Tutorial 3 for ERG2040C

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Outline

- **Review**
  - Axioms of probability

- **Examples**
  - Events with different weights (problem last time)
  - Register
  - Sample with replacement
  - Cube coloring
  - Network capacity
Review--axioms of probability

- Sample space $S$
  - The set of all possible outcomes of experiments, every element of the set is an outcome

- Event $E$
  - A subset of sample space, $E \subseteq S$

- Axioms of probability
  - $0 \leq P(E) \leq 1$
  - $P(S) = 1$
  - For mutually exclusive events, $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$
  - If $S$ is finite and each one point set is assumed to have equal probability, then $P(E) = \frac{|E|}{|S|}$
Example 1: events with different weights

- Here are two teams of ordered players
  - Players A1, A2, A3, A4, A5 from Team A
  - Players B1, B2, B3, B4, B5 from Team B
  - Each player is equally likely to win or lose against player

- First round: A1 vs. B1

- Nth round:
  - The loser of the previous round is out
  - The winner of the previous round vs. the next player of the loser’s team

- The play ends when all players in one team are out

- What’s the probability that team A wins and four players in A are out?
Solution 1:

- Use a sequence of players to represent an outcome.
  - The sequence is (from left side to right side):
    - The loser in each round (from left side to right side).
    - When the play ends, put the winner and those who do not participate in the sequence (because their previous teammates killed all the competitors).
  - For example: (red--losers, green--last winner, purple--stander-by)
    - A1, A2, A3, A4, A5, B1, B2, B3, B4, B5 means that B1 wins against A1 to A5.
    - A1, A2, A3, A4, B1, B2, B3, B4, A5, B5 means team B wins while B5 is the last winner and four of B are out.
    - The relative order of A1 to A5, and B1 to B5 do not change.
Solution 1:

- Sample space:
  - For every sequence, we can find the corresponding result
  - Every sequence corresponds to different result
  - So the sample space is the all the possible sequences, it is equal to choose 5 positions from 10, \( \binom{10}{5} \)

- What is the event:
  - Team A wins and four members are out
  - A5 is the last because A5 is the last winner
  - B5 is the penultimate because four members of A are out
  - Number of sequence \( \binom{8}{4} \)
  - Probability is \( \frac{\binom{8}{4}}{\binom{10}{5}} \) WRONG!!!
Solution 1:

- How many rounds of play?—how many losers
  - A1, A2, A3, A4, A5, B1, B2, B3, B4, B5: 5 rounds, probability: $\frac{1}{25}$
  - A1, A2, A3, A4, B1, B2, B3, B4, A5, B5: 9 rounds, probability: $\frac{1}{2^9}$
- Different sequences have different probability (weight)

Consider the event:

- For every sequence, 9 rounds (A have 4 losers and B has 5 losers)
- How many sequences: $\binom{8}{4}$
- So the probability is: $\frac{\binom{8}{4}}{2^9}$
Remark

- When the sample points are of different weights, we cannot use division to calculate the probability.
- In many cases, the weight of sample point is not equal, this will happen in the following examples.
Example 2: registers

- A register contains 8 random binary digits which are mutually independent. Each digit is a zero or a one with equal probability. Calculate the probability of following event:
  - E1: no two neighboring digits are the same
  - E2: some cyclic shift of the register contents is equal to 01100110
  - E3: the register contains exactly four zeros
  - E4: there is a run of at least six consecutive ones
Solution 2:

- Sample space S:
  \[ x_1x_2x_3x_4x_5x_6x_7x_8 : x_i \in \{0, 1\} \text{ for each } i \]  
  The total number is \( 256 = 2^8 \)

- Different events
  - \( E_1 = \{01010101, 10101010\} \) and \( P(E_1) = 2/256 \)
  - \( E_2 = \{00110011, 01100110, 11001100, 10011001\} \) and \( P(E_2) = 4/256 \)
  - \( E_3 = \{x: x_1 + ... + x_8 = 4\} \) and \( P(E_3) = \frac{\binom{8}{4}}{256} \)
  - \( E_4 = \{11111111, 11111110, 11111101, 10111111, 01111111, 00111111, 01111110, 11111100\} \) and \( P(E_4) = 8/256 \)
Example 3: sample with replacement

- An urn contains $n$ white and $m$ black balls, where $n$ and $m$ are positive numbers
  
  a) If two balls are randomly withdrawn, what is the probability that they are of the same color?
  
  b) If a ball is randomly withdrawn and when replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?
  
  c) Compare the probability which is bigger.
Solution 3:

a) If the color is white, then probability is:

\[
\frac{\binom{n}{2}}{\binom{n+m}{2}} = \frac{n(n-1)}{(n+m)(n+m-1)}
\]

If the color is black, the probability is:

\[
\frac{\binom{m}{2}}{\binom{n+m}{2}} = \frac{m(m-1)}{(n+m)(n+m-1)}
\]

The total probability is:

\[
\frac{n(n-1)+m(m-1)}{(n+m)(n+m-1)}
\]
b) Since the picked ball is replaced, the probability of choosing a white ball or a back ball stay unchanged

Two while balls: \( \left( \frac{n}{n+m} \right)^2 \)

Two black balls: \( \left( \frac{m}{n+m} \right)^2 \)

The total probability is: \( \frac{n^2 + m^2}{(n+m)^2} \)

c) \( \frac{n(n-1) + m(m-1)}{(n+m)(n+m-1)} < \frac{n^2 + m^2}{(n+m)^2} \)

The probability in the putting back case is bigger!
Example 4: cube coloring

- Suppose each corner of a cube is colored blue with probability $p$, red with probability $1 - p$. Let $E$ denote the event that at least one face of the cube has all four corners colored blue.
- Find $P[E]$
Solution 4:

- **i=4**  
  - Appear on different faces: 6  
  - Diagonal: $2\times6=12$

- **i=5**  
  - $4\times6=24$

- **i=6**  
  - Diagonal: $4\times6/2=12$
Solution 4:

- The number of blue corners must be greater than 4
- Denote $E_i$ as the event that there are exactly $i$ corners colored with blue and at least one face of the cube has four corners colored blue
- Then $P(E) = \sum_{i=4}^{8} P(E_i)$

- $P(E_4) = 6p^4(1 - p)^4$
- $P(E_5) = 24p^5(1 - p)^3$
- $P(E_6) = 24p^6(1 - p)^2$
- $P(E_7) = 8p^7(1 - p)^1$
- $P(E_8) = p^8$
Example 5:

A communication network is shown. The link capacities in megabits per second (Mbps) are given by $C_1 = C_3 = 5$, and $C_2 = C_5 = 10$, $C_4 = 8$, and are the same in each direction. Information flow from the source to the destination can be split among multiple paths. Each link fails with probability $p$ independently. Let $X$ be defined as the maximum rate (in Mbits per second) at which data can be sent from the source node to the destination node. Find $P(X \geq 8)$.
Solution 5:

- If all links are working, then the maximum communication rate is 10 Mbps
- If \( X \geq 8 \), then
  - Link 1 and link 3 must be working
  - Then we have the following conditions that \( X \geq 8 \)
    - 1, 3, 2, 4, 5 work, the probability is \((1 - p)^5\)
    - 1, 3, 2, 4 work, 5 fails, the probability is \((1 - p)^4 p\)
    - 1, 3, 2, 5 work, 4 fails, the probability is \((1 - p)^4 p\)
    - 1, 3, 5, 4 work, 2 fails, the probability is \((1 - p)^4 p\)
  - So \( P(X \geq 8) = (1 - p)^2((1 - p)^3 + 3(1 - p)^2 p)\)
- What about \( P(X \geq 5) \)?