Homework # 3 Solutions

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1. We define \((m, n)\) as the result of one roll, i.e., the first dice lands on \(m\) and the second dice lands on \(n\). Obviously, we have \(P\{(m, n)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}\) for \(m = 1, \ldots, 6\) and \(n = 1, \ldots, 6\).

\[
P\{\text{the first dice lands on } 6|\text{the sum is } i\} = \frac{P\{m = 6, n = i - 6\}}{P\{m + n = i\}}
\]

Based on Equation (1), we have:

- \(i = 2, 3, 4, 5, 6:\) \(= \frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = 0;\)
- \(i = 7:\) \(= \frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,1)\}}{P\{(6,1)\} + P\{(1,6)\} + P\{(2,5)\} + P\{(5,2)\} + P\{(3,4)\} + P\{(4,3)\}} = \frac{\frac{1}{36}}{\frac{1}{36}} = \frac{1}{6};\)
- \(i = 8:\) \(= \frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,2)\}}{P\{(6,2)\} + P\{(2,6)\} + P\{(3,5)\} + P\{(5,3)\} + P\{(4,4)\} + P\{(4,4)\}} = \frac{\frac{4}{36}}{\frac{1}{36}} = \frac{4}{3};\)
- \(i = 9:\) \(= \frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,3)\}}{P\{(6,3)\} + P\{(3,6)\} + P\{(4,5)\} + P\{(5,4)\}} = \frac{\frac{6}{36}}{\frac{1}{36}} = \frac{1}{4};\)
- \(i = 10:\) \(= \frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,4)\}}{P\{(6,4)\} + P\{(4,6)\} + P\{(5,5)\}} = \frac{\frac{5}{36}}{\frac{1}{36}} = \frac{5}{3};\)
- \(i = 11:\) \(= \frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,5)\}}{P\{(6,5)\} + P\{(5,6)\}} = \frac{\frac{6}{36}}{\frac{1}{36}} = \frac{1}{2};\)
- \(i = 12:\) \(= \frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,6)\}}{P\{(6,6)\}} = \frac{\frac{1}{36}}{\frac{1}{36}} = 1;\)

2.

\[
P\{\text{both are girls|the older one is a girl}\} = \frac{P\{\text{both are girls and the older one is a girl}\}}{P\{\text{the older one is a girl}\}} = \frac{P\{\text{the older one is a girl, and the younger one is also a girl}\}}{P\{\text{the older one is a girl}\}} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{2}
\]

3. We use \(A, B,\) and \(C\) to denote the ball chosen from urn \(A,\) urn \(B,\) and urn \(C\) respectively. We use \(w\) to represent white and \(r\) to represent red. We have:

\[
P\{\text{the ball chosen from URN A was white|exactly 2 white balls were selected}\} = \frac{P\{A = w, B = w, c = r\} + P\{A = w, B = r, c = w\} + P\{A = r, B = w, c = w\}}{P\{A = w, B = w, c = r\} + P\{A = w, B = r, c = w\} + P\{A = r, B = w, c = w\}}
\]

\[
= \frac{\frac{2}{6} \times \frac{8}{12} \times \frac{4}{4} + \frac{2}{6} \times \frac{4}{12} \times \frac{1}{4} + \frac{2}{6} \times \frac{4}{12} \times \frac{1}{4} + \frac{4}{6} \times \frac{8}{12} \times \frac{1}{4}}{\frac{2}{6} \times \frac{8}{12} \times \frac{4}{4} + \frac{2}{6} \times \frac{4}{12} \times \frac{1}{4} + \frac{2}{6} \times \frac{4}{12} \times \frac{1}{4} + \frac{4}{6} \times \frac{8}{12} \times \frac{1}{4}} = \frac{7}{11}
\]

1
4. (a) \( P\{ \text{the first 2 balls are black, and the next 2 are white} \} = \frac{7}{12} \times \frac{9}{14} \times \frac{5}{16} \times \frac{7}{18} = \frac{35}{768} \)

(b) We use \( w \) and \( b \) to denote white and black respectively, and use a string to represent the colors of the first four balls, e.g., \( wwbw \) means that the first ball selected is white, the second ball selected is white, the third ball selected is black, and the fourth ball selected is black.

\[
P\{ \text{of the first 4 balls selected, exactly 2 are black} \} = P\{bbww\} + P\{bwbw\} + P\{bwbw\} + P\{wbwb\} + P\{wwbb\} \\
= \frac{7}{12} \times \frac{9}{14} \times \frac{5}{16} \times \frac{7}{18} + \frac{7}{12} \times \frac{5}{14} \times \frac{9}{16} \times \frac{7}{18} + \frac{5}{12} \times \frac{7}{14} \times \frac{9}{16} \times \frac{7}{18} + \frac{5}{12} \times \frac{7}{14} \times \frac{9}{16} \times \frac{7}{18} + \frac{5}{12} \times \frac{7}{14} \times \frac{9}{16} \times \frac{7}{18} \\
= \frac{35}{128} \tag{4}
\]

Note: “it is replaced in the urn along with 2 other balls of the same color” means that keep the selected ball in the urn and add two additional balls with the same color in the urn.

5. We use \( D \) to denote the event “a family owns a dog”, and use \( C \) to denote the event “a family owns a cat”. We have \( P(D) = 0.36 \), \( P(C) = 0.30 \), and \( P(C|D) = 0.22 \).

(a) \( P\{ \text{A randomly selected family owns both a dog and a cat} \} = P(CD) = P(D)P(C|D) = 0.36 \times 0.22 = 0.0792 \)

(b) \( P\{ \text{a randomly selected family owns a dog given that it owns a cat} \} = P(D|C) = \frac{P(CD)}{P(C)} = \frac{0.0792}{0.30} = 0.264 \)

6. We use \( F \) to denote the event “the student is female”, and use \( CS \) to denote the event “the student is majoring in computer science”. We have \( P(F) = 0.52 \), \( P(CS) = 0.05 \), and \( P(F, CS) = 0.02 \).

(a) \( P\{ \text{a student is female| the student is majoring in computer science} \} = P(F|CS) = \frac{P(F, CS)}{P(CS)} = \frac{0.02}{0.05} = 0.40 \)

(b) \( P\{ \text{a student is majoring in computer science| the student is female} \} = P(CS|F) = \frac{P(F, CS)}{P(F)} = \frac{0.02}{0.52} \approx 0.0385 \)

\[2\]