Introduction to Wireless Networks

John C.S. Lui

Department of Computer Science & Engineering
The Chinese University of Hong Kong
www.cse.cuhk.edu.hk/∼cslui

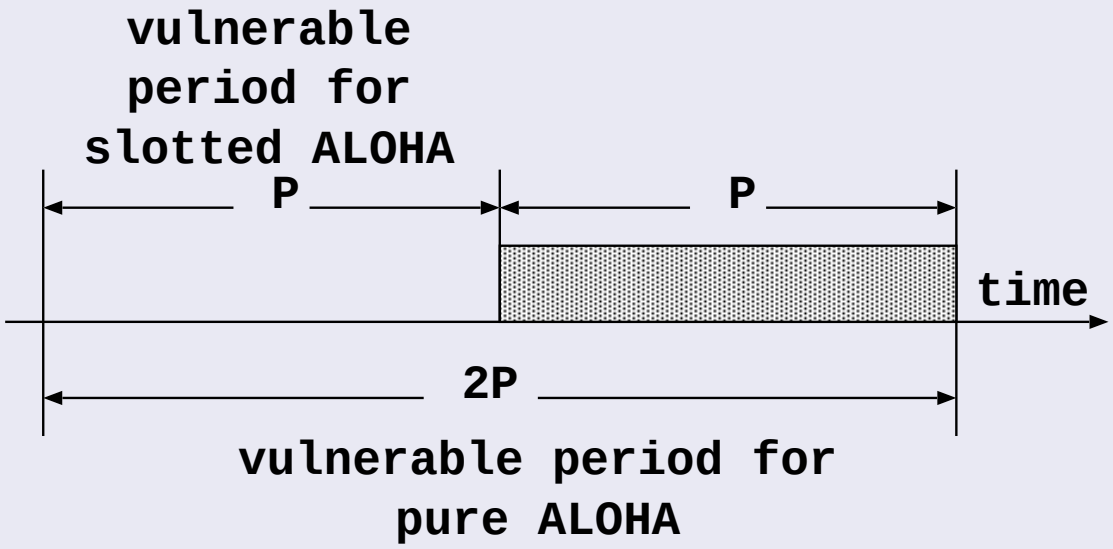
Outline

Goals

- For "pure" ALOHA, users transmit any time they desire.
- For "slotted ALOHA, the channel is to "slot" time into segments whose duration is exactly equal to transmission time of a single packet.
- The aim is to derive the
  - throughput
  - delay
  - throughput-delay tradeoff
  - stability issue

Analysis

- Let say transmission of a packet takes P secs. The vulnerable period is $2P$, as shown:



- Define:
  - S denote the throughput of the channel (e.g., average number of successful transmission per transmission period P).
  - G denote the average channel traffic (e.g., number of packet transmission attempted per P sec).

Analysis: Pure ALOHA

- Assume the total traffic (G) is Poisson and entering the channel is an independent process generated by an *infinite* population, then

$$S = G\,\text{Prob[no additional packet in vulnerable period]} = Ge^{-2G}. \tag{1}$$

  This was obtained by Abramson for ALOHA.
- What is the maximum S? Taking $\frac{\partial S}{\partial G}$ and equate to 0, maximum occurs at $G = 1/2$ and we have:

$$S^* = \frac{1}{2}e^{-1} \approx 0.184. \tag{2}$$

- Comment on the physical meaning and the efficiency of pure ALOHA.

Analysis: Slotted Aloha

- Time is "slotted" and all users are synchronized to these slot intervals.
- When a packet arrives within a slot, user delays the transmission until next time slot. Therefore, the vulnerable period is P only.
- Using similar assumption that the total traffic is a Poisson process:

$$S = Ge^{-G}. \tag{3}$$

- To obtain the maximum value of S, use similar technique, the maximum occurs at $G = 1$, we have

$$S^* = e^{-1} = 0.368. \tag{4}$$

It is twice that of pure ALOHA.

Throughput for pure and slotted ALOHA

- Consider the throughput vs. offered traffic

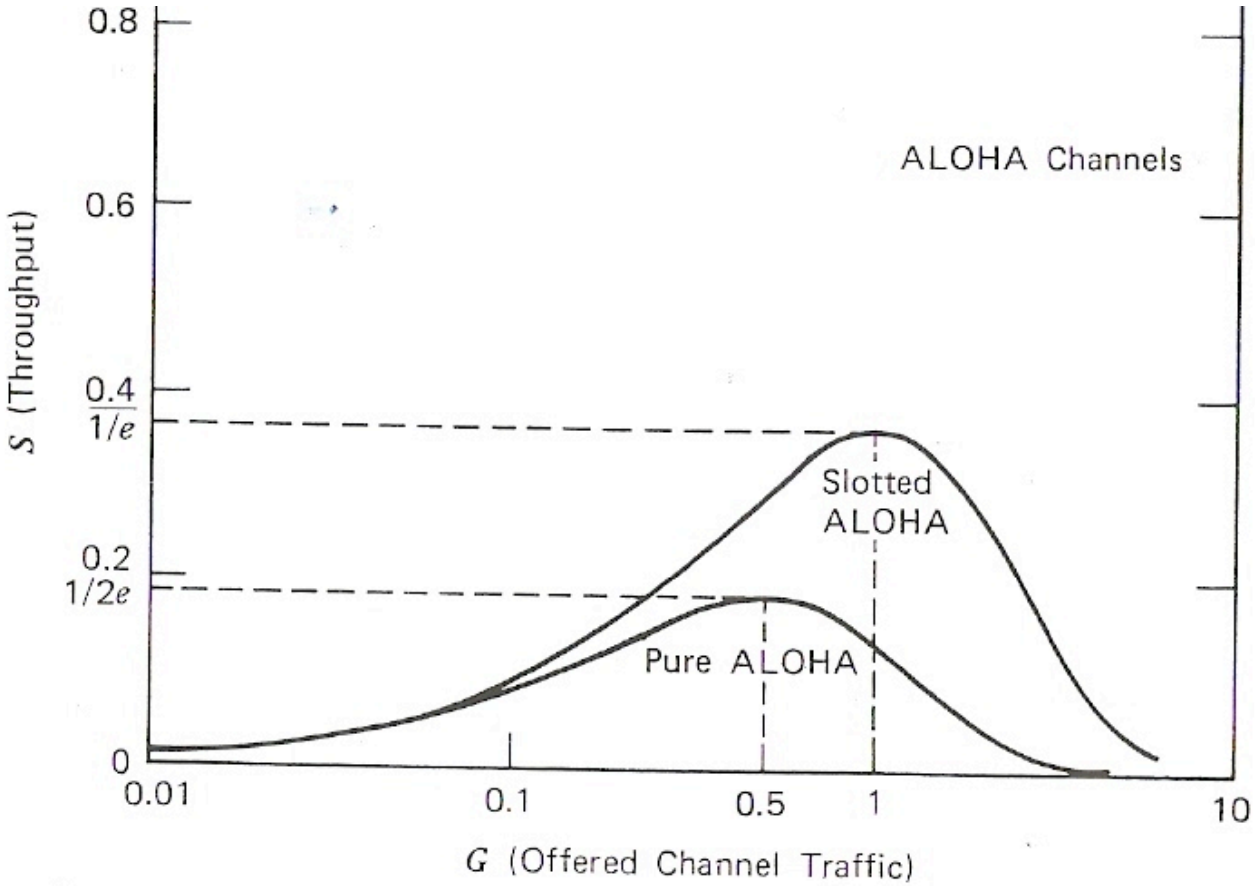


Figure 5.35 Throughput for pure and slotted ALOHA.

Analysis: continue

- Consider a *finite* population model with M independent users in a slotted ALOHA.
- User's packet transmission (old or new packet) as a sequence of independent Bernoulli trials, or G_m be the probability the mth user transmits a packet in any give slot, where $m = 1, \ldots, M$.
- G_m can be viewed as the average traffic (per slot) for the mth user. So the average offered load is $G = \sum_{i=1}^{M} G_m$.
- Let S_m be the probability that the mth user's packet is successfully transmitted. Similarly, $S = \sum_{i=1}^{m} S_m$ is the system throughput (per slot). We have:

$$S_m = G_m \prod_{i \neq m} (1 - G_i) \quad m = 1, 2, \ldots, M. \tag{5}$$

The set of M equations has a solution set, $\{S_m\}$, which defines the allowable mixtures of source rates which this channel can support.

Example 1

- Assume all users are statistically identical. Then $S_m = S/M$ and $G_m = G/M$, we have

$$S = G\left(1 - \frac{G}{M}\right)^{M-1}.$$

  Also, as $M \to \infty$, $S \to Ge^{-G}$. So as M increases, throughput of finite user model approaches to the Poisson throughput of infinite users.
- The above express, maximum S^* occurs when $M = 1$ and $S = G$.
- Define $g = \prod_{i=1}^{M}(1 - G_i)$, we have

$$S_m = \frac{G_m}{1 - G_m} g. \quad m = 1, 2, \ldots, M.$$

  This set of M equations define the set of achievable throughput $\{S_m\}$ in terms of offer load $\{G_m\}$.

Condition to achieve maximum S

- Previous M equations define a region in the $M-$dimensional space whose coordinates are $G_1, \ldots, G_M$.
- The boundary to this region defines the maximum throughput for S_m when all other $S_i (i \neq m)$ are fixed at allowable values. This can be found by setting the Jacobian (determinant) $\boldsymbol{J}$ to zero with:

$$\frac{\partial S_j}{\partial G_i} = \begin{cases} \frac{g}{1-G_i} & i = j \\ \frac{-gG_i}{(1-G_j)(1-G_i)} & i \neq j \end{cases}$$

- After some algebra, we find

$$\begin{aligned} \boldsymbol{J} &= g^{M-2} \begin{vmatrix} (1-G_1) & -G_1 & -G_1 & \cdots & -G_1 \\ -G_2 & (1-G_2) & -G_2 & \cdots & -G_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -G_M & -G_M & -G_M & \cdots & (1-G_M) \end{vmatrix} \\ &= g^{M-2}(1 - G_1 - G_2 - \cdots - G_M). \end{aligned}$$

Condition to achieve maximum S

- Applying $\boldsymbol{J} = 0$, we have the following general condition on the set of offered loads that achieves the set of maximum throughput values:

$$G = \sum_{m=1}^{M} G_i = 1.$$

- The above equation gives us our throughput performance contour.

Example 1

- Let $M = 1$, we then have $S = G = 1$.
- This is obvious the best we can do that a user never collides or destroys his own packet.

Example 2

- The M users form two groups such that m_1 of them have a rate S_1 and $m_2 = M - m_1$ of them have a rate S_2.
- $S = m_1 S_1 + m_2 S_2$ and $G = m_1 G_1 + m_2 G_2$. The M equations reduce

$$\begin{aligned} S_1 &= G_1(1-G_1)^{m_1-1}(1-G_2)^{m_2} \\ S_2 &= G_2(1-G_2)^{m_2-1}(1-G_1)^{m_1}. \end{aligned}$$

  This defines the set of permitted rates $\{S_1, S_2\}$ as a function of offer loads $\{G_1, G_2\}$.
- Imposing the optimization condition $G = 1$, we may solve for S_1 and S_2 for any given value of $G_1 \in [0, 1/m_1]$ and $G_2 = (1 - m_1 G_1)/m_2$.

Throughput Contour:

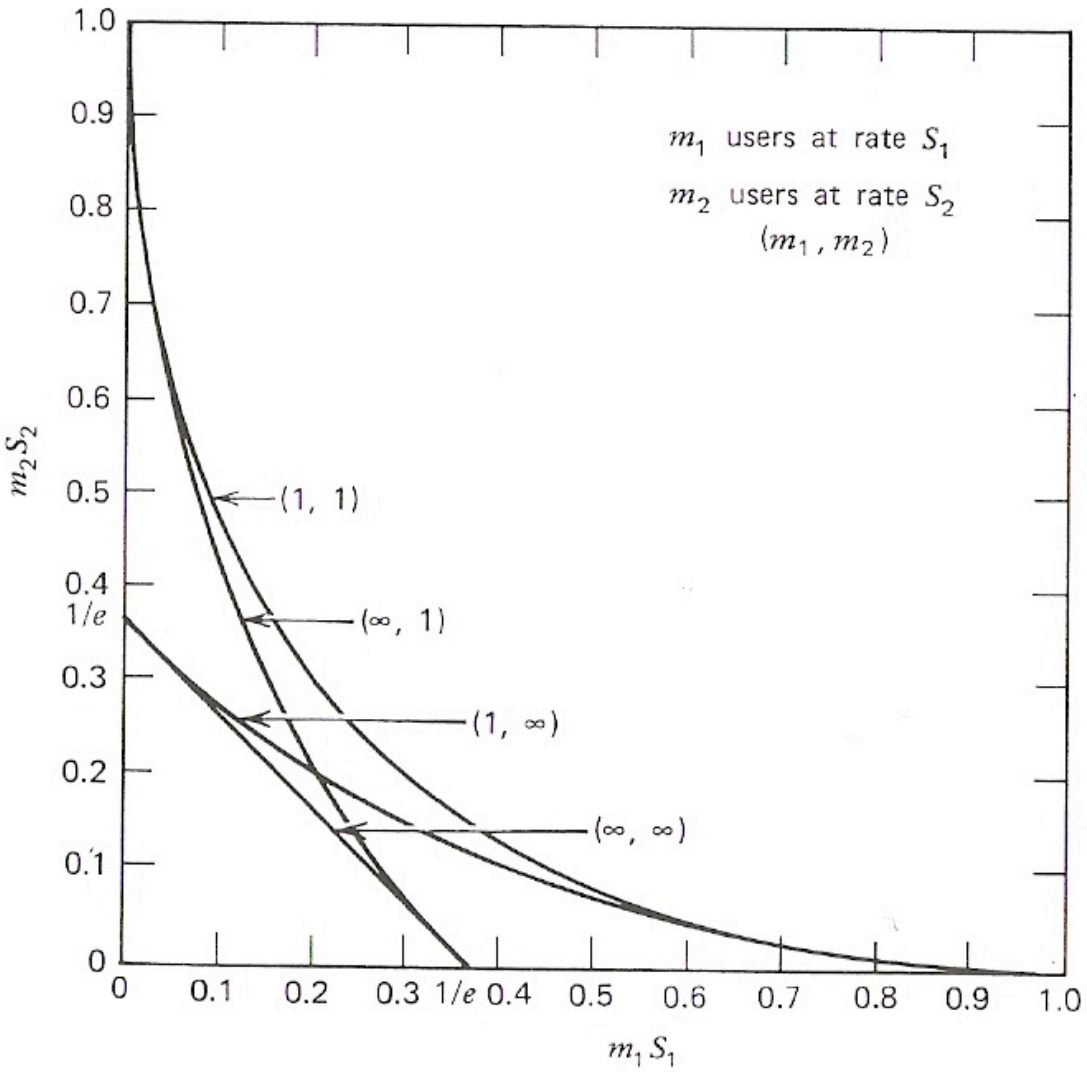


Figure 5.36 Allowable source rates for slotted ALOHA.

Throughput Contour:

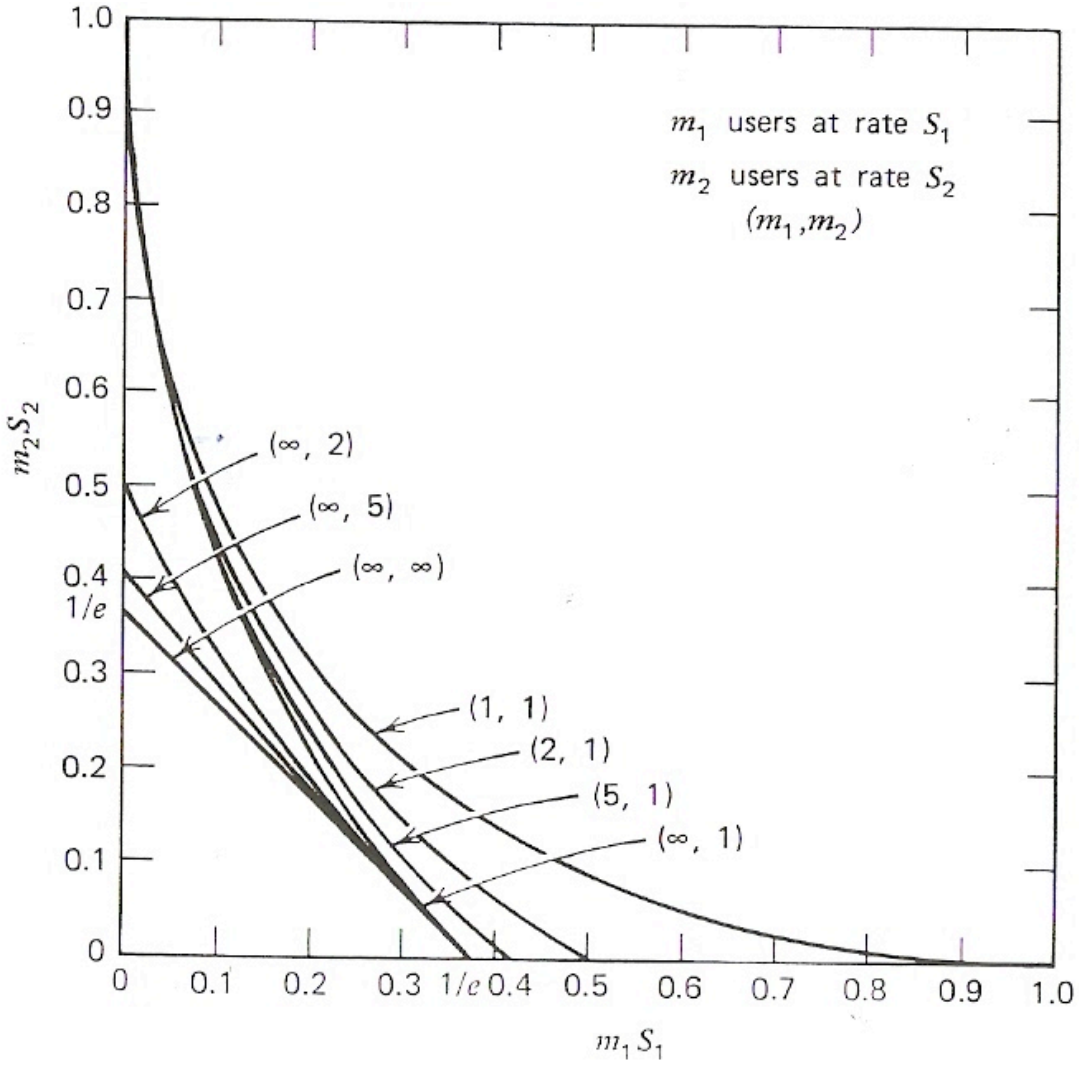


Figure 5.37 Allowable source rates for slotted ALOHA.

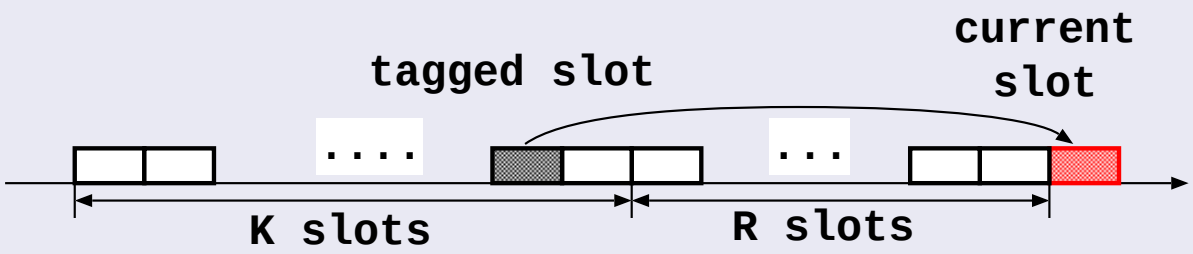
3

Slotted ALOHA with Delay

- Assume an infinite number of users who collectively form a source that generates packets independent of the channel.
- This source generates V packets per slot from the distribution $v_k = P[V = k]$ with mean of S packets per slot.
- Packet is of constant length requiring 1 slot time to transmit.
- Let R be the number of slots it takes to check whether a transmitted packet is successful or not (e.g., satellite).
- If the packet was destroyed, user retransmits and randomly choose one of the next K slots to retransmit (so as to avoid colliding again with other collided packets). Thus, the retransmission will take place either $R + 1$ to $R + K$ slots after the initial transmission.
- Total packets (new and retransmit pkt) per slot is L, where $p_k = P[L = k]$ with mean traffic of G packets per slot.
- What is the maximum throughput S, the average packet delay T?

Analysis

- Goal: derive $p_k = P[\text{exactly } k \text{ transmitted packets per slot}]$ or its $Z-$transform. Consider the following figure:



  The traffic of "current slot" is due to (a) retransmission from any of the "K-slots" and (b) new packets generate in the current slot.

- Let $q_m =$ Probability that exactly m packets which are transmitted in the current slot due to retransmission from the tagged slot, for $m = 0, 1, 2, \ldots$.
- Define Z-transforms:
  - $P(z) = \sum_{k=0}^{\infty} p_k z^k$; $q(z) = \sum_{m=0}^{\infty} q_m z^m$; $V(z) = \sum_{k=0}^{\infty} v_k z^k$.

Since transmission in slots are *independent*, let us first derive q_0:

$$q_0 = p_0 + p_1 + \sum_{k=2}^{\infty} p_k \left(1 - \frac{1}{K}\right)^k.$$

For q_1, we have:

$$q_1 = \sum_{k=2}^{\infty} p_k k \left(\frac{1}{K}\right) \left(1 - \frac{1}{K}\right)^{k-1}.$$

For the rest q_m, where $m \geq 2$, we have:

$$q_m = \sum_{k=m}^{\infty} p_k \binom{k}{m} \left(\frac{1}{K}\right)^m \left(1 - \frac{1}{K}\right)^{k-m} \quad m \geq 2.$$

The $Z-$transform of q_m, or $q(Z)$ is

$$q(Z) = q_0 + q_1 Z + \sum_{m=2}^{\infty} \left[\sum_{k=m}^{\infty} p_k \binom{k}{m} \left(\frac{1}{K}\right)^m \left(1 - \frac{1}{K}\right)^{k-m} \right] Z^m$$

Adding and subtracting $m = 0$ and $m = 1$ terms, we have

$$\begin{aligned} q(Z) &= q_0 + q_1 Z - \sum_{k=0}^{\infty} p_k \left(1 - \frac{1}{K}\right)^k - z \sum_{k=1}^{\infty} p_k k \left(\frac{1}{K}\right)^m \left(1 - \frac{1}{K}\right)^{k-1} \\ &\quad + \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} p_k \binom{k}{m} \left(\frac{1}{K}\right)^m \left(1 - \frac{1}{K}\right)^{k-m} \right] Z^m \end{aligned}$$

Substituting q_0 and q_1 and interchanging the order of summation in the double sum:

$$q(Z) = p_0 + p_1 - \left[p_0 + p_1\left(1 - \frac{1}{K}\right)\right] - z\frac{p_1}{K} + \sum_{k=0}^{\infty} p_k \sum_{m=0}^{k} \binom{k}{m} \left(\frac{z}{K}\right)^m \left(1 - \frac{1}{K}\right)^{k-m}$$

Reorganizing the binomial expansion, we get

$$q(Z) = \frac{p_1}{K}(1 - Z) + \sum_{k=0}^{\infty} p_k \left[1 - \frac{1}{K} + \frac{z}{K}\right]^k$$

Taking the definition of $P(Z)$, we have:

$$q(Z) = \frac{p_1}{K}(1 - Z) + P\left(1 - \frac{1 - Z}{K}\right).$$

Now, the number of packets to be transmitted in the current slot is the due to (1) the K slots in the tagged slot region; (2) number of newly generated packets in the current slot, expressing the relationship of these *random variables*:

$$\tilde{p} = \tilde{q} + \cdots \tilde{q} + \tilde{v}.$$

Expressing in $Z-$ transform, we have:

$$P(Z) = [q(Z)]^K \, V(Z).$$

Or

$$P(Z) = \left[\frac{p_1}{K}(1 - Z) + P\left(1 - \frac{1 - Z}{K}\right)\right]^K V(Z).$$

Since $P(Z)$ is a moment generating function, we can obtain various moments and "crank" out probabilities.

Analysis with Delay in Retransmission

Since S is the throughput of the channel (e.g., average number of successful transmission per transmission period P), and G is the average offered traffic (e.g., number of packet transmission attempted per P sec), we have:

- S/G : probability of successful packet transmission,
- G/S : average number of transmission until success.

Let E be the average number of *extra* transmissions a packet incurs (i.e., collision), we have:

$$1 + E = G/S.$$

We want to derive E. To do this, let us define:

- q : P[newly generated packet is successfully transmitted]
- q_t : P[previously collided packet is successfully transmitted]

Let x_i be the probability that a packet collides exactly i times before successful transmission, we have:

$$x_i = (1-q)(1-q_t)^{i-1} q_t \qquad i = 1, 2, \ldots,$$

So

$$E = \sum_{i=1}^{\infty} i\, x_i = \frac{1-q}{q_t}.$$

We can now express S, the channel throughput:

$$S = \frac{G}{1+E} = G\left(\frac{q_t}{q_t + 1 - q}\right). \tag{6}$$

Remark: If we can derive q and q_t, then we have S.

Derive q: Based on previous analysis and definition of q_m, we have

$$q = q_0^K e^{-S}$$

where q_0 can be expressed as:

$$\begin{aligned} q_0 &= e^{-G} + Ge^{-G} + \sum_{i=2}^{\infty} \frac{G^i}{i!} e^{-G} \left(1 - \frac{1}{K}\right)^i \\ &= e^{-G} + Ge^{-G} + e^{-G} \left[e^{G(1-1/K)} - 1 - G\left(1 - \frac{1}{K}\right)\right] \\ &= e^{-G/K} + \frac{G}{K} e^{-G}. \end{aligned}$$

Substituting this into the expression of q, we have:

$$q = \left(e^{-G/K} + \frac{G}{K} e^{-G}\right)^K e^{-S}. \tag{7}$$

Derive q_t: the probability that previously collided packet is successfully transmitted. It has three factors:

- no packet which collided with it in the tagged slot again collides with it in the current slot.
- no packets from one of the $K-1$ other possible tagged slots collides with this packet in the current slot.
- no new packet is generated in the current slot.

Let q_c be the probability of the *first* event, we have

$$q_c = \sum_{k=1}^{\infty} \left(\frac{G^k}{k!}\right) \left(\frac{e^{-G}}{1-e^{-G}}\right) \left(\frac{K-1}{K}\right)^k = \frac{e^{-G/K} - e^{-G}}{1-e^{-G}}.$$

For second event, the probability is q_0^{K-1}.
For third event, the probability is e^{-S}.
We have

$$q_t = q_c q_0^{K-1} e^{-S} = \left[\frac{e^{-G/K} - e^{-G}}{1-e^{-G}}\right] \left[e^{-G/K} + \frac{G}{K} e^{-G}\right]^{K-1} e^{-S}. \quad (8)$$

Note that Equations (6), (7) and (8) need to be solved *numerically* so as to derive S, q and q_t.

To derive T, the average number of slots to successfully deliver a packet, we know that:

- the time needed for one successful transmission $(R+1)$ slots,
- the extra transmission due to collision.

Therefore, T is:

$$T = (R+1) + \left[\frac{K+1}{2} + (R+1)\right] E = (R+1) + \frac{1-q}{q_t}\left[R+1+\frac{K-1}{2}\right]. \quad (9)$$

Remark: We now have the performance measures of S and T as a function of K, R and G.

To illustrate (a) S vs. G; (b) effect of K; (c) effect on T.

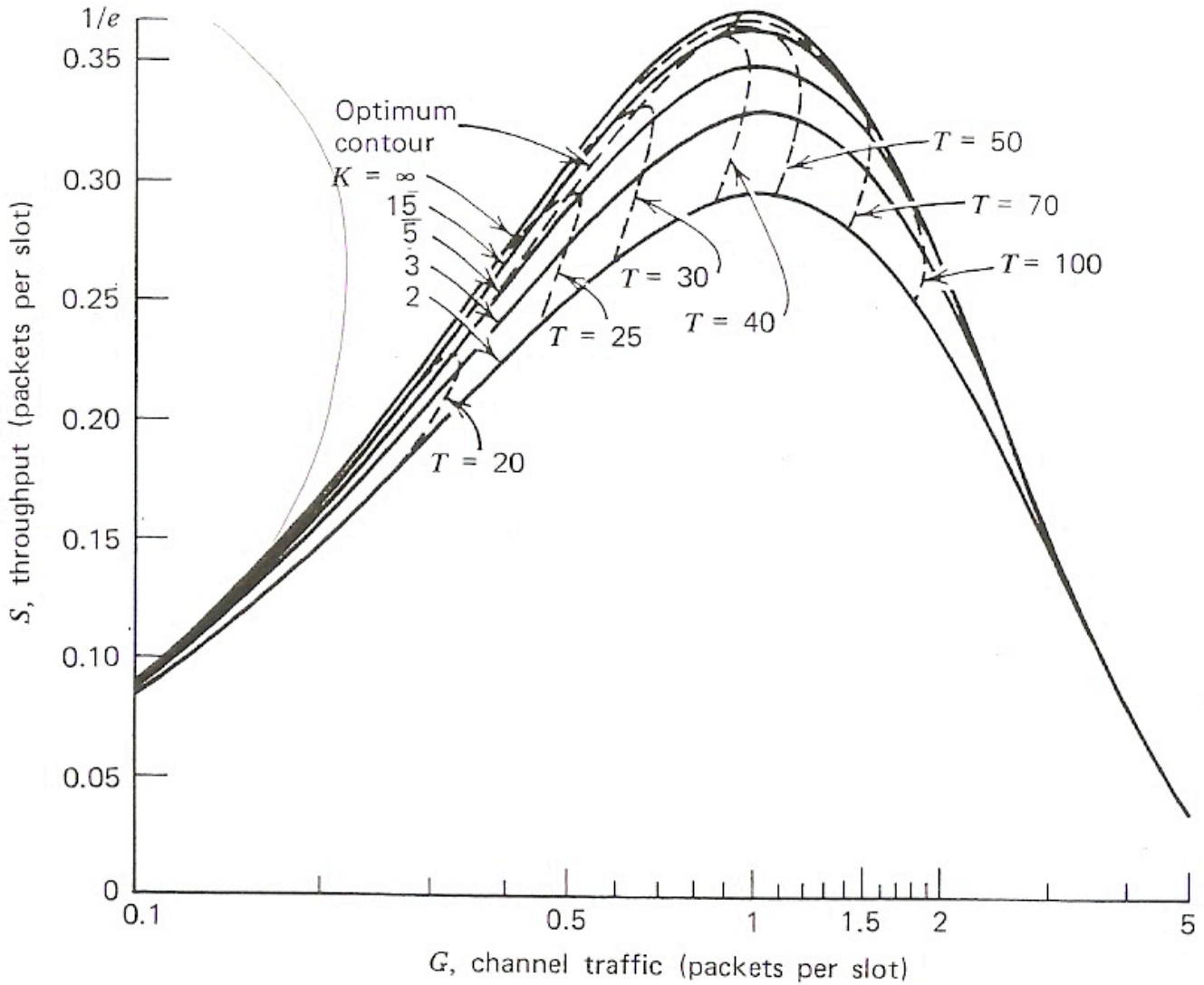


Figure 5.38 Throughput as a function of channel traffic.

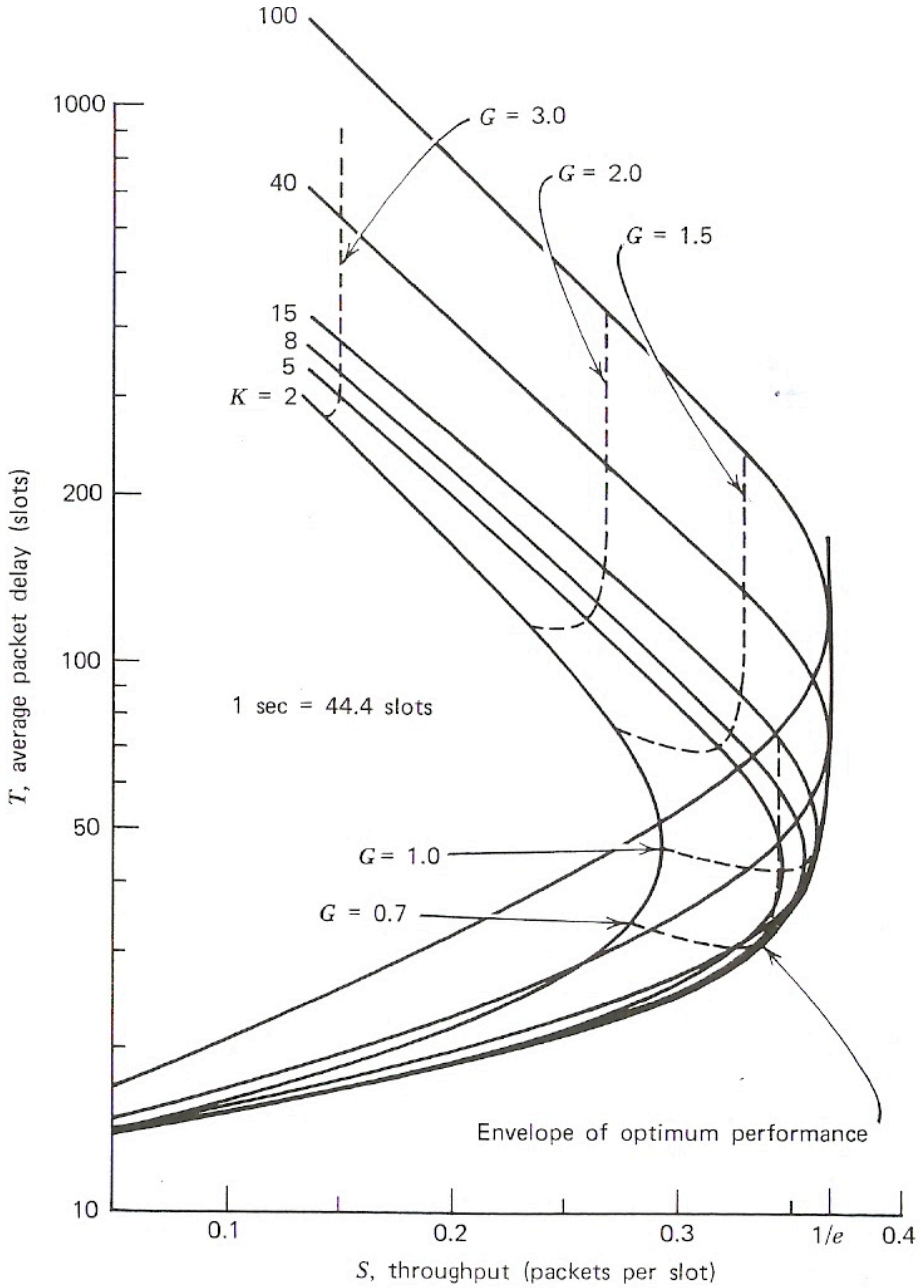


Figure 5.39 Delay-throughput tradeoff.

Two groups of users: a group of "small" users ($m_1 = \infty$) and one large user ($m_2 = 1$), with $S_\infty = \lim_{m_1 \to \infty} m_1 S_1$ and their $G_\infty = m_1 G_1$. The large user throughput is S_2 and offered load is G_2. In her $T = (S_\infty T_\infty + S_2 T_2)/S$.

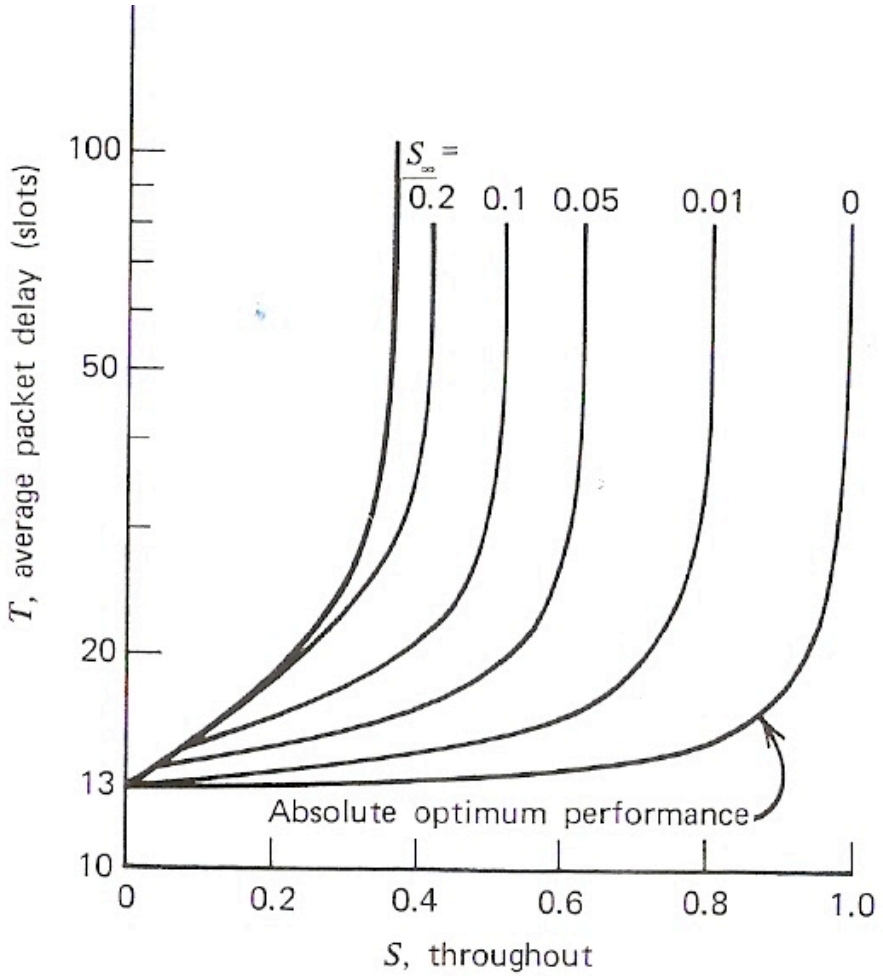


Figure 5.40 Optimum delay-throughput tradeoffs.

Model

Previous analysis assumes equilibrium conditions. Now consider

- Slotted ALOHA with large (but finite) number (M) of active terminals.
- Each terminal has one buffer space and only when the buffer is empty a new packet can be generated with probability σ per slot.
- Let $N(t)$ be the r.v. representing the total number of nonempty terminals (or channel backlog) and $S(t)$ be the combined input rates of packets into all terminals at time t. E.g., $N(t) = n$, then $[N(t), S(t)] = [n, (M - n)\sigma]$. This is a *linear feedback model*.
- Packets that collide are retransmitted (after a round-trip propagation delay of R slots) in one of the next K slots, each being chosen at random with probability $1/K$.
- The retransmission takes place on the average $R + (K + 1)/2$ slots after the previous transmission.

Analysis

- The model is hard to analyze. We approximate the backlogged packet independently retransmits with probability p where

$$p = \frac{1}{R + (K+1)/2}.$$

- Define S_{out} be the throughput rate of the channel, which is the probability of exact one successful packet transmission in a slot. If $[N(t), S(t)] = [n, (M-n)\sigma]$, then

$$S_{out} = (1-p)^n (M-n)\sigma(1-\sigma)^{M-n-1} + np(1-p)^{n-1}(1-\sigma)^{M-n} \quad (10)$$

- In the limit when $M \to \infty$ and $\sigma \to 0$ such that $M\sigma = S < \infty$, we have the *infinite population model* in which packets are generated over the channel at the Poisson rate S. We rewrite the equation to

$$S_{out} = (1-p)^n S e^{-S} + np(1-p)^{n-1} e^{-S}. \quad (11)$$

This expression is very accurate even for finite M if $\sigma << 1$ and if we replace $S = M\sigma$ by $S = (M-n)\sigma$.

For a fixed K, we show the behavior of S_{out} as function of the channel load $[n, S]$

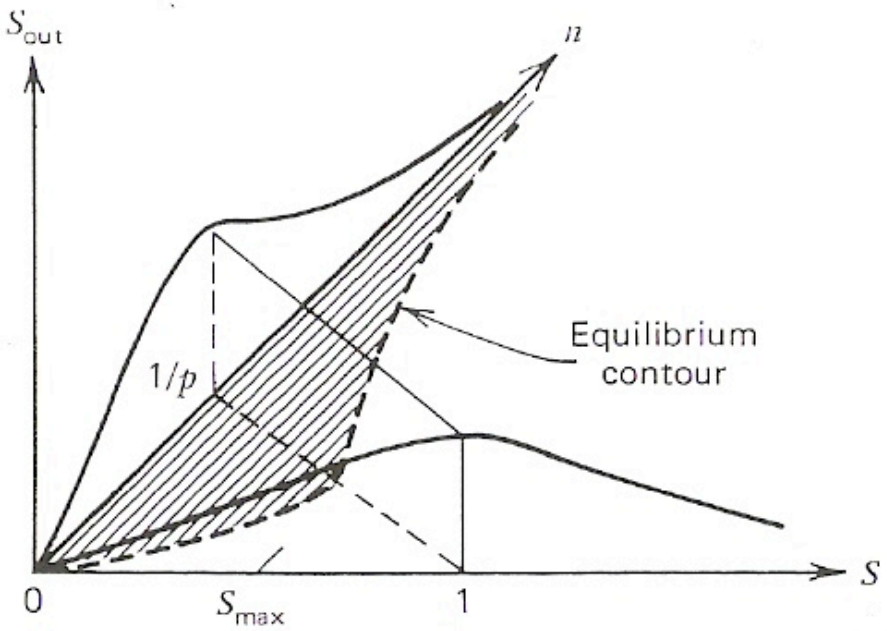


Figure 5.42 Channel throughput rate as a function of load and backlog.

- There is an equilibrium contour in the (n, S) "phase plane" on which the input rate S is equal to the system throughout S_{out}.
- In the shaded region, $S_{out} > S$; elsewhere $S_{out} < S$ (the system capacity is exceeded!).

The area of the shaded region (where $S_{out} > S$) may be increased by increasing K

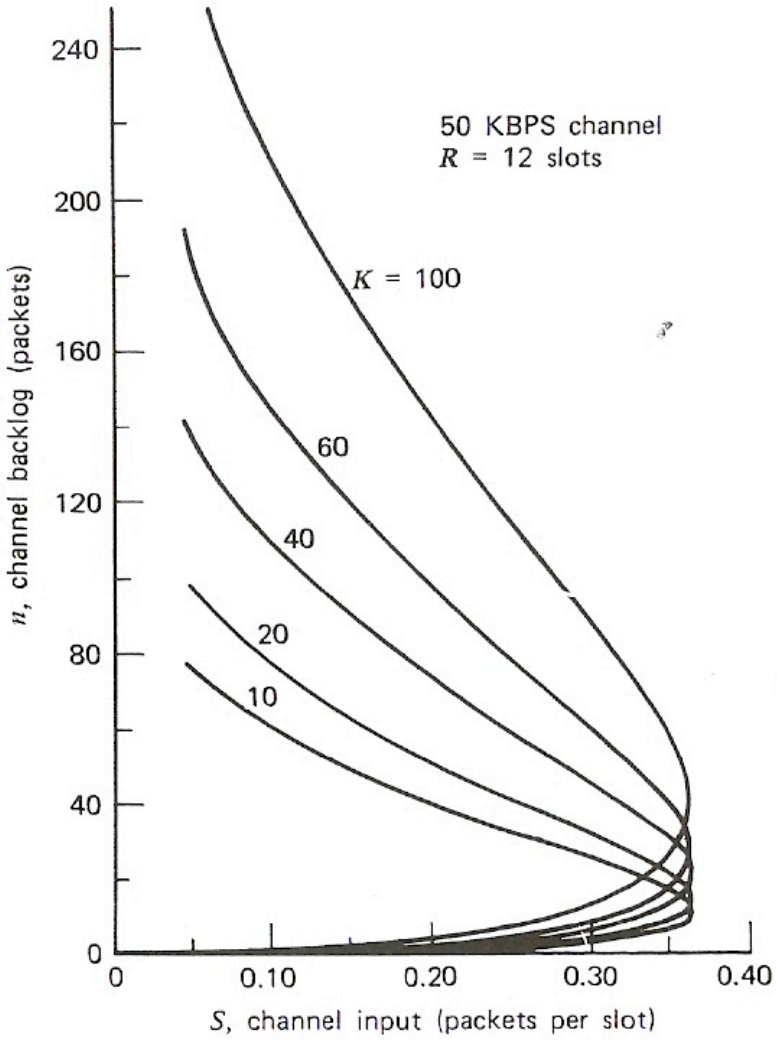


Figure 5.43 Equilibrium contours.

We define the channel load line in the (n, S) plane as the line $n = M - (S/\sigma)$. A channel is said to be *"stable"* when its load line intersects the equilibrium contour in exactly one place.

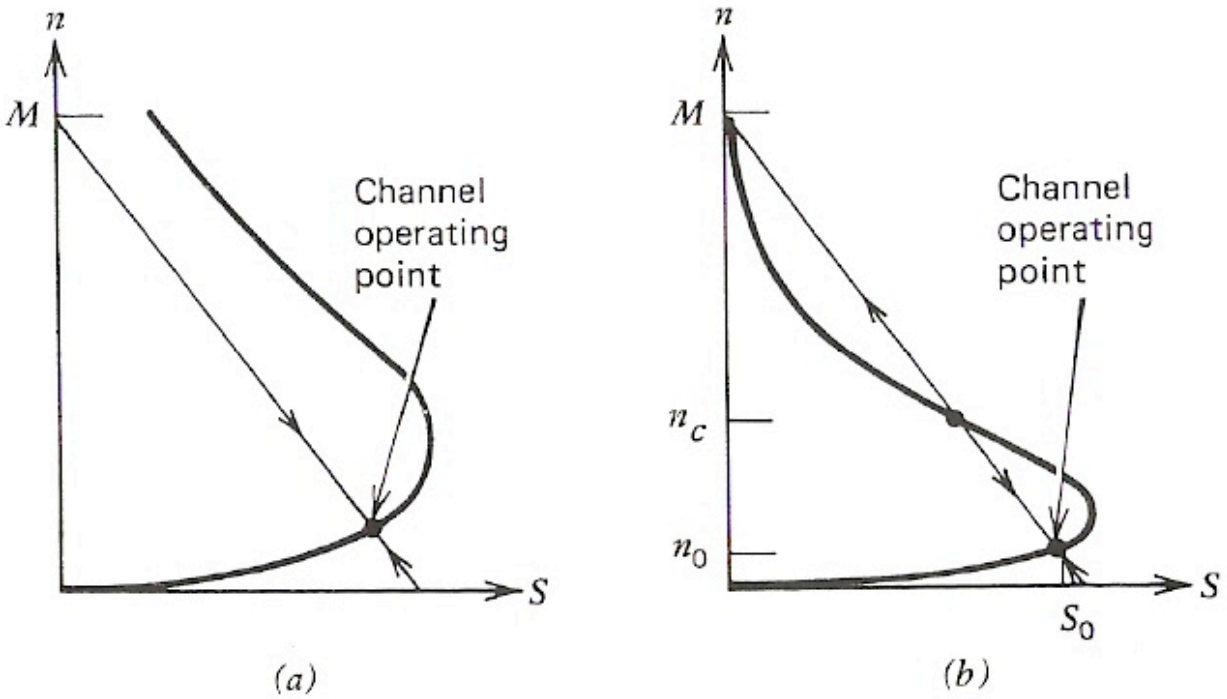


Figure 5.46 Channel stability. (*a*) A stable channel. (*b*) An unstable channel.