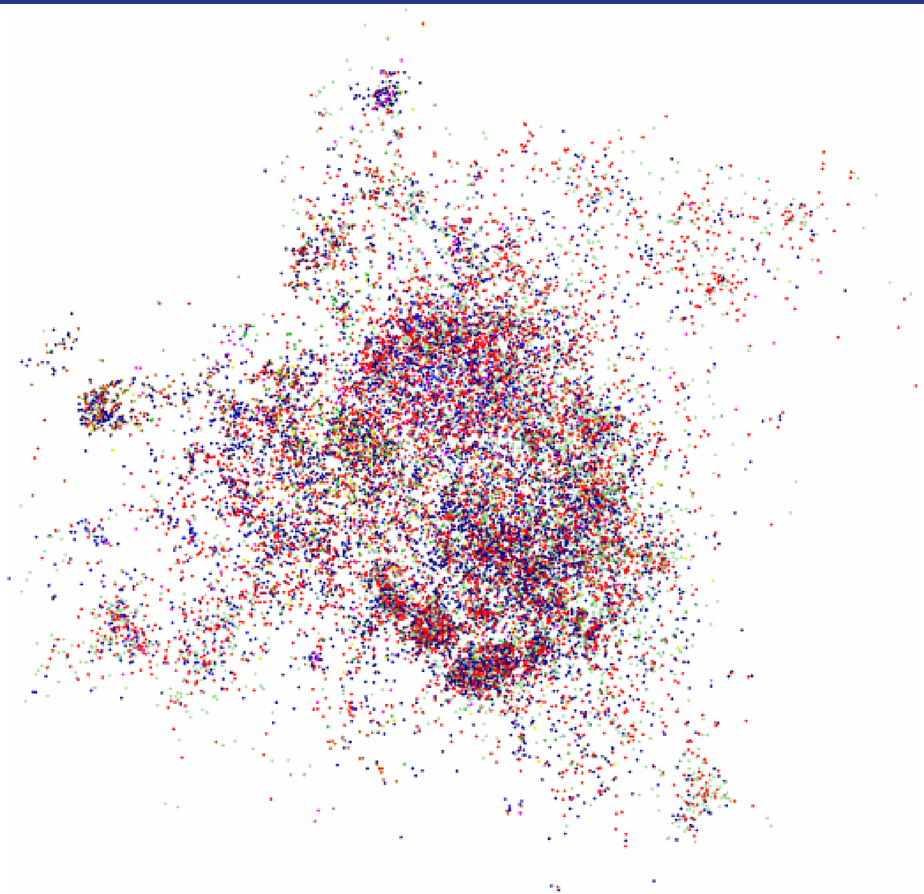
Network Upgrade Game



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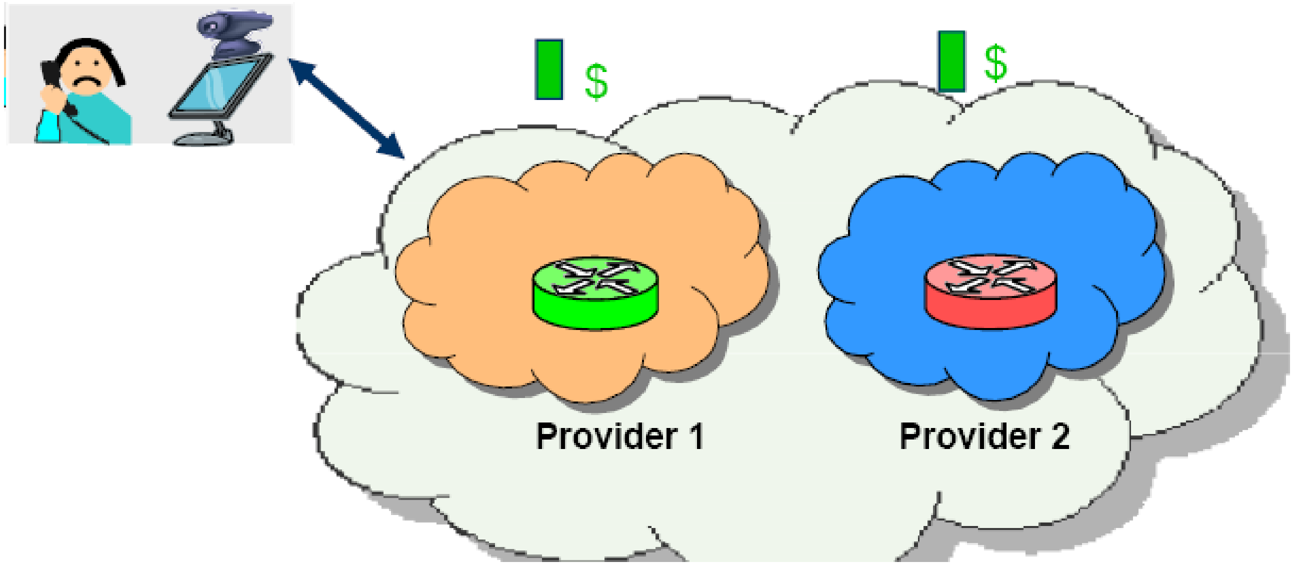
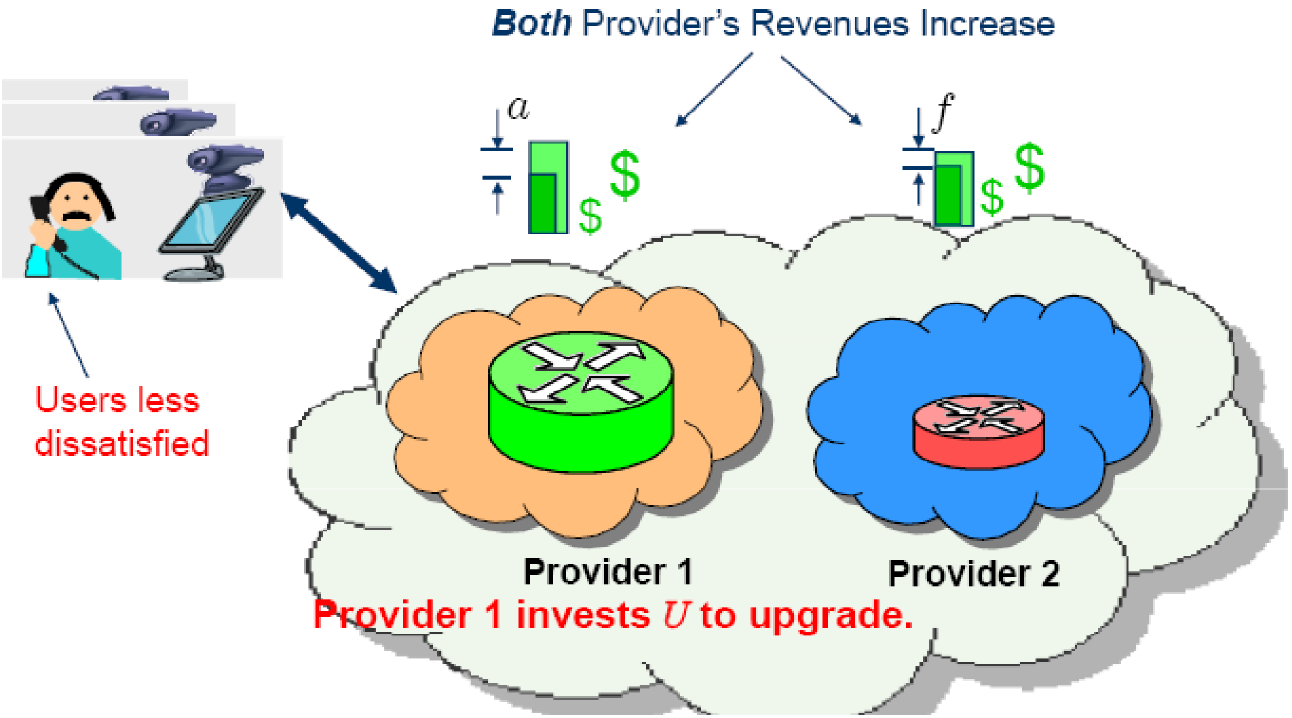
Outline

- Two ISPs
- N-ISPs
- Declining Upgrade Cost Model

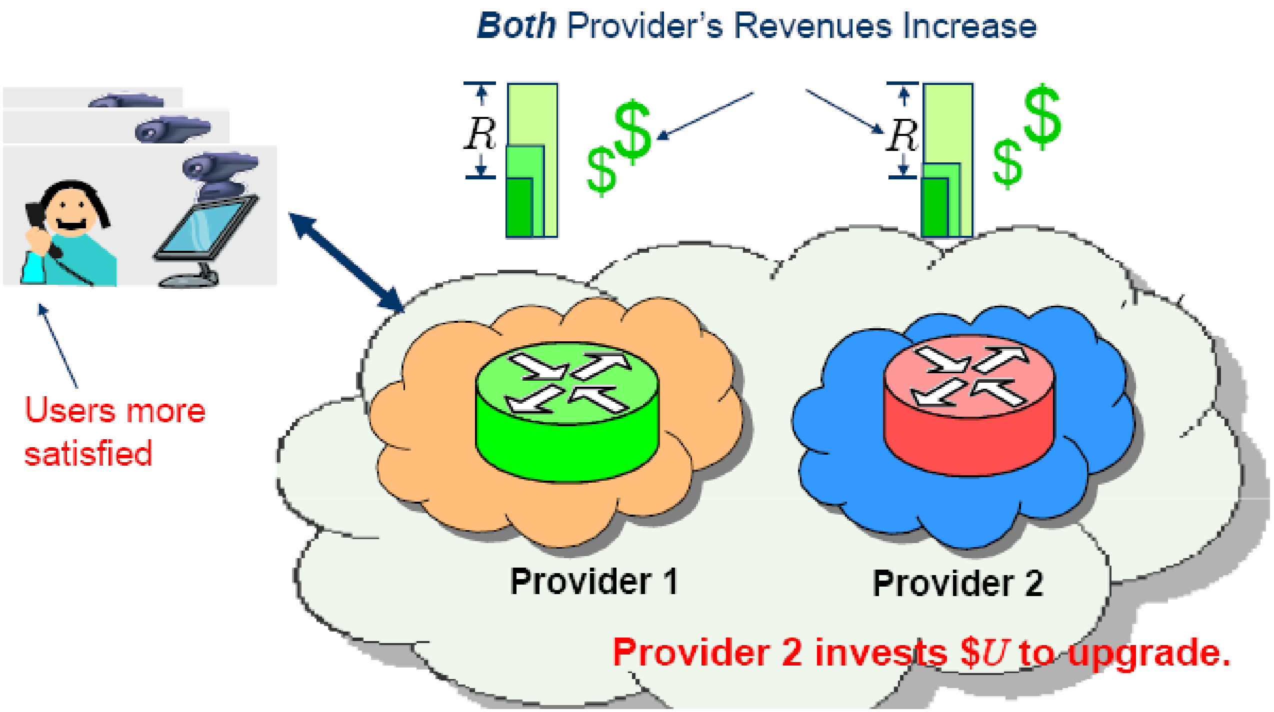
Motivation



Network after one upgrade



Network after both upgrade



Contribution

- Formulate this upgrade decision process as a repeated game.
- Obtain several sub-game perfect equilibria(SPE) under different conditions.
- Analyze the effect of declining upgrading cost in a continuous model.

Two ISPs model

- In every time period, an ISP makes a decision according to its strategy.
- Discount factor is δ .
- Game model for single period:

	Upgrade	Not Upgrade
Upgrade	R,R	a,f
Not Upgrade	f,a	0,0

SPE and Conditions

- **Upgrade Immediately:** Both ISP choose to upgrade in the current period. This is SPE if

$$U \le \frac{R-f}{1-\delta}$$

- **Delayed Upgrade:** If another ISP upgrade before period n, it upgrade in the next period. Else it waits until the period n to upgrade. This is SPE if

$$\frac{a}{1-\delta^n} + \frac{(\delta-\delta^n)R}{(1-\delta)(1-\delta^n)} \le U \le \frac{R-f}{1-\delta}$$

- **Never Upgrade:** Using this strategy, the ISPs will never upgrade. It is SPE if:

$$U > \frac{a}{1-\delta} \text{ and } U > \frac{R-f}{1-\delta}$$

SPE and Conditions cont.

- **No First Upgrade:** Both do not upgrade until the other upgrades in the previous period. It's SPE if:

$$a + \frac{\delta R}{1-\delta} \leq U \leq \frac{R-f}{1-\delta}$$

- **Asymmetric Free-ride:** One upgrades immediately and the other never upgrade. This is SPE if

$$\frac{R-f}{1-\delta} \leq U \leq \frac{a}{1-\delta}$$

SPE and Conditions cont.

- **Mixed Free-ride:** If no one has upgraded, both upgrade with probability α . If one has upgraded, the other remains not-upgraded. It's SPE if:

$$\frac{R-f}{1-\delta} \leq U \leq \frac{a}{1-\delta}$$

- **Mixed Upgrade:** If no one has upgraded, both upgrade with probability α . If one has upgraded, the other also upgrades. It's SPE if:

$$a+\frac{\delta R}{1-\delta} \leq U \leq \frac{R-f}{1-\delta}$$

Proof

- **Upgrade Immediately:** Both ISP choose to upgrade in the current period. This is SPE if

$$U \leq \frac{R-f}{1-\delta}$$

- Proof: Let ISP B upgrade immediately.

Action of A	Payoff of A
Upgrade at 0	$-U + \frac{R}{1-\delta}$
Never upgrade	$\frac{f}{1-\delta}$
Upgrade at n	In between

Proof

- **Delayed Upgrade:** If another ISP upgrade before period n, it upgrade in the next period. Else it waits until the period n to upgrade. This is SPE if

$$\frac{a}{1-\delta^n}+\frac{(\delta-\delta^n)R}{(1-\delta)(1-\delta^n)}\le U\le\frac{R-f}{1-\delta}$$

- **Proof:** Let ISP B use delayed upgrade:

Action of A	Payoff of A
Upgrade at 0	$a-U+\frac{\delta R}{1-\delta}$
Never upgrade	$\frac{\delta^n f}{1-\delta}$
Upgrade at n	$\delta^n\left(\frac{R}{1-\delta}-U\right)$
Before or after n	Less or in between

Proof

- **Mixed Free-ride:** If no one has upgraded, both upgrade with probability α . If one has upgraded, the other remains not-upgraded. It's SPE if:

$$\frac{R-f}{1-\delta} \leq U \leq \frac{a}{1-\delta}$$

- **Proof:** Let ISP B use Mixed Free-ride. When no one has upgraded, the expected payoff of A for upgrading is

$$J_0(\alpha) = \alpha \frac{R}{1-\delta} + (1-\alpha)\frac{a}{1-\delta} - U$$

- The expected payoff J of A for not upgrading is

$$J_1(\alpha) = \alpha \frac{f}{1-\delta} + (1-\alpha)\delta J_1(\alpha)$$

$$J_0(0) - J_1(0) = \frac{a}{1-\delta} - U \geq 0 \qquad J_0(1) - J_1(1) = \frac{R-f}{1-\delta} - U \leq 0$$

- There exist $0 < \alpha^* < 1$ such that $J_0(\alpha) = J_1(\alpha)$ and A is indifferent between two choices.

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N-ISPs

- Define
  - $f(j)$: free-rider benefit if *j* ISPs upgraded
    - *f(0)* = 0
  - $a(j)$: early-adopter benefit if j ISPs upgraded
    - $a(N) = R$
- Unique SPE: every ISP upgrading in the first period, if:

$$U \leq \frac{R - f(N-1)}{1-\delta}$$

  - if stick to upgrade: $\frac{R}{1-\delta} - U$

$$>$$

  - if never upgrade: $\frac{f(N-1)}{1-\delta}$

Upgrade Immediately (Strict Dominance)

- Claim: upgrade is a better decision in any situation, if

$$U < \min_{j=0,\ldots,N-1} \left[\frac{a(j+1) - f(j)}{1-\delta} \right]$$

  - prove by induction

- Initial: suppose all but 2 ISPs have upgraded, say A & B.

A \ B	upgrade	freeride
upgrade	$\frac{a(N)}{1-\delta} - U$	$\frac{a(N-1)}{1-\delta} - U$
freeride	$\frac{f(N-1)}{1-\delta}$	$\frac{f(N-2)}{1-\delta}$

  - Upgrade Incentive: $\frac{a(N)}{1-\delta} - U \geq \frac{f(N-1)}{1-\delta}, \quad \frac{a(N-1)}{1-\delta} - U \geq \frac{f(N-2)}{1-\delta}$
  - Dominant: $U < \min\left(\frac{a(N) - f(N-1)}{1-\delta}, \frac{a(N-1) - f(N-2)}{1-\delta}\right)$

Upgrade Immediately (Strict Dominance)

- Assume all but (i) ISPs upgrade, and the claim is correct.
- Now, suppose all but (i+1) ISPs upgrade,
  - if $k \in \{0,\dots,i\}$ upgrade in current period,

$$\begin{cases} a(N-i+k)+\delta\dfrac{R}{1-\delta}-U, & \text{A upgrades now} \\ f(N-i+k-1)+\delta\dfrac{R}{1-\delta}-\delta U, & \text{A waits} \end{cases}$$

—— $\delta\dfrac{R}{1-\delta}$ is a upbound

  - Upgrade Incentive: $U<\dfrac{a(N-i+k)-f(N-i+k-1)}{1-\delta},\ k\in 0,\dots,i$
- So, induction argument shows Every One Upgrade is a SPE iff:

$$U<\min_{j=0,\dots,N-1}\left[\frac{a(j+1)-f(j)}{1-\delta}\right]$$

upgrade is better than free-rider!

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Declining Upgrade Cost Model

- Continuous Time Model, 2 ISPs here.

A \ B	upgrade	free ride
upgrade	R, R	a, f
free ride	f, a	0, 0

- Revenues discount factor: $e^{-\delta t}$
  - if both/one upgrade at time 0, the total revenue, is:

$$\int_0^{+\infty} e^{-\delta t} R = \frac{R}{\delta}, \int_0^{+\infty} e^{-\delta t} a = \frac{a}{\delta}, \int_0^{+\infty} e^{-\delta t} f = \frac{f}{\delta}$$

- Cost decline at factor: $e^{-\gamma t}$
  - Includes: declining upgrade costs and discounting factor, $\gamma > \delta$
  - The upgrade cost at time t' is:

$$e^{-\gamma t'} U$$

Two Critical Time

- t_f^*: the time upgrade is a better decision than free-ride,

$$t_f^* = arg \max_{t \in \mathbb{R}^+} \left[e^{-\delta t} \frac{(R-f)}{\delta} - Ue^{-\gamma t} \right] = \left[\frac{1}{\gamma - \delta} \log \left(\frac{\gamma U}{R-f} \right) \right]$$

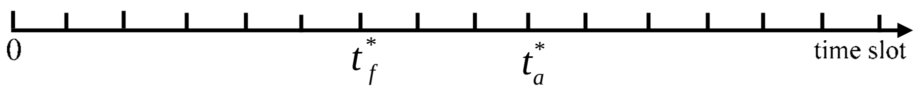
- t_a^*: the time the first to upgrade is a better decision

$$t_a^* = arg \max_{t \in \mathbb{R}^+} \left[e^{-\delta t} \frac{a}{\delta} - Ue^{-\gamma t} \right] = \left[\frac{1}{\gamma - \delta} \log \left(\frac{\gamma U}{a} \right) \right]$$

- Claim: both ISPs have upgraded after time t_f^* , proof as follows.

Theorem

- Theorem 3:



  - Suppose $t_a^* \geq t_f^*$, the only SPE is two ISPs both upgrade at t_f^*
  - Proof: $R - f \geq a$
    - Step 1: A upgrade at time before t_f^*, B will upgrade at time t_f^*

free rider revenue

$$arg \max_{t \geq t'} \left[e^{-\delta t} \frac{R}{\delta} + (e^{-\delta t'} - e^{-\delta t}) \frac{f}{\delta} - Ue^{-\gamma t} \right] = \arg \max_{t \geq t'} \left[e^{-\delta t} \frac{R - f}{\delta} - Ue^{-\gamma t} \right] = t_f^*$$

upgrade revenue

cost

    - Step 2: A upgrade after time t_f^*, B follows immediately

decrease function if t >= t_f^*

$$arg \max_{t \geq t'} \left[e^{-\delta t} \frac{R}{\delta} + (e^{-\delta t'} - e^{-\delta t}) \frac{f}{\delta} - Ue^{-\gamma t} \right] = \arg \max_{t \geq t'} \left[e^{-\delta t} \frac{R - f}{\delta} - Ue^{-\gamma t} \right] = t'$$

Theorem 3 (cont.)

- Step 3:
  - if A consider upgrade before time t_f^* , derive his payoff as he can induce B's behavior:

$$\arg\max_{t \le t_f^*}\left[e^{-\delta t_f^*}\frac{R}{\delta}+(e^{-\delta t}-e^{-\delta t_f^*})\frac{a}{\delta}-Ue^{-\gamma t}\right]=\arg\max_{t \le t_f^*}\left[e^{-\delta t}\frac{a}{\delta}-Ue^{-\gamma t}\right]=\min(t_a^*,t_f^*)=t_f^*$$

  - if A consider upgrade after time t_f^* , maximize his payoff as he can induce B's behavior:

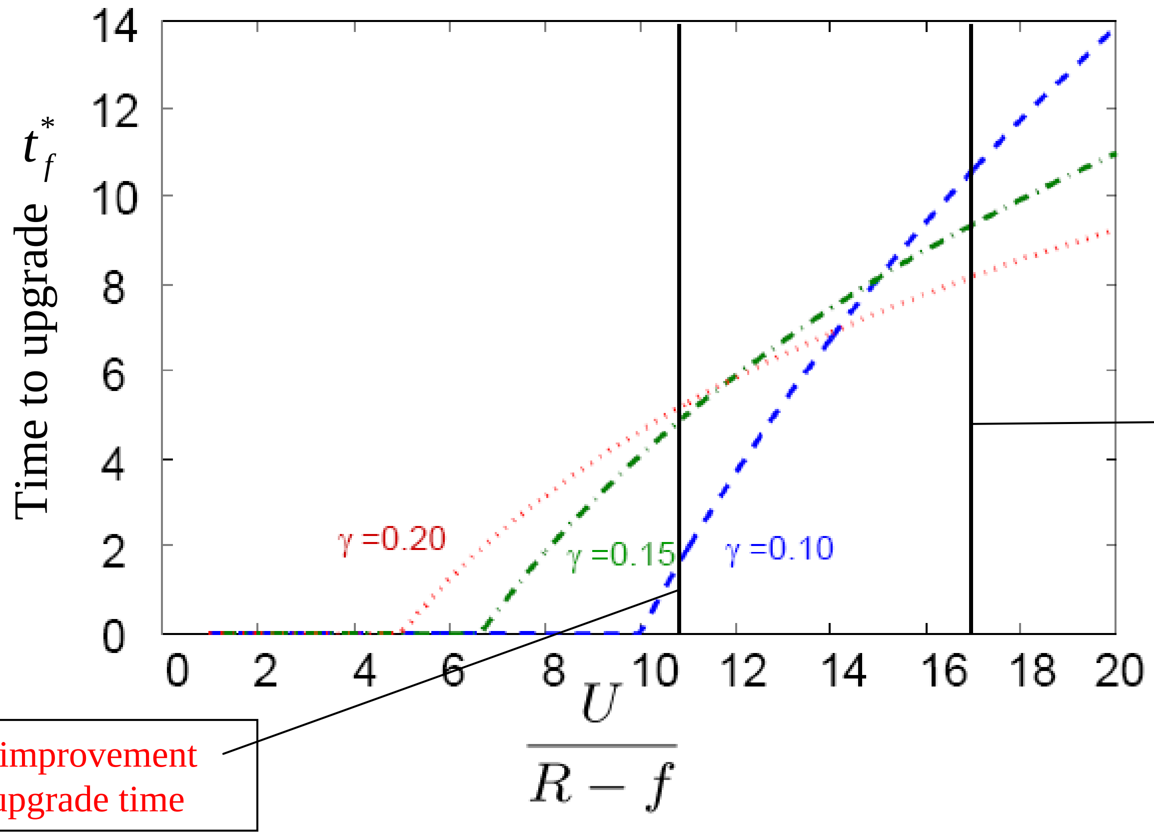
$$\arg\max_{t \ge t_f^*}\left[e^{-\delta t}\frac{R}{\delta}-Ue^{-\gamma t}\right]=\min\left[t_f^*,\frac{1}{\gamma-\delta}\log(\frac{\gamma U}{R})\right]=t_f^*$$

  - So, upgrade at time t_f^* is always a best decision when $t_a^* \ge t_f^*$

Theorem

- Theorem 4:
  - Suppose $t_a^* < t_f^*$, the only SPE is:
    - one ISPs upgrade at t_a^*
    - the other one upgrade at t_f^*
  - Similar proof with Theorem 3.
  - Intuition: compete for the time t_a^*, if fail wait to upgrade until time t_f^*

Time to Upgrade t_f^*



technique improvement delay the upgrade time

technique improvement speed up the upgrade process

Summary

- Discrete Time Model:
  - Obtain several sub-game perfect equilibrium(SPE) under different conditions.
  - Discuss the free riding effect to the upgrading decisions
- Continuous Time Model:
  - Cost declining model, discuss the effect of technique improvement to the upgrade decision.
  - Technique improvement may delay the upgrading time

- Thank you!
  - Questions?