price differentiation induces a unique Nash equilibrium and any non-dictatorial resource allocation can be implemented as a Nash equilibrium under price differentiation. We further characterize the optimality condition under which the social welfare is maximized. Based on this optimality condition, we design a feedback price control mechanism that takes observable system parameters as input and adapts to the optimal Nash equilibrium that maximizes the social welfare.

1. Introduction

Resource allocation [1,2] for computing resources, e.g., CPU cycles, and network resources, e.g., bandwidth capacity, have been studied extensively during the last decade. They are important for achieving the performance goals of computer systems that involve multiple users competing for common divisible resources. For example, in the networking context, providing Quality of Service (QoS) among heterogeneous application flows has been a long-standing research problem that concerns about bandwidth capacity allocation. One of the biggest challenges of these problems in practice is that the characteristics of users and their applications are naturally unknown to the system. Furthermore, users are often autonomous and selfish; therefore, if requested to report their private information, e.g., preference to allocated resources and induced utility, they might misreport so as to maximize their own utilities.

To mitigate the above incentive problem, resource pricing [3–6] mechanisms have been introduced to manage resource allocation. Pricing comes naturally when the resource owner cares about revenue. For example, in cloud computing [7], cloud providers apply various pricing schemes to sell resources, e.g., the on-demand instance and reserved instance pricing of Amazon EC2 [8]. In this work, our objective is to use pricing as a control mechanism to maximize social welfare of the system, i.e., the total utility of all individual users. This objective is aligned with that of prior work in *congestion pricing* and *optimal flow control* [1,5,6] and is very common in mechanism design [9] and auction design [10].

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Among various proposed pricing and allocation mechanisms, the Kelly mechanism [1] stands out as a simple and desirable solution. Under this mechanism, users bid for resources by submitting the amount they are willing to pay and the resource is proportionally allocated based on the bids of the users. Kelly et al. [1,11] showed that when used as a *congestion pricing* mechanism, it achieves a *proportional fairness* allocation among the users. Furthermore, when the user population is large and each user's impact on the market price of the resource is negligible, the resource allocation under the Kelly mechanism maximizes the social welfare of the system. However, because the number of users is often bounded in practice, oligopolistic competitions of resources among the users happen more often. Johari and Tsitsiklis [2] analyzed the resource competition game under the Kelly mechanism and found that the resulting resource allocation Nash equilibrium might induce an efficiency loss up to 25% of the maximum social welfare.

In this work, we design a price differentiation on the Kelly mechanism, which can be regarded as a generalization. Our generalization not only inherits the scalability property of the Kelly mechanism, by choosing an appropriate price differentiation, it can also close up the efficiency gap. Our contributions include the following.

- We design (Section 2.2) a novel price differentiation mechanism to generalize the Kelly mechanism.
- We analyze (Section 3) the resource competition game induced by the generalized mechanism and show the following.
  - Under any price differentiation, the competition game has a unique Nash equilibrium (Theorem 1).
  - Non-dictatorial resource allocations can be achieved as a unique Nash equilibrium under price differentiation and a bijective mapping between the price and allocation domains exists (Theorem 4).
- We characterize the condition under which price differentiation maximizes the social welfare (Theorem 5).
- We propose a feedback price control mechanism (Section 4) that adapts to the maximum social welfare.

Our new mechanism extends the flexibility of the Kelly mechanism in a way that allows autonomous resource owners to apply different price differentiation schemes so as to achieve individual objectives, e.g., making tradeoffs between user fairness (in terms of price differentiation) and system efficiency (in terms of social welfare). Because the mechanism depends on a congestion pricing principle and the allocations are implemented as Nash equilibrium solutions, it is also adaptive and robust. We believe that our new generalization of the Kelly mechanism provides better controls for the resource owner to achieve different performance goals of the system.

2. Resource allocation

In this section, we give some background about the Kelly mechanism and further generalize it by using an embedded price differentiation. In the next section, we will explore the properties of the generalized mechanisms.

2.1. The Kelly mechanism

We consider a set $\mathcal{N} = \{1, \ldots, N\}$ of rational users bidding for a share of divisible resource of capacity C. We assume that more than one user compete for the resource, i.e., $N = |\mathcal{N}| > 1$. Each user i has a valuation function $v_i(\cdot)$, where $v_i(d_i)$ defines the monetary utility to user i when she is given d_i amount of the resource.

A common objective in resource allocation is to maximize the social welfare. Under our context, it is to maximize the sum of the valuations of all the users as the following optimization problem:

$$\begin{aligned} \max \quad & \sum_{i\in\mathcal{N}} v_i(d_i) \\ \text{subject to} \quad & \sum_{i\in\mathcal{N}} d_i \le C \quad \text{and} \quad d_i \ge 0 \ \forall i \in \mathcal{N}. \end{aligned} \tag{1}$$

We define the above convex and compact constraint set as

$$\mathcal{D} = \left\{ \mathbf{d} \mid \sum_{i\in\mathcal{N}} d_i \le C, \text{ and } d_i \ge 0, \ \forall i \in \mathcal{N} \right\}.$$

In the Kelly mechanism [1], each user i submits a bid $w_i \ge 0$, which equals the payment for obtaining a share d_i of the resource. We denote u_i as the utility of each user i, defined in a *quasi-linear* [12] environment as

$$u_i(d_i) = v_i(d_i) - w_i,$$

which is the valuation of the allocated resource $v_i(d_i)$ less the payment w_i. The Kelly mechanism allocates the full capacity C among all users and the resource share d_i of each user i is proportional to her bid w_i. Mathematically, given a nonzero bid vector $\mathbf{w} = (w_1, w_2, \ldots, w_N)$, the resource allocation vector $\mathbf{d} = (d_1, d_2, \ldots, d_N)$ is defined by

$$d_i = D_i(\mathbf{w}) = \frac{w_i}{\sum_{j=1}^{N} w_j} C, \quad \forall i \in \mathcal{N}, \tag{2}$$

where $D_i(\cdot)$ denotes the proportional allocation function for user i under the Kelly mechanism.

As a result of the Kelly mechanism, each user is charged the same unit price of the resource μ such that $\mu d_i = w_i$ for all users. This implicit unit price μ can be calculated as

$$\mu = \frac{\sum_{j=1}^{N} w_j}{C}.$$

2.2. The generalized Kelly mechanism

Rather than implementing a nondiscriminatory price μ under the Kelly mechanism, we consider a price differentiation among users. Our motivation of designing the price differentiation is to achieve different efficiency points for the social welfare defined as the objective function of (1). Under our generalization, we consider a strict positive price vector $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ as a parameter of the mechanism. Each user i submits a bid $t_i \geq 0$ to compete for the resource and the allocation rule is the same proportional rule defined in Eq. (2):

$$d_i = D_i(\mathbf{t}) = \frac{t_i}{\sum_{j=1}^{N} t_j} C, \quad \forall i \in \mathcal{N}. \tag{3}$$

The difference of our generalization from the Kelly mechanism is that each user i pays $p_i t_i$ amount of money for $D_i(\mathbf{t})$ amount of shared resource, and therefore, obtains a utility of

$$u_i(\mathbf{t}, \mathbf{p}) = v_i(d_i) - p_i t_i = v_i(D_i(\mathbf{t})) - p_i t_i.$$

This generalized mechanism can be imagined as a process where users buy divisible tickets to compete for the resource. We denote t_i as the number of tickets bought by user i and p_i as the monetary price of each ticket for user i. Like the Kelly mechanism, it fully allocates the capacity C among all users and the resource share $d_i = D_i(\mathbf{t})$ to each user i is proportional to the number of tickets bought: t_i. Although we do not differentiate tickets in resource allocation, the unit ticket price to users could be different. In particular, the Kelly mechanism is a special case of the generalization where $\mathbf{p} = \mathbf{1}$.

Compared to the Kelly mechanism, the generalized mechanism achieves a similar virtual unit price ν in terms of tickets (measured as tickets per unit of resource) defined as

$$\nu = \frac{\sum_{j=1}^{N} t_j}{C}.$$

Consequently, the effective/real unit price for resource among users will be proportional to the price vector $\mathbf{p}$, because each user i's real price becomes $p_i\nu$ (measured as abstract monetary units per unit of resource). Notice that although a pre-determined price is assigned to each user, the generalized mechanism inherits the simplicity/scalability of the Kelly mechanism in two ways: (1) the strategy space of the mechanism is still simply one-dimensional; and (2) only a single virtual price feedback, i.e., ν, is required to be sent to all users.

2.3. Resource competition game $(\mathcal{N}, v, \mathbf{p})$

Kelly's original work [1] considered the *competitive equilibrium*: a pair of a strategy profile and a single unit price of resource $(\mathbf{w}, \mu)$ that satisfies the following conditions:

$$v_i\left(\frac{w_i}{\mu}\right) - w_i \geq v_i\left(\frac{\hat{w}_i}{\mu}\right) - \hat{w}_i, \quad \forall \hat{w}_i \geq 0,\ i \in \mathcal{N} \text{ and } \mu = \frac{\sum_{j=1}^{N} w_j}{C}.$$

The competitive equilibrium assumes that a same price μ is given to all users so that they choose the demand of resource to maximize individual utilities. This same price in equilibrium is also a *market clearing* price for which the summation of users' demand equals the available capacity C. Kelly proved that the solution to the above equations solves the social welfare optimization problem defined in (1).

In a competitive equilibrium, users are assumed to be price-takers. This assumption holds when the number of users is large and each user's impact on the market price is negligible. However, in practice, we might have a few number of users forming an oligopolistic competition, and therefore, if we implement the proportional share allocation, strategic users will know that their strategies are going to change the implicit price μ as well as the resource allocation to them.

To consider rational and price-anticipating users, we regard the generalized Kelly mechanism as a resource competition game, through which each user i uses her strategy t_i to maximize the individual utility of u_i. More precisely, given a price vector $\mathbf{p}$ for the set of players $\mathcal{N}$, each user $i \in \mathcal{N}$ tries to choose strategy t_i that maximizes the utility $u_i(t_i; \mathbf{t}_{-i}, \mathbf{p}) = v_i(D_i(\mathbf{t})) - p_i t_i$, where $\mathbf{t}_{-i}$ denotes the strategy profile of users other than i. A strategy profile $\mathbf{t}^*$ is a *Nash equilibrium* of the resource competition game if for any user i, the following is satisfied:

$$u_i(t_i^*; \mathbf{t}_{-i}^*, \mathbf{p}) \geq u_i(\hat{t}_i; \mathbf{t}_{-i}^*, \mathbf{p}), \quad \forall \hat{t}_i \geq 0. \tag{4}$$

Under our generalization, we define the feasible set of price vectors $\mathcal{P}$ to be strictly positive, i.e.,

$$\mathcal{P} = \{\mathbf{p} | \, p_i > 0, \ \forall i \in \mathcal{N}\}.$$

We require each component p_i to be positive, because any user with $p_i \leq 0$ can always increase its strategy t_i to increase utility, and therefore no Nash equilibrium exists. For a set $\mathcal{N}$ of players and their valuation functions $v = \{v_i(\cdot) : i \in \mathcal{N}\}$, given any $\mathbf{p} \in \mathcal{P}$, we denote the generalized resource competition game as a triple $(\mathcal{N}, v, \mathbf{p})$.

Hajek and Gopalakrishnan [13] showed that the Kelly mechanism (i.e., $\mathbf{p} = \mathbf{1}$) induces a unique Nash equilibrium. Johari and Tsitsiklis [14] showed that the worst efficiency loss of the Nash equilibrium relative to the social efficient solution to the problem defined by (1) is 25%.

3. Price differentiation

In this section, we study how the price differentiation generalization of the Kelly mechanism affects the resulting equilibrium of the resource competition game.

3.1. Uniqueness of the Nash equilibrium

Inherited from the uniqueness property of the Nash equilibrium under the Kelly mechanism, we first show that for any price vector $\mathbf{p} \in \mathcal{P}$, the corresponding resource competition game $(\mathcal{N}, v, \mathbf{p})$ also induces a unique Nash equilibrium. The result is parallel to Theorem 2.2 of Johari [15], originated from Hajek and Gopalakrishnan [13]. Similar to the assumptions made in prior work [14], we make the following assumption for the valuation functions.

Assumption 1. Each $v_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is concave, strictly increasing, and continuously differentiable.

Next, we present the uniqueness result for the Nash equilibrium of our generalized competition game $(\mathcal{N}, v, \mathbf{p})$. We present the proofs of the theorems in the Appendix.

Theorem 1 (*Uniqueness of the Nash Equilibrium*). *Under Assumption 1 and for any $\mathbf{p} \in \mathcal{P}$, the resource competition game $(\mathcal{N}, v, \mathbf{p})$ has a unique Nash equilibrium $\mathbf{t} \geq 0$, under which at least two components of $\mathbf{t}$ are positive.*

In this case, the resource allocation vector $\mathbf{d}^*$ defined by

$$d_i^* = D_i(\mathbf{t}) = \frac{t_i}{\sum_{j=1}^{N} t_j} C, \quad \forall i \in \mathcal{N}, \tag{5}$$

is the unique solution to the optimization problem:

$$\begin{aligned} &\textit{max} && \sum_{i=1}^{N} \hat{v}_i(d_i) \\ &\textit{subject to} && \mathbf{d} \in \mathcal{D}, \end{aligned} \tag{6}$$

where

$$\hat{v}_i(d_i) = \frac{1}{p_i}\left[\left(1 - \frac{d_i}{C}\right) v_i(d_i) + \frac{1}{C}\int_0^{d_i} v_i(z)dz\right]. \tag{7}$$

Theorem 1 states that for any $\mathbf{p} \in \mathcal{P}$, there is a unique Nash equilibrium. Thus, we denote $\mathbf{t^p}$ as the unique Nash equilibrium of the game $(\mathcal{N}, v, \mathbf{p})$ that satisfies

$$u_i(t_i^{\mathbf{p}}; \mathbf{t}_{-i}^{\mathbf{p}}, \mathbf{p}) \geq u_i(\hat{t}_i; \mathbf{t}_{-i}^{\mathbf{p}}, \mathbf{p}), \quad \forall \hat{t}_i \geq 0.$$

Accordingly, we denote the mapping $d_i : \mathcal{P} \mapsto \mathcal{D}$ to be the unique resource allocation for user i under the unique Nash equilibrium, defined as $d_i(\mathbf{p}) = D_i(\mathbf{t^p})$.

3.2. Structural properties of the Nash equilibrium

Although each game $(\mathcal{N}, v, \mathbf{p})$ induces a unique Nash equilibrium $\mathbf{t^p}$, different price vectors $\mathbf{p}$ and $\mathbf{q}$ might induce the same Nash equilibrium $\mathbf{t^p} = \mathbf{t^q}$, resulting in the same resource allocation $\mathbf{d}(\mathbf{p}) = \mathbf{d}(\mathbf{q})$, and even the same user utilities, i.e., $u_i(\mathbf{t^p}, \mathbf{p}) = u_i(\mathbf{t^q}, \mathbf{q})$ for all $i \in \mathcal{N}$. We classify the equivalent price vectors by the following two theorems.

Theorem 2 (*Linear Equivalence*). *Given any* $\mathbf{p} \in \mathcal{P}$ *and* $\mathbf{q} = k\mathbf{p}$ *for some* $k > 0$, *the games* $(\mathcal{N}, v, k\mathbf{p})$ *and* $(\mathcal{N}, v, \mathbf{q})$ *result in the same resource allocation under their unique Nash equilibria, i.e.,* $\mathbf{d}(\mathbf{q}) = \mathbf{d}(\mathbf{p})$. *Moreover,* $\mathbf{t}^{\mathbf{q}} = \frac{1}{k}\mathbf{t}^{\mathbf{p}}$ *and* $u_i(\mathbf{t}^{\mathbf{q}}, \mathbf{q}) = u_i(\mathbf{t}^{\mathbf{p}}, \mathbf{p})$ *for all* $i \in \mathcal{N}$.

Theorem 2 states that when the price vector is scaled by a positive constant k, the resource allocation does not change in equilibrium. Consequently, the strategy profile scales by $1/k$ and keeps the user payments and utilities unchanged in equilibrium. A consequence of this theorem is that any equal price mechanism will result in the same resource allocation and utilities for the users as those under the Kelly mechanism.

Theorem 3 (*No-Share Equivalence*). *For any* $\mathbf{p} \in \mathcal{P}$, *we denote* $\mathcal{E}^{\mathbf{p}}$ *as the set of users who get no resource in equilibrium, defined as* $\mathcal{E}^{\mathbf{p}} = \{i | t_i^{\mathbf{p}} = 0\}$. *Let the set* $\mathcal{Q}^{\mathbf{p}}$ *be*

$$\mathcal{Q}^{\mathbf{p}} = \{\mathbf{q} \mid q_i = p_i, \ \forall i \notin \mathcal{E}^{\mathbf{p}}; \ q_i \ge p_i, \ \forall i \in \mathcal{E}^{\mathbf{p}}\}.$$

Then for any $\mathbf{q} \in \mathcal{Q}^{\mathbf{p}}$, *the Nash equilibrium* $\mathbf{t}^{\mathbf{q}}$ *equals* $\mathbf{t}^{\mathbf{p}}$. *Also,* $\mathbf{d}(\mathbf{q}) = \mathbf{d}(\mathbf{p})$ *and* $u_i(\mathbf{t}^{\mathbf{q}}, \mathbf{q}) = u_i(\mathbf{t}^{\mathbf{p}}, \mathbf{p})$ *for all* $i \in \mathcal{N}$.

Theorem 3 states that when we only increase the prices of the users who bid zero, these users will remain to bid zero and prefer not to get any resource. Also, there is no change in resource allocation or user utility.

Next, we explore the conditions under which the mapping between the price differentiation vector $\mathbf{p}$ and the resource allocation $\mathbf{d}(\mathbf{p})$ is one-to-one. To exclude the equivalent "no-share" prices under Theorem 3, we focus on the domain $\tilde{\mathcal{D}}$ of non-zero allocations to the users, defined as follows:

$$\tilde{\mathcal{D}} = \left\{ \mathbf{d} \mid \sum_{i \in \mathcal{N}} d_i = C, \ \text{and } d_i > 0, \ \forall i \in \mathcal{N} \right\}.$$

Accordingly, we define the domain of non-zero allocation price vectors as $\tilde{\mathcal{P}} = \{\mathbf{p} \mid \mathbf{d}(\mathbf{p}) \in \tilde{\mathcal{D}}\}$.

To further exclude the linearly dependent price vectors mentioned by Theorem 2, we consider any constant $c > 0$ and define $\tilde{\mathcal{P}}_c = \{\mathbf{p} \in \tilde{\mathcal{P}}f \mid \sum_{j=1}^{N} p_j = c\}$.

Theorem 4 (*Bijective Mapping*). *For any* $c > 0$, *the mapping* $\mathbf{d} : \tilde{\mathcal{P}}_c \mapsto \tilde{\mathcal{D}}$ *is continuous and bijective. In particular, the inverse mapping* $P : \tilde{\mathcal{D}} \mapsto \tilde{\mathcal{P}}_c$ *is*

$$\tilde{p}_i = P_i(\mathbf{d}) = \frac{v_i'(d_i)(C - d_i)}{\sum_{j \in \mathcal{N}} v_j'(d_j)(C - d_j)} c. \tag{8}$$

The corresponding unique Nash equilibrium is

$$\mathbf{t}^{\tilde{\mathbf{p}}} = \frac{\sum_{j \in \mathcal{N}} v_j'(d_j)(C - d_j)}{cC} \mathbf{d}.$$

Theorem 4 implies that any resource allocation solution $\mathbf{d} \in \tilde{\mathcal{D}}$ can be implemented as a unique Nash equilibrium in the price domain $\tilde{\mathcal{P}}_c$ for any $c > 0$. To illustrate, we consider an example of three users competing for a divisible resource with capacity $C = 3$. The user valuation functions are $v_1(d_1) = \frac{d_1}{2}$, $v_2(d_2) = \sqrt{d_2 + \frac{1}{2}}$, $v_3(d_3) = \ln(d_3 + 1)$. Fig. 1 shows the social welfare under any resource allocation $\mathbf{d} \in \tilde{\mathcal{D}}$, where the maximum social welfare is achieved at the allocation $\mathbf{d}^* = (1.5, 0.5, 1.0)$. Fig. 2 shows the social welfare achieved under the Nash equilibrium $\mathbf{t}^{\mathbf{p}}$ for any $\mathbf{p} \in \tilde{\mathcal{P}}_1$. In particular, we illustrate that the corresponding optimal price vector is $\mathbf{p}^* = P(\mathbf{d}^*) = (\frac{1}{4}, \frac{5}{12}, \frac{1}{3})$ based on Eq. (8). Also, we show the set $\tilde{\mathcal{P}}$ by highlighting the three boundary segments drawn in solid, dash and dotted curves, which correspond to the boundary segments $d_2 = 0$, $d_3 = 0$ and $d_1 = 0$ of $\tilde{\mathcal{D}}$, respectively.

By Theorem 1, at least two components of any Nash equilibrium $\mathbf{t}^{\mathbf{p}}$ are positive, which implies that any dictatorial allocation, i.e., $d_i = C$ for some $i \in \mathcal{N}$ and $d_j = 0$ for all $j \neq i$, is not implementable via a Nash equilibrium. We denote $\hat{\mathcal{D}}$ as the set of non-dictatorial allocations, defined by

$$\hat{\mathcal{D}} = \left\{ \mathbf{d} \mid \sum_{i \in \mathcal{N}} d_i = C, \ \text{and } d_i \in [0, C), \ \forall i \in \mathcal{N} \right\}.$$

Theorem 4 implies that any $\mathbf{d} \in \tilde{\mathcal{D}}$ can be implemented as a Nash equilibrium via a price vector $\mathbf{p} \in \tilde{\mathcal{P}}$. In fact, any non-dictatorial allocation $\hat{\mathbf{d}} \in \hat{\mathcal{D}} \setminus \tilde{\mathcal{D}}$ can also be implemented via Nash equilibria. For any target $\hat{d}_j = 0$, we can keep increasing p_j until $\hat{d}_j$ reaches zero in equilibrium. Any further increase in p_j will result in the same resource allocation by Theorem 3. By doing so, we virtually exclude user j from participating in the competition game. For the remaining users with positive allocation targets, i.e., $\hat{p}_j > 0$, we can apply Theorem 4 again to find the corresponding prices as if the competition game is restricted to themselves.

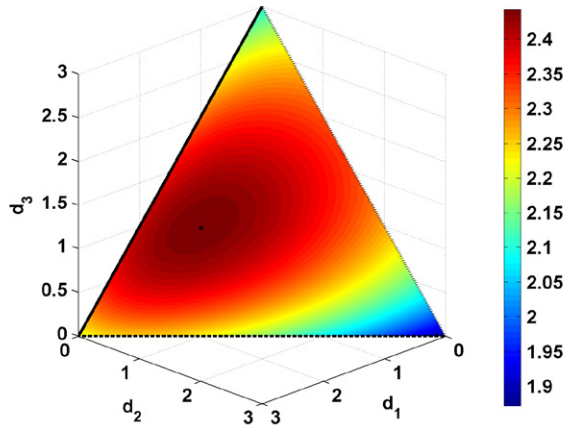

Fig. 1. Social welfare under the allocation domain $\tilde{\mathcal{D}}$. The maximum welfare is attained at $\mathbf{d}^* = (1.5, 0.5, 1.0)$.

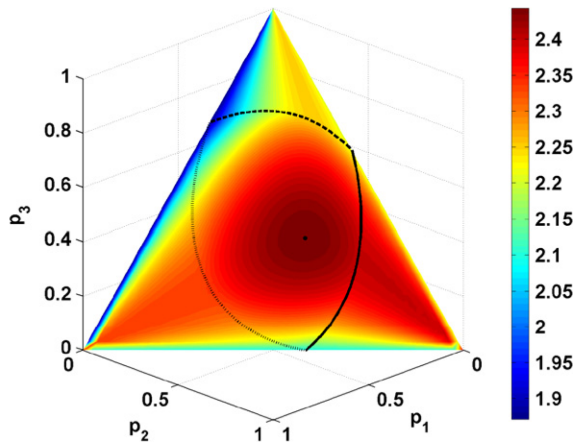

Fig. 2. Social welfare achieved as the Nash equilibrium under price differentiations (boundary $\tilde{P}_1$ showed).

4. Optimization via price control

We have shown that any non-dictatorial resource allocation $\hat{\mathbf{d}} \in \hat{\mathcal{D}}$ can be achieved as a Nash equilibrium under the generalized Kelly mechanism. Thus, in theory, we can close the 25% efficiency gap by choosing an appropriate price vector that maximizes the social welfare, defined in (1). However, the central planner might not know what the optimal allocation $\mathbf{d}^*$ is in the first place. In general, the challenge is that we need to know the valuation functions $v_i(\cdot)$ of users, which are private information and may not be revealed by the users.

In this section, we derive the condition for a resource allocation $\mathbf{d}$ to be optimal for the social welfare and then design a feedback control mechanism on the prices so as to drive the users to converge to the global optimality.

Theorem 5 (*Conditions of Optimality*)**.** *A price vector $\mathbf{p} \in \tilde{\mathcal{P}}$ induces a resource allocation $\mathbf{d}(\mathbf{p})$ that maximizes the social welfare defined by the optimization problem (1), if and only if for any positive $d_i(\mathbf{p})$ and $d_j(\mathbf{p})$, the following condition is satisfied:*

$$p_i : p_j = C - d_i(\mathbf{p}) : C - d_j(\mathbf{p}). \tag{9}$$

In particular, when $N = 2$, the optimality is achieved when both users pay the same amount, i.e., $p_1 t_1^{\mathbf{p}} = p_2 t_2^{\mathbf{p}}$.

Theorem 5 implies that for any pair of users i and j with positive allocation in the optimal resource allocation, i.e., $d_i^* > 0$ and $d_j^* > 0$, the optimal price ratio $p_i^* : p_j^*$ should equal the ratio of $C - d_i^* : C - d_j^*$. This result not only gives us a way to verify the optimality without knowing hidden valuations of the users, but it also guides us to design a feedback price control mechanism under which the resource allocation under the Nash equilibrium converges the maximum social welfare of the system.

We design a control system such that it adjusts the prices to achieve the optimal social welfare without the knowledge of users' valuation function $v_i(\cdot)$. We denote $\mathbf{p}(t)$ as the system price differentiation at time t. Based on Theorem 5, we know

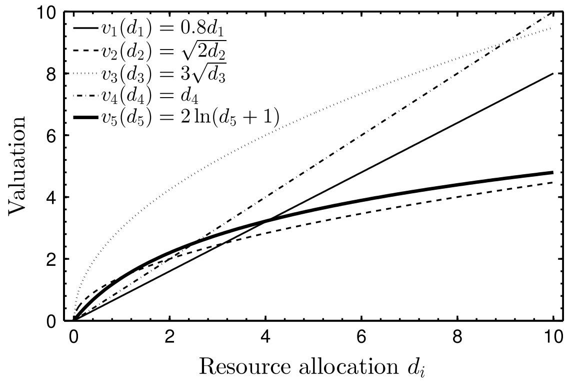

Fig. 3. The valuation functions of users.

that an optimal allocation $\mathbf{p}^* \in \tilde{\mathscr{P}}$ satisfies the following conditions

$$\frac{C - d_i(\mathbf{p}^*)}{p_i^*} = \frac{C - d_j(\mathbf{p}^*)}{p_j^*} = \frac{(N-1)C}{\sum_{k\in\mathscr{N}} p_k^*}, \quad \forall i, j \in \mathscr{N}.$$

This further implies that for each $d_i^* > 0$, we must have

$$\frac{C - d_i^*}{N-1} = \frac{p_i^*}{\sum_{k\in\mathscr{N}} p_k^*} C.$$

Motivated by the above optimality condition, we design a feedback price control mechanism that updates the prices every Δt amount of time by the following equation:

$$p_i(t + \Delta t) = p_i(t) + \left(\frac{C - d_i(\mathbf{p}(t))}{N-1} - \frac{Cp_i(t)}{\sum_{j\in\mathscr{N}} p_j(t)} \right) \Delta t. \tag{10}$$

To fully capture the behavior of our price differentiation and feedback control mechanism under a dynamic environment where arrivals and departures of users occur, we perform a simulation study of five users, whose valuation functions are shown in Fig. 3.

Without loss of generality, we control the price vector within the domain $\tilde{\mathscr{P}}_1$ and start with the equal pricing as the Kelly mechanism, i.e., $\mathbf{p}(0)_i = 1/N(0)$ for all user i, where $N(0)$ is the number of competing users at time $t = 0$.

We assume that the resource capacity is $C = 10$ and users 1, 2 and 3 stay in the system competing for the resource throughout the whole simulation. User 4 arrives at the system at time $t = 1$ and departs at time $t = 3$. User 5 arrives at the system at time $t = 2$ and stays until the end of simulation. Upon a user arrival at time t to the system, we set the new user's price as $1/N(t)$, where $N(t)$ includes the new user in the system. For the remaining users, we decrease their prices proportionally by a factor $\frac{N(t)-1}{N(t)}$, so that we maintain the new price vector to be in the domain $\tilde{\mathscr{P}}_1$; however, the price ratios are kept the same among all old users. Similarly, when a user departure happens in the system at time t, we increase the price of all users proportionally, so that the price vector $\mathbf{p}(t)$ lies in $\tilde{\mathscr{P}}_1$ throughout the simulation.

We take the simulation time unit as a day. For every 10 min, we randomly choose a user for updating its bidding strategy; for every hour, i.e., $\Delta t = 1/24$ (day), the system updates the price differentiation based on Eq. (10). In order to capture the user behavior more realistically under price differentiations, we simulate the user's response to prices as follows. When a user i updates its bidding strategy, it calculates its best response t_i^* and sets its new bid as $t_i' = t_i + K_t(t_i^* - t_i)$, where the step size K_t is chosen to be 0.2. At time $t = 0$, each user i starts with bidding $t_i = 1$ simultaneously. Subsequently, when a new user arrives in the system, it starts $t_i = K_t t_i^*$, as if its last bit was zero.

We conduct a simulation from time $t = 0$–4.1. In Fig. 4, we plot the instantaneous social welfare of the system under our feedback price control mechanism and that under the Kelly mechanism, together with the maximum achievable social welfare for comparison. In Figs. 5–8, we plot the instantaneous prices $p_i(t)$, the allocated resources $d_i(t)$, the bidding strategies $t_i(t)$, and the marginal valuations $v_i'(t)$ of the users, respectively.

From Fig. 4, we can observe that when more users join the system, the potential maximum social welfare increases, and vice-versa. Throughout the simulation, we observe that under the Kelly mechanism, when users adapt to their optimal strategies, the social welfare converges to some suboptimal value; however, under our feedback price control, users' optimal strategies and the resulting resource allocation drive the social welfare to converge to the global maximum.

In Fig. 4, we use the Kelly mechanism for a fair comparison in which the price vector always equals $(\frac{1}{N}, \ldots, \frac{1}{N})$, and choose the same user to change strategy as our approach. As we randomly let users change strategy, the social welfare

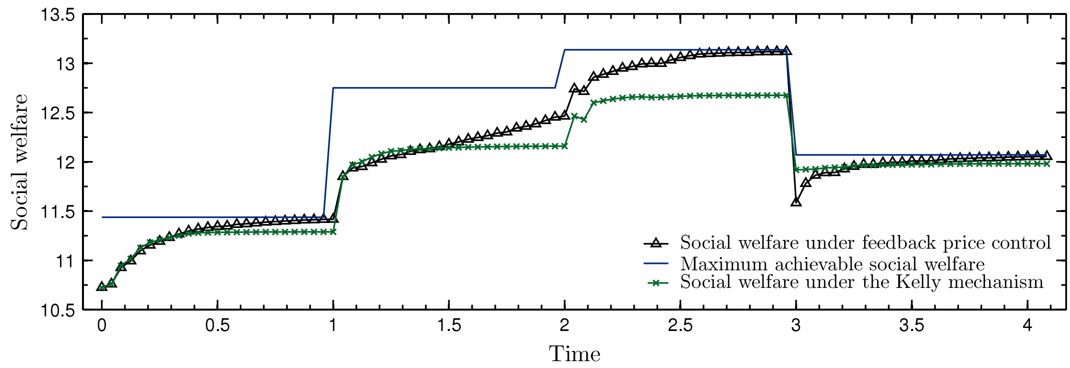

Fig. 4. Social welfare dynamics under feedback price control.

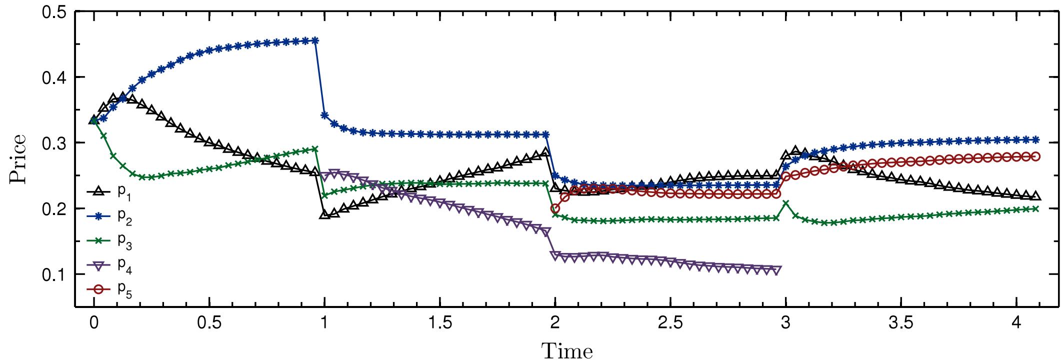

Fig. 5. The dynamics of prices.

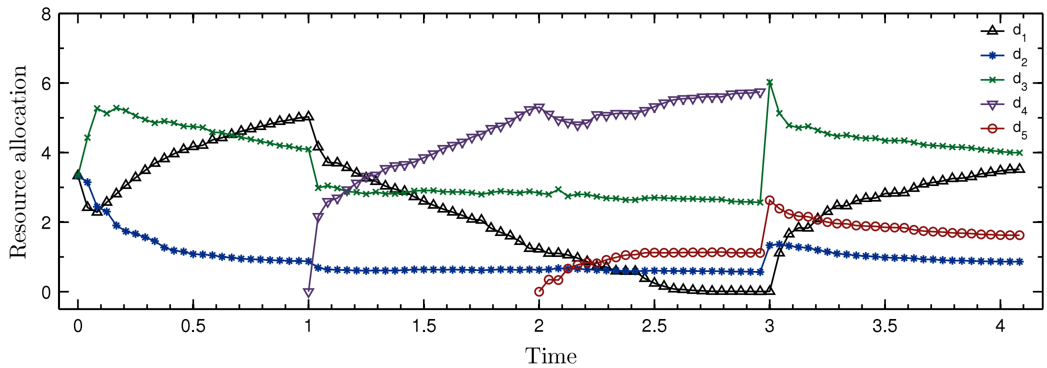

Fig. 6. The dynamics of resource allocation.

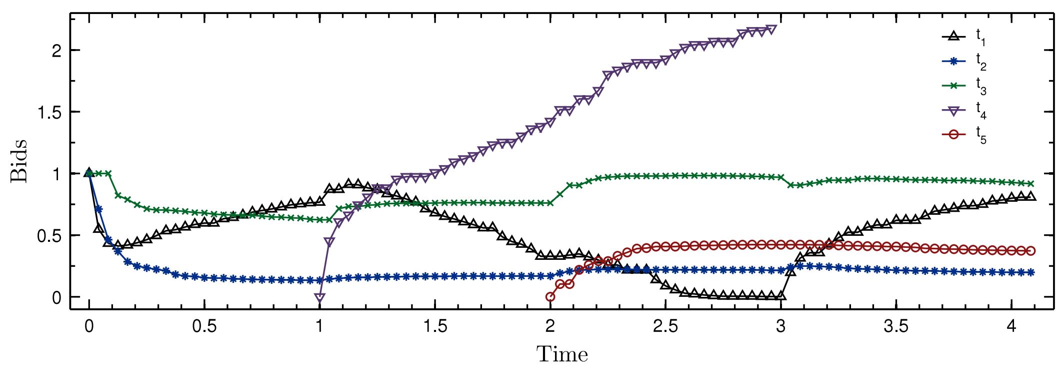

Fig. 7. Bids t_i of all users.

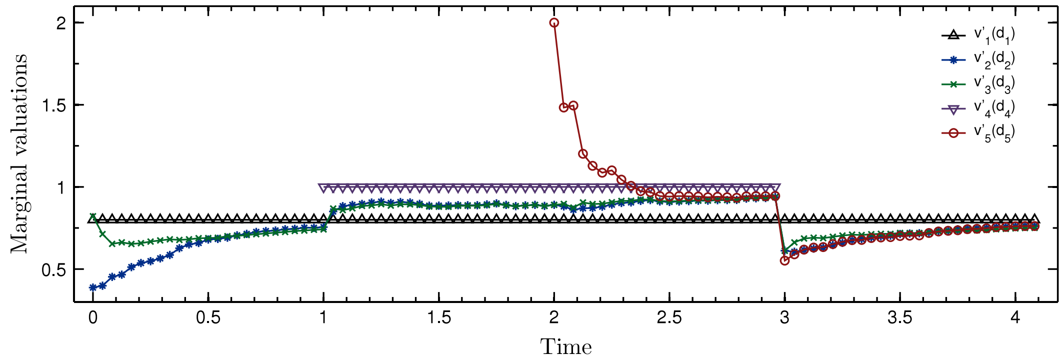

Fig. 8. The marginal valuations $v_i'(d_i)$ of all the users.

under Kelly will go to an upper bound after users' strategy reaches the Nash equilibrium. However, the social welfare of our approach goes to the optimal after several rounds of adjustment on the price vector. At time $t = 1$ and $t = 2$, the optimal social welfare increases because a new user comes into the system. At time $t = 3$, as user 4 leaves the system, both our approach and Kelly mechanism will lose social welfare because the system is out of Nash equilibrium, but our approach will adjust to social optimal again after a while.

Next, we investigate the price dynamics under the feedback control mechanism. In Figs. 5 and 6, during the first day, the prices and resource allocations adjusted to the optimal social welfare point. During the second day, the resource is mainly exchanged between user 1 and user 4, this is because $v_4'(d_4) > v_1'(d_1)$, then allocating resources to user 4 will always have more valuation than user 1, which leads user 1's resource allocation d_1 goes to 0. At time $t = 2$, user 5 arrives, and finally gets resource $d_5 = 1$ as $v_5'(d_5) = \frac{1}{d_5+1} = v_4'(d_4) = 1$. At time $t = 3$, user 4 departs; other user's allocation increased proportionally at first, but as user 1's price gradually decreases, user 1's allocation will increase until optimal social welfare is reached. In Fig. 8, we observe that the marginal valuations v_i' for each user who has resource allocation $d_i > 0$ will go to a same value.

5. Related work

Pricing [3,5,4] and resource allocation [1,15] for computing and network resources, e.g., bandwidth, have been studied extensively during the last decade. Among various proposed pricing and allocation mechanisms, the Kelly mechanism stands out as a simple and practical mechanism that achieves multiple desirable properties [14]: (1) the strategy space of the users is "simple", i.e., one-dimensional; (2) the feedback from the system to users is a single price per unit of resource; (3) for price-taking users, the competitive equilibrium optimizes the social welfare of the network system; and (4) for price-anticipating users, the resource competition game has a unique Nash equilibrium. As mentioned in [14], the stream of research of divisible resource allocation and pricing can be compared as their *strategic flexibility* and *pricing flexibility*. The Kelly mechanism is an example of low strategic flexibility (one-dimensional strategy space) and low pricing flexibility (single price), which has a bounded efficiency loss [2]. It was argued in [14] that if we increase the strategic flexibility while preserving the single price restriction, the efficiency loss can be arbitrarily large.

To achieve efficiency, many mechanisms have been designed by introducing pricing flexibility into the proportional allocation mechanism. Maheswaran and Basar [16,17] modified the proportional allocation rule by adding a parameter ϵ to the aggregate bids in the denominator and designed explicit price functions for players with different valuation functions. Nguyen and Vojnovic [18] introduced weights to the proportional allocation function and studied the revenue maximization problem for the resource provider. In our mechanism, the proportional allocation remains the same; however, the players are charged at different rates. Our objective is to maximize social welfare rather than revenue.

Another major line of extending pricing flexibility is to apply the celebrated Vickrey–Clarke–Groves (VCG) [19–21] mechanism with the proportional allocation rule. Yang and Hajek [22,23] and Johari and Tsitsiklis [14] independently designed VCG-type of mechanism with one-dimensional bids from the players. Dimakis, Jain and Walrand [24] considered the VCG mechanism in multiple divisible goods' allocation for a two-dimensional strategy space. Stoenescu and Ledyard [25] designed a mechanism that can implement efficient allocation in the sense of Pareto optimality as a Nash equilibrium in a two-dimensional strategy space. We introduce the pricing flexibility as a built-in parameter, integrated into the proportional allocation mechanism itself; therefore, we view our mechanism as a generalization of the Kelly mechanism instead of an add-on pricing mechanism to the players.

6. Future work and conclusion

In this work, we generalized the Kelly mechanism by designing a price differentiation mechanism. In particular, we consider that price-anticipating users compete under oligopolistic competitions and we model the resource competition game under price differentiations. We show that a unique Nash equilibrium exists under any positive price differentiation. Based on an optimality condition of the Nash equilibrium, we further develop a feedback price control mechanism to achieve the maximum social welfare of the system, which closes the 25% efficiency gap of the Kelly mechanism. The breakthrough of this result is that we can use the prices as a control knob to incentivize selfish users to reach a Nash equilibrium where the social welfare is optimal. In particular, our approach only takes observable parameters, e.g., the prices and bids of the users, as input and does not require the knowledge of the hidden/private valuation functions of the users. From a practical point of view, our generalization also inherits the simplicity of the Kelly mechanism; for example, we keep the strategy space of the users to be one-dimensional and a single resource price ν (in terms of tickets) needs to be broadcast to the users.

Two future directions of this work are as follows.

- Revenue Maximization: This paper's focus is mainly on maximizing the social welfare. However, revenue maximization could be an interesting orthogonal direction where resource owners are profit seeks themselves. Related questions include: what are the prices that maximize the total revenue and how does that affect the efficiency (social welfare) of the system and the fairness among the users?
- Multi-Resource Allocation: In a large scale market-oriented resource platform, resources of heterogeneous types might be available to users. Moreover, users' valuation might depend on a bundle of resources, e.g., CPU time and bandwidth. One interesting open question is to see if the price differentiation mechanism can be extended to a multi-resource scenario.

In conclusion, the generalized Kelly mechanism enables resource owners to make tradeoffs between different system objectives, e.g., fairness among users, social welfare, and revenue. We believe that it shed some light on designing pricing mechanisms that suit a large range of market-based computing and networking platforms.

Appendix. Proof of theorems

Lemma 1. *The strategy profile* $\mathbf{t}$ *is a Nash equilibrium of the resource competition game* $(\mathcal{N}, v, \mathbf{p})$ *if and only if at least two components of* $\mathbf{t}$ *are positive, and for each i, the following conditions hold:*

$$\frac{1}{p_i} v_i'\left(\frac{t_i C}{\sum_{j=1}^{N} t_j}\right)\left(1 - \frac{t_i}{\sum_{j=1}^{N} t_j}\right) = \frac{\sum_{j=1}^{N} t_j}{C}, \quad \text{if } t_i > 0; \tag{A.1}$$

$$\frac{1}{p_i} v_i'(0) \le \frac{\sum_{j=1}^{N} t_j}{C}, \quad \text{if } t_i = 0. \tag{A.2}$$

Proof of Lemma 1. The proof follows the same argument of Hajek and Gopalakrishnan [13] and Johari [15] and takes the price vector $\mathbf{p}$ as a new parameter.

Step 1: *If* $\mathbf{t}$ *is a Nash equilibrium, at least two components of* $\mathbf{t}$ *are positive.* Suppose we have a strategy profile $\mathbf{t}$ such that only $t_i > 0$ for some user i and $t_j = 0$ for all $j \neq i$. This strategy profile cannot be an equilibrium, because user i can always be better off by reducing the bid t_i slightly. However, $\mathbf{t} = \mathbf{0}$ cannot be an equilibrium as well, because any user can be better off by bidding an infinitesimal amount and obtains the whole bandwidth. Thus, in equilibrium, $\mathbf{t}$ must have at least two positive components.

Step 2: *For any* $\mathbf{p} > 0$ *and* $\mathbf{t} \ge 0$ *with at least two positive components, the function* $u_i(t_i; \mathbf{t}_{-i}, \mathbf{p})$ *is strictly concave and continuously differentiable in* t_i*, for* $t_i \ge 0$. Because $\mathbf{t}$ has at least two positive components, the utility function u_i can be written as

$$u_i(t_i; \mathbf{t}_{-i}, \mathbf{p}) = v_i\left(\frac{t_i}{t_i + \sum_{j\neq i} t_j} C\right) - p_i t_i.$$

Because $t_i/(t_i + \sum_{j\neq i} t_j)$ is a strictly increasing function of t_i (for $t_i \ge 0$) and $v_i(\cdot)$ is a strictly increasing, concave, and differentiable function by assumption, by extracting a linear function, $p_i t_i$, u_i is also a strictly increasing, concave, and differentiable function in t_i.

Step 3: Let $\mathbf{t}$ be a Nash equilibrium. By Steps 1 and 2, $\mathbf{t}$ has at least two positive components and $u_i(t_i; \mathbf{t}_{-i}, \mathbf{p})$ is strictly concave and continuously differentiable in $t_i \ge 0$. Thus t_i must be the unique maximizer of $u_i(t_i; \mathbf{t}_{-i}, \mathbf{p})$ over $t_i \ge 0$, and satisfy the first order optimality conditions:

$$\frac{\partial}{\partial t_i} u_i(t_i; \mathbf{t}_{-i}, \mathbf{p}) \begin{cases} = 0, & \text{if } t_i > 0; \\ \le 0, & \text{if } t_i = 0. \end{cases}$$

By multiplying $\sum_{j=1}^{N} t_j/C$, the above conditions become the conditions (A.1)–(A.2).

Conversely, if we have a strategy profile $\mathbf{t}$ with at least two positive components, by Step 2, we know that $u_i(t_i; \mathbf{t}_{-i}, \mathbf{p})$ is strictly concave and continuously differentiable in $t_i \geq 0$. The conditions (A.1)–(A.2) imply that t_i maximizes $u_i(t_i; \mathbf{t}_{-i}, \mathbf{p})$ over $t_i \geq 0$. Thus $\mathbf{t}$ is a Nash equilibrium. □

Proof of Theorem 1. We first show that $\mathbf{d}^*$ is the unique solution to the optimization problem (6). The proof uses Lemma 1 and follows the same argument of Johari [15].

Step 1: *The function* $\hat{v}_i$ *defined in* (7) *is strictly concave and strictly increasing over* $0 \leq d_i \leq C$. By differentiating $\hat{v}_i$, we obtain $\hat{v}_i'(d_i) = \frac{1}{p_i} v_i'(d_i)(1 - d_i/C)$. Since v_i is concave and strictly increasing, we know that $v_i'(d_i) > 0$, and that v_i' is non-increasing. Because $1 - d_i/C$ is decreasing over $0 \leq d_i \leq C$, we conclude that $\hat{v}_i'$ is nonnegative and strictly decreasing in d_i over $0 \leq d_i \leq C$, as required.

Step 2: *There exists a unique* $\mathbf{d}$ *and scalar* ρ *such that*

$$\hat{v}_i'(d_i) = \frac{1}{p_i} v_i'(d_i)\left(1 - \frac{d_i}{C}\right) = \rho, \quad \text{if } d_i > 0; \tag{A.3}$$

$$\hat{v}_i'(0) = \frac{1}{p_i} v_i'(0) \leq \rho, \quad \text{if } d_i = 0; \tag{A.4}$$

$$\sum_{j=1}^{N} d_j = C. \tag{A.5}$$

The vector $\mathbf{d}$ is the unique optimal solution to the optimization problem (6)–(7). By Step 1, we know that the optimization problem (6)–(7) has a unique optimal solution. This optimal solution $\mathbf{d}$ is uniquely identified by the optimality conditions (A.3)–(A.4) and the constraint $\sum_{i=1}^{N} d_i \leq C$. Because each $\hat{v}_i$ is strictly increasing, the constraint must be tight and satisfy (A.5). Finally, because at least one d_i is strictly positive, ρ is uniquely determined by Eq. (A.3).

Step 3: *If* $(\mathbf{d}, \rho)$ *satisfies* (A.3)–(A.5), *then* $\mathbf{t} = \rho\mathbf{d}$ *is a Nash equilibrium.* First, we show that at least two components of $\mathbf{d}$ are strictly positive and $\rho > 0$. From Eq. (A.5), we know that at least one component of $\mathbf{d}$ is strictly positive. If only one component $d_i > 0$, we know that $d_i = C$, and from Eq. (A.3), $\rho = 0$. However, since $v_i'(0) > 0$, condition (A.4) cannot hold. Thus at least two components of $\mathbf{d}$ are strictly positive and $\rho > 0$ follows from Eq. (A.3).

By Lemma 1, we only need to check the conditions (A.1)–(A.2). Using Eq. (A.5), we rewrite $\mathbf{t} = \rho\mathbf{d}$ as $\rho = \sum_{i=1}^{N} t_i/C$ and $d_i = t_i C / \sum_{i=1}^{N} t_i$. By substituting the above into (A.3)–(A.4), we obtain the conditions (A.1)–(A.2), and therefore, $\mathbf{t}$ is a Nash equilibrium.

Step 4: *If* $\mathbf{t}$ *is a Nash equilibrium, then the corresponding resource allocation* $\mathbf{d}$ *and* $\rho = \sum_{i=1}^{N} t_i/C$ *are the unique solution to* (A.3)–(A.5). We can reverse the argument of Step 3. The uniqueness of $(\mathbf{d}, \rho)$ follows by Step 2.

Step 5: *There exists a unique Nash equilibrium* $\mathbf{t}$, *and the resource allocation* $\mathbf{d}^*$ *defined by* (5) *is the unique optimal solution to* (6)–(7). Existence follows by Steps 2 and 3, and uniqueness follows by Step 4 (since the mapping from $\mathbf{t}$ to $(\mathbf{d}, \rho)$ is one-to-one). Finally, $\mathbf{d}^*$ is an optimal solution to (6)–(7) followed by Steps 2 and 4. □

Proof of Theorem 2. Let $\mathbf{t} = \frac{1}{k}\mathbf{t}^{\mathbf{p}}$. We want to prove that $\mathbf{t}$ is a Nash equilibrium of $(\mathcal{N}, v, \mathbf{q})$. Since $\mathbf{t}^{\mathbf{p}}$ is a Nash equilibrium, by Lemma 1, at least two components of $\mathbf{t}^{\mathbf{p}}$ are strictly positive, and therefore, so as the vector $\mathbf{t}$.

By Lemma 1, we know that the Nash equilibrium $\mathbf{t}^{\mathbf{p}}$ satisfies the following conditions:

$$\frac{1}{p_i} v_i'\left(\frac{t_i^{\mathbf{p}} C}{\sum_{j=1}^{N} t_j^{\mathbf{p}}}\right)\left(1 - \frac{t_i^{\mathbf{p}}}{\sum_{j=1}^{N} t_j^{\mathbf{p}}}\right) = \frac{\sum_{j=1}^{N} t_j^{\mathbf{p}}}{C}, \quad \text{if } t_i^{\mathbf{p}} > 0;$$

$$\frac{1}{p_i} v_i'(0) \leq \frac{\sum_{j=1}^{N} t_j^{\mathbf{p}}}{C}, \quad \text{if } t_i^{\mathbf{p}} = 0.$$

By substituting $p_i = \frac{1}{k} q_i$ and $t_i^{\mathbf{p}} = k t_i$ into the above, we obtain the following conditions:

$$\frac{1}{\frac{1}{k} q_i} v_i'\left(\frac{k t_i C}{\sum_{j=1}^{N} k t_j}\right)\left(1 - \frac{k t_i}{\sum_{j=1}^{N} k t_j}\right) = \frac{\sum_{j=1}^{N} k t_j}{C}, \quad \text{if } k t_i > 0;$$

$$\frac{1}{\frac{1}{k} q_i} v_i'(0) \leq \frac{\sum_{j=1}^{N} k t_j}{C}, \quad \text{if } k t_i = 0.$$

After dividing k on both sides, we obtain the following:

$$\frac{1}{q_i} v_i'\left(\frac{t_i C}{\sum_{j=1}^N t_j}\right)\left(1-\frac{t_i}{\sum_{j=1}^N t_j}\right)=\frac{\sum_{j=1}^N t_j}{C}, \quad \text{if } t_i>0;$$

$$\frac{1}{q_i} v_i'(0) \le \frac{\sum_{j=1}^N t_j}{C}, \quad \text{if } t_i=0.$$

Since $\mathbf{t}$ has at least two strictly positive components and satisfies the above stationarity conditions, by Lemma 1, we deduce that $\mathbf{t}=\frac{1}{k}\mathbf{t^p}$ is a Nash equilibrium of $(\mathcal{N}, v, \mathbf{q})$. By Theorem 1, $\mathbf{t}$ is also the unique Nash equilibrium $\mathbf{t^q}$.

Since $\mathbf{t^q}=\frac{1}{k}\mathbf{t^p}$, the proportional share rule of (3) gives the same resource allocation, i.e., $d(\mathbf{t^q})=d(\mathbf{t^p})$. Each user i's payment under $(\mathcal{N}, v, \mathbf{q})$ is $q_i t_i^{\mathbf{q}} = k p_i \frac{1}{k} t_i^{\mathbf{p}} = p_i t_i^{\mathbf{p}}$, which is the same as the payment under $(\mathcal{N}, v, \mathbf{p})$. Since users have the same valuation and payment under both games, they achieve the same amount utility as well. □

Proof of Theorem 3. Let $\mathbf{t}=\mathbf{t^p}$. We want to prove that $\mathbf{t}$ is a Nash equilibrium of $(\mathcal{N}, v, \mathbf{q})$. Since $\mathbf{t^p}$ is a Nash equilibrium, by Lemma 1, at least two components of $\mathbf{t^p}$ are strictly positive, and therefore, so as the vector $\mathbf{t}$.

To prove that $\mathbf{t}$ is a Nash equilibrium of $(\mathcal{N}, v, \mathbf{q})$, by Lemma 1, we need to show the following conditions:

$$\frac{1}{q_i} v_i'\left(\frac{t_i C}{\sum_{j=1}^N t_j}\right)\left(1-\frac{t_i}{\sum_{j=1}^N t_j}\right)=\frac{\sum_{j=1}^N t_j}{C}, \quad \text{if } t_i>0;$$

$$\frac{1}{q_i} v_i'(0) \le \frac{\sum_{j=1}^N t_j}{C}, \quad \text{if } t_i=0.$$

Since $q_i = p_i$ for all $t_i^{\mathbf{p}} > 0$, the first equation is the same as the stationarity condition of (A.1) for $\mathbf{t^p}$ being a Nash equilibrium. Since $q_i \ge p_i$ for all $t_i^{\mathbf{p}}=0$, we want to show

$$\frac{1}{q_i} v_i'(0) \le \frac{1}{p_i} v_i'(0) \le \frac{\sum_{j=1}^N t_j}{C}, \quad \text{if } t_i=0.$$

The above condition is the same as the stationarity condition of (A.2) for $\mathbf{t^p}$ being a Nash equilibrium. By Theorem 1, $\mathbf{t}$ is also the unique Nash equilibrium $\mathbf{t^q}$ of $(\mathcal{N}, v, \mathbf{p})$.

Since $\mathbf{t^q}=\mathbf{t^p}$, both mechanisms achieve the same resource allocation, i.e., $d(\mathbf{t^q})=d(\mathbf{t^p})$. Each user i's payment under $(\mathcal{N}, v, \mathbf{q})$ is

$$q_i t_i^{\mathbf{q}} = q_i t_i^{\mathbf{p}} = \begin{cases} p_i t_i^{\mathbf{p}}, & \text{if } t_i^{\mathbf{p}}>0; \\ 0, & \text{if } t_i^{\mathbf{p}}=0. \end{cases}$$

This is the same as the payment under $(\mathcal{N}, v, \mathbf{p})$. Since users have the same valuation and payment under both games, they achieve the same amount utility as well. □

Proof of Theorem 4. First, we do not restrict $\mathbf{p}$ to be in the set $\tilde{\mathcal{P}}_c$. We want to show that if $\mathbf{p}$ is defined by

$$p_i = h_i(d_i) = v_i'(d_i)\left(1-\frac{d_i}{C}\right), \quad i=1,\ldots,N, \tag{A.6}$$

the Nash equilibrium strategy profile $\mathbf{t^p}=\mathbf{d}$.

Let $\mathbf{t}=\mathbf{d}$ and $\mathbf{p}$ be defined as in Eq. (A.6). Since $\sum_{i=1}^N t_i = \sum_{i=1}^N d_i = C$, the resource allocation to the strategy profile $\mathbf{t}$ becomes $\mathbf{d}(\mathbf{t})=\mathbf{d}$ under the proportional share rule (3). Now, we want to prove that $\mathbf{t}$ is a Nash equilibrium of $(\mathcal{N}, v, \mathbf{p})$. Since $\mathbf{t}=\mathbf{d}\in\tilde{\mathcal{D}}$, at least two components of $\mathbf{t}$ are strictly positive.

To prove that $\mathbf{t}$ is a Nash equilibrium of $(\mathcal{N}, v, \mathbf{q})$, by Lemma 1, we need to show the following conditions:

$$\frac{1}{p_i} v_i'\left(\frac{t_i C}{\sum_{j=1}^N t_j}\right)\left(1-\frac{t_i}{\sum_{j=1}^N t_j}\right)=\frac{\sum_{j=1}^N t_j}{C}, \quad \text{if } t_i>0;$$

$$\frac{1}{p_i}v_i'(0) \leq \frac{\sum_{j=1}^{N} t_j}{C}, \quad \text{if } t_i = 0.$$

Since $\sum_{i=1}^{N} t_i = C$, the right hand sides of the above equal 1. By substituting $t_i = d_i$ and $p_i = v_i'(d_i)(1 - d_i/C)$ into the above, the left hand sides equal 1 too. By Theorem 1, $\mathbf{t} = \mathbf{d}$ is the unique Nash equilibrium $\mathbf{t}^{\mathbf{p}}$ of $(\mathcal{N}, v, \mathbf{p})$.

Finally, by combining the above result and the linearity of Theorem 2, we derive the bijective mapping result. □

Proof of Theorem 5. To show the optimality condition of Eq. (9), we start with the "only if" part. Because $\mathbf{d} > 0$, the optimality condition for $\mathbf{d}$ being optimal is $v_i'(d_i^*) = v_j'(d_j^*)$ for all $i, j \in \mathcal{N}$. Suppose $\mathbf{p}$ induces a Nash equilibrium that achieves $\mathbf{d}$. Then the Nash equilibrium condition of (A.1) can be rewritten as

$$\frac{1}{p_i}v_i'(d_i^*)(C - d_i^*) = \sum_{j=1}^{N} t_j^{\mathbf{p}}, \quad \forall i = 1, \ldots, N.$$

Therefore $p_i : p_j = v_i'(d_i^*)(C - d_i^*) : v_j'(d_j^*)(C - d_j^*) = C - d_i^* : C - d_j^*$ for all $i, j = 1, \ldots, N$.

Then we solve the "if" part. By Theorem 4, we know that $\mathbf{p}$ defined by Eq. (A.6), i.e., $p_i = h_i(d_i^*) = v_i'(d_i^*)(1 - \frac{d_i^*}{C})$ for all $i = 1, \ldots, N$, induces d^*. Since $v_i'(d_i^*)$ is the same for any user i, we know that $\mathbf{p}$ satisfies condition (9). Then any price vector $\hat{\mathbf{p}}$ that satisfies condition (9) can be expressed as $\hat{\mathbf{p}} = k\mathbf{p}$ for some positive constant k. By Theorem 2, we know that $\hat{\mathbf{p}}$ achieves $\mathbf{d}^*$ as well. □

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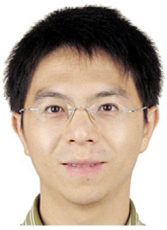

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