Introduction to Routing and Congestion Control

John C.S. Lui

Department of Computer Science & Engineering
The Chinese University of Hong Kong
www.cse.cuhk.edu.hk/∼cslui

Outline

Overview

- The role of routing: decide which routes through the network should be used to get the data from sources to destinations.
- The role of congestion control: determine the data rates for the users so that fairness among users is achieved, and a reasonable operating point is found on the tradeoffs between throughput, delay or loss.

Introduction

- Suppose that Ω is a subset of $\mathbf{R}^n$ for some $n \geq 1$ and Ω is a convex set.
- A convex set: any pair of points in Ω, the line segment connecting them is also in Ω, or whenever$x, x' \in \Omega$, then $ax + (1-a)x' \in \Omega$ for $0 \leq a \leq 1$.
- A function f on Ω is a *convex function* if along any line segment in Ω, f is less than or equal to the value of the linear function agreeing with f at the endpoints, or

$$f(ax + (1-a)x') \leq af(x) + (1-a)f(x')$$

  for $x, x' \in \Omega$ and $0 \leq a \leq 1$.
- A concave function f is a function f such that $-f$ is a convex function.
- Minimizing a convex function can be translate to maximizing a concave function.

Definitions

- The gradient, ∇f, of a function f, is a vector valued faction as:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

  We assume ∇f exists and continuous.

- If $x \in \Omega$ and if v is a vector such that $x + \epsilon v \in \Omega$ for small enough $\epsilon > 0$, then the directional derivative of f at x in the direction of v:

$$\lim_{\epsilon \longrightarrow 0} \frac{f(x + \epsilon v) - f(x)}{\epsilon} = (\nabla f(x)) \cdot v = \sum_i \frac{\partial f(x)}{\partial x_i} v_i$$

- We have $x^* \in argmin_{x \in \Omega} f(x)$ if x^* minimizes f over Ω.

Proposition

Suppose f is a convex function on a convex set Ω with a continuous gradient function ∇f. Then $x^* \in argmin_{x \in \Omega} f(x)$ if and only if $(\nabla f(x^*)) \cdot (x - x^*) \geq 0$ for all $x \in \Omega$.

Model

The flow rates of users are taken as given, the problem is to determine which routes should be taken. We define

- J is the set of links.
- R is the set of routes. Each route r is associated with a subset of links. We write $j \in r$ to denote link j is in route r. In general, we allow two different routes to use the same set of links.
- A is the link-route incidence matrix with $A_{j,r} = 1$ if $j \in r$ and $A_{j,r} = 0$ otherwise.
- S is the set of users. Each user s is associated with a subset of the set of routes. We write $r \in s$ to denote that route r serves user s.
- H is the user-route incidence matrix, $H_{s,r} = 1$ if $r \in s$ and $H_{s,r} = 0$ otherwise.

Model

- y_r is the flow on route r.
- f_j is the flow on link j : $f_j = \sum_{r:j \in r} y_r$, or in vector form, $f = Ay$.
- x_s is the total flow available to user s : $x_s = \sum_{r \in s} y_r$, or in vector form, $x = Hy$. The vector $x = (x_s : s \in S)$ is fixed for the routing problem, and variable in the joint congestion control and routing problem.
- $D_j(f_j)$: is the cost of carrying flow j on link j. The function D_j is assumed to be a convex function, continuous differentiable, and increase function $\mathbf{R}_+$. The cost associated with flow rate f_j on link j is $D_j(f_j)$.

The Routing Problem

$ROUTE(x, H, A, D)$:

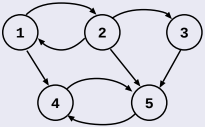
$$\min F(y) = \sum_{j \in J} D_j \left(\sum_{r: j \in r} y_r \right)$$

subject to

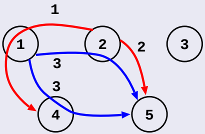
$$x = Hy, \qquad \text{over } y \geq 0.$$

- We say that vector y is a feasible flow meeting demand x if $y \geq 0$ and $x = Hy$.
- A possible choice of D_j is $D_j(f_j) = \frac{f_j}{C_j - f_j} + f_j d_j$ where C_j is the capacity of link j and d_j is the propagation delay of link j.
- If we use an $M/M/1$ model, $D_j(f_j)$ represents the mean number of packets in transit of link j. So the total cost $F(y)$ represents the mean number of packets in transits in the network. Then $F(y)/\sum_s x_s$ is the average delay of packets entering the network.

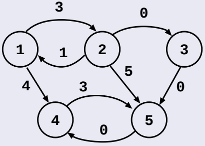
Example



(a): network



(b): route flows



(c): link flows

- Set of links $J = \{(1,2),(1,4),(2,1),(2,3),(2,5),(3,5),(4,5),(5,4)\}$.
- Set of users $S = \{(1,5),(2,4),(2,5)\}$
- We assume user $(1,5)$ is served by routes $(1,2,5),(1,4,5),(1,2,3,5)$, user $(2,4)$ is served by routes $(2,1,4),(2,5,4)$, user $(2,5)$ is served by routes $(2,5),(2,3,5)$.

Example: continue

- Flow rates on each route: $y_{(1,2,5)} = 3.0$, $y_{(1,4,5)} = 3.0$, $y_{(1,2,3,5)} = 0$, $y_{(2,1,4)} = 1.0$, $y_{(2,5,4)} = 0$, $y_{(2,5)} = 2.0$, $y_{(2,3,5)} = 0$.
- This implies that flow rates for each source: $x_{(1,5)} = 6.0$, $x_{(2,4)} = 1.0$, $x_{(2,5)} = 2.0$.
- The link rates: $f_{(1,2)} = 3$, $f_{(1,4)} = 4$, $f_{(2,1)} = 1$,...etc, as seen in the figure.
- The result cost is:

$$F(y)=D_{1,2}(3)+D_{1,4}(4)+D_{2,1}(1)+D_{2,3}(0)+D_{2,5}(5)+D_{3,5}(0)+D_{4,5}(3)+D_{5,4}(0).$$

- Considering changing the flow rates on a route. If $y_{1,2,5}$ were increase to $y_{1,2,5} + \epsilon$ for some $\epsilon > 0$, then the cost $F(y)$ would increase by:

$$\begin{aligned}\Delta F &= D_{1,2}(3+\epsilon) - D_{1,2}(3) + D_{2,5}(5+\epsilon) - D_{2,5}(5) \\ &= \epsilon(D'_{1,2}(3) - D'_{2,5}(5)) + O(\epsilon).\end{aligned}$$

Example: continue

- Similarly, if we decrease $y_{1,4,5}$ by ϵ, the cost would decrease by

$$\epsilon(D^{'}_{1,4}(4) - D^{'}_{4,5}(3)) + O(\epsilon).$$

- If $y_{1,2,5}$ were increased by ϵ and If $y_{1,4,5}$ were simultaneously decreased by ϵ (of ϵ of the flow were *deviated* to route $(1,2,5)$ from route $(1,4,5)$), the *flow deviation cost* would be:

$$\epsilon(D^{'}_{1,2}(3) + D^{'}_{2,5}(5) - D^{'}_{1,4}(4) - D_{4,5}(3)) + O(\epsilon)$$

- first order approximation, the change in cost due to the flow deviation is ϵ times the first derivative length $D'_{1,2}(3) + D'_{2,5}(5)$ of route $(1,2,5)$ minus the first derivative length $D'_{1,4}(4) + D'_{4,5}(3)$ of route $(1,4,5)$.
- The flow deviation will decrease the cost if flow is deviated from a *longer* route to a *shorter* route, where length of a route is the sum of the first derivative link lengths, $D_j(f_j)$.

Important Observation

- The above example implies that a *necessary condition* for y to be optimal is that for any s and any $r \in s$, the rate y_r is strictly positive only if r has the minimum first derivative length among the routes serving s.
- Since
    - the set of feasible rate vectors j meeting the demand x is a convex set,
    - the cost function F is convex,
    - It follows from convex optimization that the first order necessary condition for optimality is also sufficient for optimality.

Proposition

A feasible flow $\boldsymbol{y} = (y_r : r \in R)$ meeting demand $\boldsymbol{x}$ minimizes the cost over all such flows, if and only if there exists a vector $(\lambda_s : s \in S)$ so that $\sum_{j \in r} D_j'(f_j) \geq \lambda_s$, with *equality* if $y_r > 0$, whenever $r \in s \in S$.

Comment

Given a flow vector $\boldsymbol{y}$, let δ_r denote the first derivative length of route r, or $\delta_r = \sum_{j \in r} D_j(\sum_{r' : j \in r'} y_r')$. If the rate requirement of a user s is $x_s > 0$, then the parameter λ_s must satisfy $\lambda_s = \min_{r \in s} \delta_r$.

The Flow Deviation Algorithm

One iteration is described. At the beginning of an iteration, a feasible flow vector $\boldsymbol{y}$ is given. We find the next such vector $\bar{\boldsymbol{y}}$.

- Compute the link flow f_j and first derivative lengths $D_j(f_j)$ for all links $j \in J$.
- For each user s, find a route r_s^* serving s with the minimum first derivative link length.
- Let $\alpha \in [0, 1]$. For each user s, deviate a fraction α of the flow from each of the other routes serving s to r_s^*, that is

$$\bar{y}_r = \begin{cases} (1-\alpha)y_r & \text{if } r \in s \text{ and } r \neq r_s^* \\ y_r + \alpha\left(\sum_{r':r\neq r'\in s} y_{r'}\right) & \text{if } r = r_s^* \end{cases}$$

  Adjust α to minimize $F(\bar{\boldsymbol{y}})$. The resulting $\bar{\boldsymbol{y}}$ is the solution of one iteration.