Introduction to Wireless Networks

John C.S. Lui

Department of Computer Science & Engineering
The Chinese University of Hong Kong
www.cse.cuhk.edu.hk/~cslui
Outline

1. Aloha and Slotted Aloha
2. Slotted ALOHA with Delay
3. Stability in ALOHA
Goals

- For "pure" ALOHA, users transmit any time they desire.
- For "slotted ALOHA, the channel is to "slot" time into segments whose duration is exactly equal to transmission time of a single packet.
- The aim is to derive the
  - throughput
  - delay
  - throughput-delay tradeoff
  - stability issue
Analysis

Let say transmission of a packet takes $P$ secs. The vulnerable period is $2P$, as shown:

Define:

- $S$ denote the throughput of the channel (e.g., average number of successful transmission per transmission period $P$).
- $G$ denote the average channel traffic (e.g., number of packet transmission attempted per $P$ sec).
Analysis: Pure ALOHA

- Assume the total traffic ($G$) is Poisson and entering the channel is an independent process generated by an *infinite* population, then

$$S = G \text{Prob}[\text{no additional packet in vulnerable period}] = Ge^{-2G}.$$  \hspace{1cm} (1)

This was obtained by Abramson for ALOHA.

- What is the maximum $S$? Taking $\frac{\partial S}{\partial G}$ and equate to 0, maximum occurs at $G = 1/2$ and we have:

$$S^* = \frac{1}{2} e^{-1} \approx 0.184.$$  \hspace{1cm} (2)

- Comment on the physical meaning and the efficiency of pure ALOHA.
Analysis: Slotted Aloha

- Time is "slotted" and all users are synchronized to these slot intervals.
- When a packet arrives within a slot, user delays the transmission until next time slot. Therefore, the vulnerable period is $P$ only.
- Using similar assumption that the total traffic is a Poisson process:

$$S = Ge^{-G}.$$  \hspace{1cm} (3)

- To obtain the maximum value of $S$, use similar technique, the maximum occurs at $G = 1$, we have

$$S^* = e^{-1} = 0.368.$$ \hspace{1cm} (4)

It is twice that of pure ALOHA.
Consider the throughput vs. offered traffic

Figure 5.35 Throughput for pure and slotted ALOHA.
Analysis: continue

Consider a *finite* population model with \( M \) independent users in a slotted ALOHA.

User’s packet transmission (old or new packet) as a sequence of independent Bernoulli trials, or \( G_m \) be the probability the \( m^{th} \) user transmits a packet in any give slot, where \( m = 1, \ldots, M \).

\( G_m \) can be viewed as the average traffic (per slot) for the \( m^{th} \) user. So the average offered load is \( G = \sum_{i=1}^{M} G_m \).

Let \( S_m \) be the probability that the \( m^{th} \) user’s packet is successfully transmitted. Similarly, \( S = \sum_{i=1}^{m} S_m \) is the system throughput (per slot). We have:

\[
S_m = G_m \prod_{i \neq m} (1 - G_i) \quad m = 1, 2, \ldots, M.
\]  

(5)

The set of \( M \) equations has a solution set, \( \{ S_m \} \), which defines the allowable mixtures of source rates which this channel can support.
Example 1

- Assume all users are statistically identical. Then $S_m = S/M$ and $G_m = G/M$, we have

$$S = G \left(1 - \frac{G}{M}\right)^{M-1}.$$ 

Also, as $M \to \infty$, $S \to Ge^{-G}$. So as $M$ increases, throughput of finite user model approaches to the Poisson throughput of infinite users.

- The above express, maximum $S^*$ occurs when $M = 1$ and $S = G$.

- Define $g = \prod_{i=1}^{M}(1 - G_i)$, we have

$$S_m = \frac{G_m}{1 - G_m} g. \quad m = 1, 2, \ldots, M.$$ 

This set of $M$ equations define the set of achievable throughput $\{S_m\}$ in terms of offer load $\{G_m\}$. 
Condition to achieve maximum $S$

- Previous $M$ equations define a region in the $M$–dimensional space whose coordinates are $G_1, \ldots, G_M$.

- The boundary to this region defines the maximum throughput for $S_m$ when all other $S_i(i \neq m)$ are fixed at allowable values. This can be found by setting the Jacobian (determinant) $J$ to zero with:

  \[
  \frac{\partial S_j}{\partial G_i} = \begin{cases} 
  \frac{g}{1-G_i} & i = j \\
  \frac{-gG_i}{(1-G_j)(1-G_i)} & i \neq j 
  \end{cases}
  \]

- After some algebra, we find

  \[
  J = g^{M-2} \begin{vmatrix} 
  (1-G_1) & -G_1 & -G_1 & \cdots & -G_1 \\
  -G_2 & (1-G_2) & -G_2 & \cdots & -G_2 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  -G_M & -G_M & -G_M & \cdots & (1-G_M) 
  \end{vmatrix} = g^{M-2}(1 - G_1 - G_2 - \cdots - G_M).
  \]
Condition to achieve maximum $S$

- Applying $J = 0$, we have the following general condition on the set of offered loads that achieves the set of maximum throughput values:

$$G = \sum_{m=1}^{M} G_i = 1.$$ 

- The above equation gives us our throughput performance contour.

Example 1

- Let $M = 1$, we then have $S = G = 1$.
- This is obvious the best we can do that a user never collides or destroys his own packet.
Example 2

The $M$ users form two groups such that $m_1$ of them have a rate $S_1$ and $m_2 = M - m_1$ of them have a rate $S_2$.

$S = m_1S_1 + m_2S_2$ and $G = m_1G_1 + m_2G_2$. The $M$ equations reduce

$$S_1 = G_1(1 - G_1)^{m_1-1}(1 - G_2)^{m_2}$$
$$S_2 = G_2(1 - G_2)^{m_2-1}(1 - G_1)^{m_1}.

This defines the set of permitted rates $\{S_1, S_2\}$ as a function of offer loads $\{G_1, G_2\}$.

Imposing the optimization condition $G = 1$, we may solve for $S_1$ and $S_2$ for any given value of $G_1 \in [0, 1/m_1]$ and $G_2 = (1 - m_1G_1)/m_2$. 
Aloha and Slotted Aloha

Throughput Contour:

$m_1$ users at rate $S_1$
$m_2$ users at rate $S_2$
$(m_1, m_2)$

Figure 5.36 Allowable source rates for slotted ALOHA.
Throughput Contour:

Figure 5.37  Allowable source rates for slotted ALOHA.
Assume an infinite number of users who collectively form a source that generates packets independent of the channel.

This source generates $V$ packets per slot from the distribution $\nu_k = P[V = k]$ with mean of $S$ packets per slot.

Packet is of constant length requiring 1 slot time to transmit.

Let $R$ be the number of slots it takes to check whether a transmitted packet is successful or not (e.g., satellite).

If the packet was destroyed, user retransmits and randomly choose one of the next $K$ slots to retransmit (so as to avoid colliding again with other collided packets). Thus, the retransmission will take place either $R + 1$ to $R + K$ slots after the initial transmission.

Total packets (new and retransmit pkt) per slot is $L$, where $p_k = P[L = k]$ with mean traffic of $G$ packets per slot.

What is the maximum throughput $S$, the average packet delay $T$?
Goal: derive $p_k = P[\text{exactly } k \text{ transmitted packets per slot}]$ or its $Z-$transform. Consider the following figure:

The traffic of "current slot" is due to (a) retransmission from any of the "K-slots" and (b) new packets generate in the current slot.

Let $q_m = \text{Probability that exactly } m \text{ packets which are transmitted in the current slot due to retransmission from the tagged slot, for } m = 0, 1, 2, \ldots$

Define Z-transforms:

- $P(z) = \sum_{k=0}^{\infty} p_k z^k$; $q(z) = \sum_{m=0}^{\infty} q_m z^m$; $V(z) = \sum_{k=0}^{\infty} v_k z^k$. 
Since transmission in slots are *independent*, let us first derive $q_0$:

$$q_0 = p_0 + p_1 + \sum_{k=2}^{\infty} p_k \left(1 - \frac{1}{K}\right)^k.$$

For $q_1$, we have:

$$q_1 = \sum_{k=2}^{\infty} p_k k \left(\frac{1}{K}\right) \left(1 - \frac{1}{K}\right)^{k-1}.$$

For the rest $q_m$, where $m \geq 2$, we have:

$$q_m = \sum_{k=m}^{\infty} p_k \binom{k}{m} \left(\frac{1}{K}\right)^m \left(1 - \frac{1}{K}\right)^{k-m} \quad m \geq 2.$$
The $Z$–transform of $q_m$, or $q(Z)$ is

$$q(Z) = q_0 + q_1 Z + \sum_{m=2}^{\infty} \left[ \sum_{k=m}^{\infty} p_k \binom{k}{m} \left( \frac{1}{K} \right)^m \left( 1 - \frac{1}{K} \right)^{k-m} \right] Z^m$$

Adding and subtracting $m = 0$ and $m = 1$ terms, we have

$$q(Z) = q_0 + q_1 Z - \sum_{k=0}^{\infty} p_k \left( 1 - \frac{1}{K} \right)^k - z \sum_{k=1}^{\infty} p_k k \left( \frac{1}{K} \right)^m \left( 1 - \frac{1}{K} \right)^{k-1}$$

$$+ \sum_{m=0}^{\infty} \left[ \sum_{k=m}^{\infty} p_k \binom{k}{m} \left( \frac{1}{K} \right)^m \left( 1 - \frac{1}{K} \right)^{k-m} \right] Z^m$$
Substituting \(q_0\) and \(q_1\) and interchanging the order of summation in the double sum:

\[
q(Z) = p_0 + p_1 - \left[ p_0 + p_1 \left( 1 - \frac{1}{K} \right) \right] - z \frac{p_1}{K} + \sum_{k=0}^{\infty} p_k \sum_{m=0}^{k} \binom{k}{m} \left( \frac{z}{K} \right)^m \left( 1 - \frac{1}{K} \right)^{k-m}
\]

Reorganizing the binomial expansion, we get

\[
q(Z) = \frac{p_1}{K} (1 - Z) + \sum_{k=0}^{\infty} p_k \left[ 1 - \frac{1}{K} + \frac{z}{K} \right]^k
\]

Taking the definition of \(P(Z)\), we have:

\[
q(Z) = \frac{p_1}{K} (1 - Z) + P \left( 1 - \frac{1 - Z}{K} \right).
\]
Now, the number of packets to be transmitted in the current slot is the due to (1) the $K$ slots in the tagged slot region; (2) number of newly generated packets in the current slot, expressing the relationship of these random variables:

$$\tilde{p} = \tilde{q} + \cdots \tilde{q} + \tilde{v}.$$  

Expressing in $Z$-transform, we have:

$$P(Z) = [q(Z)]^K V(Z).$$

Or

$$P(Z) = \left[ \frac{p_1}{K} (1 - Z) + P \left( 1 - \frac{1 - Z}{K} \right) \right]^K V(Z).$$

Since $P(Z)$ is a moment generating function, we can obtain various moments and “crank” out probabilities.
Analysis with Delay in Retransmission

Since $S$ is the throughput of the channel (e.g., average number of successful transmission per transmission period $P$), and $G$ is the average offered traffic (e.g., number of packet transmission attempted per $P$ sec), we have:

- $S / G$ : probability of successful packet transmission,
- $G / S$ : average number of transmission until success.

Let $E$ be the average number of extra transmissions a packet incurs (i.e., collision), we have:

$$1 + E = \frac{G}{S}.$$ 

We want to derive $E$. To do this, let us define:

- $q$ : $P$[newly generated packet is successfully transmitted]
- $q_t$ : $P$[previously collided packet is successfully transmitted]
Let $x_i$ be the probability that a packet collides exactly $i$ times before successful transmission, we have:

$$x_i = (1 - q)(1 - q_t)^{i-1} q_t \quad i = 1, 2, \ldots,$$

So

$$E = \sum_{i=1}^{\infty} i x_i = \frac{1 - q}{q_t}.$$

We can now express $S$, the channel throughput:

$$S = \frac{G}{1 + E} = G \left( \frac{q_t}{q_t + 1 - q} \right).$$  \hspace{1cm} (6)

**Remark:** If we can derive $q$ and $q_t$, then we have $S$. 
Derive $q$: Based on previous analysis and definition of $q_m$, we have

$$q = q_0^K e^{-S}$$

where $q_0$ can be expressed as:

$$q_0 = e^{-G} + Ge^{-G} + \sum_{i=2}^{\infty} \frac{G^i}{i!} e^{-G} \left(1 - \frac{1}{K}\right)^i$$

$$= e^{-G} + Ge^{-G} + e^{-G} \left[e^{G(1-1/K)} - 1 - G \left(1 - \frac{1}{K}\right)\right]$$

$$= e^{-G/K} + \frac{G}{K} e^{-G}.$$ 

Substituting this into the expression of $q$, we have:

$$q = \left(e^{-G/K} + \frac{G}{K} e^{-G}\right)^K e^{-S}. \quad (7)$$
Derive $q_t$: the probability that previously collided packet is successfully transmitted. It has three factors:

- no packet which collided with it in the tagged slot again collides with it in the current slot.
- no packets from one of the $K-1$ other possible tagged slots collides with this packet in the current slot.
- no new packet is generated in the current slot.

Let $q_c$ be the probability of the first event, we have

$$q_c = \sum_{k=1}^{\infty} \left( \frac{G^k}{k!} \right) \left( \frac{e^{-G}}{1 - e^{-G}} \right) \left( \frac{K-1}{K} \right)^k = \frac{e^{-G/K} - e^{-G}}{1 - e^{-G}}.$$

For second event, the probability is $q_0^{K-1}$.

For third event, the probability is $e^{-S}$.

We have

$$q_t = q_c q_0^{K-1} e^{-S} = \left[ \frac{e^{-G/K} - e^{-G}}{1 - e^{-G}} \right] \left[ e^{-G/K} + \frac{G}{K} e^{-G} \right]^{K-1} e^{-S}. \quad (8)$$
Note that Equations (6), (7) and (8) need to be solved \textit{numerically} so as to derive $S$, $q$ and $q_t$.

To derive $T$, the average number of slots to successfully deliver a packet, we know that:
- the time needed for one successful transmission $(R+1)$ slots,
- the extra transmission due to collision.

Therefore, $T$ is:

$$T = (R+1) + \left[ \frac{K+1}{2} + (R+1) \right] E = (R+1) + \frac{1-q}{q_t} \left[ R+1 + \frac{K-1}{2} \right]. \ (9)$$

\textbf{Remark:} We now have the performance measures of $S$ and $T$ as a function of $K$, $R$ and $G$. 

To illustrate (a) $S$ vs. $G$; (b) effect of $K$; (c) effect on $T$.

![Graph showing throughput as a function of channel traffic]

Figure 5.38  Throughput as a function of channel traffic.
Figure 5.39  Delay-throughput tradeoff.
Two groups of users: a group of "small" users \((m_1 = \infty)\) and one large user \((m_2 = 1)\), with \(S_\infty = \lim_{m_1 \to \infty} m_1 S_1\) and their \(G_\infty = m_1 G_1\). The large user throughput is \(S_2\) and offered load is \(G_2\). In her
\[
T = \left( S_\infty T_\infty + S_2 T_2 \right) / S.
\]

**Figure 5.40** Optimum delay-throughput tradeoffs.
Model

Previous analysis assumes equilibrium conditions. Now consider

- Slotted ALOHA with large (but finite) number \((M)\) of active terminals.
- Each terminal has one buffer space and only when the buffer is empty a new packet can be generated with probability \(\sigma\) per slot.
- Let \(N(t)\) be the r.v. representing the total number of nonempty terminals (or channel backlog) and \(S(t)\) be the combined input rates of packets into all terminals at time \(t\). E.g., \(N(t) = n\), then \([N(t), S(t)] = [n, (M - n)\sigma]\). This is a linear feedback model.
- Packets that collide are retransmitted (after a round-trip propagation delay of \(R\) slots) in one of the next \(K\) slots, each being chosen at random with probability \(1/K\).
- The retransmission takes place on the average \(R + (K + 1)/2\) slots after the previous transmission.
Analysis

- The model is hard to analyze. We approximate the backlogged packet independently retransmits with probability \( p \) where

\[
p = \frac{1}{R + (K + 1)/2}.
\]

- Define \( S_{out} \) be the throughput rate of the channel, which is the probability of exact one successful packet transmission in a slot. If \([\mathcal{N}(t), \mathcal{S}(t)] = [n, (M-n)\sigma]\), then

\[
S_{out} = (1 - p)^n(M - n)\sigma (1 - \sigma)^{M-n-1} + np(1 - p)^{n-1}(1 - \sigma)^{M-n} \quad (10)
\]

- In the limit when \( M \to \infty \) and \( \sigma \to 0 \) such that \( M\sigma = S < \infty \), we have the infinite population model in which packets are generated over the channel at the Poisson rate \( S \). We rewrite the equation to

\[
S_{out} = (1 - p)^n Se^{-S} + np(1 - p)^{n-1} e^{-S}.
\]

This expression is very accurate even for finite \( M \) if \( \sigma << 1 \) and if we replace \( S = M\sigma \) by \( S = (M - n)\sigma \).
For a fixed $K$, we show the behavior of $S_{out}$ as function of the channel load $[n, S]$

There is an equilibrium contour in the $(n, S)$ "phase plane" on which the input rate $S$ is equal to the system throughout $S_{out}$.

In the shaded region, $S_{out} > S$; elsewhere $S_{out} < S$ (the system capacity is exceeded!).

Figure 5.42  Channel throughput rate as a function of load and backlog.
The area of the shaded region (where $S_{out} > S$) may be increased by increasing $K$.

Figure 5.43 Equilibrium contours.
We define the channel load line in the \((n, S)\) plane as the line \(n = M - (S/\sigma)\). A channel is said to be "stable" when its load line intersects the equilibrium contour in exactly one place.

Figure 5.46  Channel stability. (a) A stable channel. (b) An unstable channel.