Introduction to Game Theory
Evolution Games Theory: Population Games

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Outline

Introduction

- So far we have considered "classical game theory", where outcome depends on the choice of rational individuals, and each individual uses a strategy that is the "best response" to other players' choice.
- If we map the same concept to "*population*", symmetric Nash equilibria $(\sigma_i^*, \sigma_{i-1}^*)$, has an alternative interpretation. In a population where everyone uses σ^*, the best thing to do is to follow the crowd. So if everyone using σ^*, it will remain that way.
- Some interesting (and important) questions:
  - What happens if the population is close to, but not at, the NE?
  - Will the population evolve toward the equilibrium?
  - Will the population move away from the equilibrium?
- Evolutionary Game Theory considers a population decision makers wherein the *frequency* with which a particular decision is made can be time varying. It is a theory started from Biology.

Under the evolutionary game, one type of end point (if any) is called an evolutionary stable strategy (ESS).

Definition

Consider an infinite population of individuals that can use a set of pure strategies, $\boldsymbol{S}$. A population profile is a vector $\boldsymbol{x}$ that gives a probability $x(s)$ with which each strategy $s \in \boldsymbol{S}$ is played in the population.

Note that the population profile needs not correspond to a strategy adopted by any members of the population !!

Example

- A population can use $\boldsymbol{S} = \{s_1, s_2\}$.
- If every member of the population randomizes by playing each of the pure strategies with probability $\frac{1}{2}$, then $\boldsymbol{x} = (\frac{1}{2}, \frac{1}{2})$. In this case, the population profile $\boldsymbol{x}$ is identical to the mixed strategy adopted by all members.
- If half of the population adopt the strategy s_1 and other half adopt strategy s_2. We have $\boldsymbol{x} = (\frac{1}{2}, \frac{1}{2})$, and this is NOT the same as the strategy adopted by **any** member of the population.

Exercise

- Give three ways in which a population with profile $\boldsymbol{x} = (\frac{5}{12}, \frac{7}{12})$ might arise.
  1. population using a mixed strategy of $(5/12, 7/12)$,
  2. $5/12(7/12)$ of the population using the first (second) strategy.
  3. $1/3(2/3)$ of the population using the mixed strategy of $(3/4, 1/4)$ $((1/4, 3/4))$.
- Consider an strategy set $\boldsymbol{S} = \{s_1, s_2, s_3\}$. If the population consists of 40% of individual using the strategy $(1/2, 0, 1/2)$ and 60% using $(1/4, 3/4, 0)$, what is the population profile?
  - $\boldsymbol{x} = \frac{4}{10}(\frac{1}{2}, 0, \frac{1}{2}) + \frac{6}{10}(\frac{1}{3}, \frac{3}{4}, 0) = (\frac{7}{20}, \frac{9}{20}, \frac{4}{20})$.

Definition

Consider a particular individual in the population with profile $\boldsymbol{x}$. If that individual uses a strategy σ, the individual's payoff is denoted as $\pi(\sigma, \boldsymbol{x})$. The payoff of this strategy is

$$\pi(\sigma, \boldsymbol{x}) = \sum_{s \in \boldsymbol{S}} p(s)\pi(s, \boldsymbol{x}).$$

These payoffs may represent the number of descendants that each type of individual has. Therefore, the payoffs determine the *evolution* of the population.

Example

- Consider a population of N animals. Each animal is programmed to use one or two strategies s_1 and s_2.
- Suppose 50% of the animal use each of the strategies, i.e., $\boldsymbol{x} = (\frac{1}{2}, \frac{1}{2})$.
- Given the current $\boldsymbol{x}$, we have:

$$\pi(s_1, \boldsymbol{x}) = 6 \quad \text{and} \quad \pi(s_2, \boldsymbol{x}) = 4.$$

- In the next generation, there will be $6N/2$ individuals using s_1 and $4N/2$ individuals using s_2. The new population profile is

$$\boldsymbol{x} = (0.6, 0.4).$$

Types of Population Games

In general, there are two types of population game: (1) *games against the field* and (2) *games with pairwise contests*.

Definition

A game against the field is one in which there is no specific "opponent" for a given individual - their payoff depends on what everyone in the population is doing.

Definition

A pairwise contest game describes a situation in which a given individual plays against an opponent that has been randomly selected (by nature) from the population and the payoff depends just on what both individual do.

Comment

- Under the pairwise contests, the payoff can be written as

$$\pi(\sigma, \boldsymbol{x}) = \sum_{s \in \boldsymbol{S}} \sum_{s' \in \boldsymbol{S}} p(s)x(s')\pi(s, s').$$

  for the suitably defined pairwise payoff $\pi(s, s')$.
- What we are interested in is the end points of the evolution of the population. Or we want to find under what "*conditions*" the population is stable.
- Let $\boldsymbol{x}^*$ be the profile generated by a population of individuals who all adopt strategy σ^*, i.e., $\boldsymbol{x}^* = \sigma^*$. A necessary condition for the evolutionary stability is

$$\sigma^* \in argmax_{\sigma \in \sum} \{\pi(\sigma, \boldsymbol{x}^*)\}.$$

  At the equilibrium, the strategy adopted by individuals must be the best response to the population profile.

Theorem

Let σ^* be a strategy that generates a population profile $\boldsymbol{x}^*$. Let $\boldsymbol{S}^*$ be the support for σ^*. If the population is stable, then $\pi(s, \boldsymbol{x}^*) = \pi(\sigma^*, \boldsymbol{x}^*), \forall s \in \boldsymbol{S}^*$.

Note that this is the stability theorem for evolution game.

Proof

- If the set $\boldsymbol{S}^*$ contains only one strategy, the it is trivially true.
- Suppose $|\boldsymbol{S}^*| > 1$. If the theorem is not true, then at least one strategy gives a higher payoff than $\pi(\sigma^*, \boldsymbol{x}^*)$. Let s' be the action that gives the greatest such payoff, then

$$\begin{aligned}
\pi(\sigma^*, \boldsymbol{x}^*) &= \sum_{s \in \boldsymbol{S}^*} p^*(s)\pi(s, \boldsymbol{x}^*) \\
&= \sum_{s \neq s'} p^*(s)\pi(s, \boldsymbol{x}^*) + p^*(s')\pi(s', \boldsymbol{x}^*) \\
&< \sum_{s \neq s'} p^*(s)\pi(s', \boldsymbol{x}^*) + p^*(s')\pi(s', \boldsymbol{x}^*) = \pi(s', \boldsymbol{x}^*).
\end{aligned}$$

This contradicts the original assumption that the population is stable.

Comment on the above theorem

If σ^* is unique best response to $\boldsymbol{x}^*$, then the evolution of the population clearly stops. But if it is not unique, so there are some other strategies that do equally well in the population with profile $\boldsymbol{x}^*$, then the population could drift in the direction of other strategy and its corresponding population profile. We want to understand this situation.

Definition

Consider a population where initially all the individuals adopt some strategy σ^*. Suppose a mutation occurs and a small proportion ϵ of individuals use some other strategy σ. The new population is called the post-entry population and will be denoted as $\boldsymbol{x}_\epsilon$.

Example

Consider a population with $\boldsymbol{S} = \{s_1, s_2\}$ and $\sigma^* = (\frac{1}{2}, \frac{1}{2})$. Suppose the mutant strategy is $\sigma = (\frac{3}{4}, \frac{1}{4})$, then

$$\boldsymbol{x}_\epsilon = (1-\epsilon)\sigma^* + \epsilon\sigma = (1-\epsilon)\left(\frac{1}{2}, \frac{1}{2}\right) + \epsilon\left(\frac{3}{4}, \frac{1}{4}\right) = \left(\frac{1}{2} + \frac{\epsilon}{4}, \frac{1}{2} - \frac{\epsilon}{4}\right).$$

Stability of ESS

A mixed strategy σ^* is an evolutionary stable strategy (ESS) if there exists an $\bar{\epsilon}$ such that for every $0 < \epsilon < \bar{\epsilon}$ and every $\sigma \neq \sigma^*$

$$\pi(\sigma^*, \boldsymbol{x}_\epsilon) > \pi(\sigma, \boldsymbol{x}_\epsilon).$$

Physical meaning: a strategy σ^* is an ESS if mutants that adopt any other strategy σ leave fewer offspring in the post-entry population, provided that the proportion of mutants is sufficiently small.

Example of a Game Against the field

Why the population ratio (male and female) is 50 : 50?

- The proportion of males (females) in the population is μ $(1 - \mu)$.
- Each female mates once and produce n children.
- Males mates, on average, $(1 - \mu)/\mu$ times.
- Only female genes affect the sex ratio of offspring.
- Assume females available strategies are (a) s_1: produce male offspring; (b) s_2: produce female offspring. The general strategy $\sigma = (p, 1 - p)$ produces a proportion p of male offspring.
- The current population profile is $\boldsymbol{x} = (\mu, 1 - \mu)$
- What is the Evolutionary Stable Strategy (ESS)?

Solution

- Since payoff of children is n, let's consider the number of grandchildren. Given the population profile $\boldsymbol{x} = (\mu, 1-\mu)$, the payoffs are

$$\pi(s_1, \boldsymbol{x}) = n^2 \left(\frac{1-\mu}{\mu} \right) \quad ; \quad \pi(s_2, \boldsymbol{x}) = n^2.$$

- The expected payoff for the strategy σ is

$$\pi(\sigma, \boldsymbol{x}) = n^2 \left(\frac{1-\mu}{\mu} \right) p + n^2(1-p).$$

Because n is independent of the strategy chosen, we can set $n = 1$ (since we are only interested in the population ratio).

Solution: continue

To find an ESS, consider the following cases:

- If $\mu < 1/2$, then using s_1 will have more grandchildren which eventually cause μ to increase. So s_1 is not an ESS.
- If $\mu > 1/2$, then using s_2 have more grandchildren causing μ to fall. So s_2 is not an ESS.
- Is $\sigma^* = (\frac{1}{2}, \frac{1}{2})$ a potential ESS? Let use the ESS Stability Theorem, which states that

$$\pi(s_1, \boldsymbol{x}^*) = \pi(s_2, \boldsymbol{x}^*) = \pi(\sigma^*, \boldsymbol{x}^*).$$

  So if the population profile is $\boldsymbol{x}^* = (\frac{1}{2}, \frac{1}{2})$, then $\sigma^* = (\frac{1}{2}, \frac{1}{2})$ is an ESS (note that this is only a necessary condition).

Solution: continue

Let us show the "sufficient" condition that $\sigma^* = (\frac{1}{2}, \frac{1}{2})$ is indeed an ESS.

- Let $\sigma = (p, 1-p)$ be another strategy, then

$$\boldsymbol{x}_\epsilon = (1-\epsilon)\sigma^* + \epsilon\sigma$$

$$\text{so,} \quad \mu_e = (1-\epsilon)\frac{1}{2} + \epsilon p = \frac{1}{2} + \epsilon\left(p - \frac{1}{2}\right).$$

- The ESS condition is $\pi(\sigma^*, \boldsymbol{x}_\epsilon) > \pi(\sigma, \boldsymbol{x}_\epsilon)$ where

$$\begin{aligned}
\pi(\sigma^*, \boldsymbol{x}_\epsilon) &= \frac{1}{2} + \frac{1}{2}\left(\frac{1-\mu_\epsilon}{\mu_\epsilon}\right) \\
\pi(\sigma, \boldsymbol{x}_\epsilon) &= (1-p) + p\left(\frac{1-\mu_\epsilon}{\mu_\epsilon}\right).
\end{aligned}$$

Solution: continue

- The difference

$$\begin{aligned}\pi(\sigma^*, \boldsymbol{x}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) &= (p - \frac{1}{2}) + (\frac{1}{2} - p)\left(\frac{1-\mu_\epsilon}{\mu_\epsilon}\right) \\ &= (\frac{1}{2} - p)\left[\frac{1-\mu_\epsilon}{\mu_\epsilon} - 1\right] \\ &= (\frac{1}{2} - p)\left[\frac{1-2\mu_\epsilon}{\mu_\epsilon}\right].\end{aligned}$$

  If the difference is positive for $\sigma = (p, 1-p)$, with $p \neq 1/2$, then σ^* is an ESS.
- If $p < 1/2$, this implies $\mu_\epsilon < 1/2$, and this implies $\pi(\sigma^*, \boldsymbol{x}_\epsilon) > \pi(\sigma, \boldsymbol{x}_\epsilon)$.
- If $p > 1/2$, this implies $\mu_\epsilon > 1/2$, and this implies $\pi(\sigma^*, \boldsymbol{x}_\epsilon) > \pi(\sigma, \boldsymbol{x}_\epsilon)$.
- So σ^* is an ESS.

Exercise

Consider a "simplified" Internet. There are two operating systems available: L and W. A user of window W has a basic utility of 1, but L is a better operating system so a user of L has a basic utility of 2. If two computers have the same operating system, then they can communicate over the network. A user's utility rises linearly with the proportion of computers that can be communicated with, up to a maximum increment of 2. Let x be the proportion of $W-$users, then $\pi(W, x) = 1 + 2x$ and $\pi(L, x) = 2 + 2(1 - x)$. What are the ESSs in this population game?

Solution

- Potential ESSs are:
  - σ_W: everyone uses W, then $x = 1$ and $\pi(W, 1) > \pi(L, 1)$.
  - σ_L: everyone uses L, then $x = 0$ and $\pi(L, 0) > \pi(W, 0)$.
  - σ_m: mixed strategy in which W is used $3/4$ of the time, then $x = 3/4$ and $\pi(W, 3/4) = \pi(L, 3/4)$.
- Now $\boldsymbol{x}_\epsilon = (p^* + \epsilon(p - p^*), 1 - p^* - \epsilon(p - p^*))$. So

$$
\begin{aligned}
\delta_\pi &= \pi(\sigma^*, \boldsymbol{x}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) \\
&= p^*\pi(W, \boldsymbol{x}_\epsilon) + (1 - p^*)\pi(L, \boldsymbol{x}_\epsilon) - p\pi(W, \boldsymbol{x}_\epsilon) - (1 - p)\pi(L, \boldsymbol{x}_\epsilon) \\
&= (p^* - p)\left(\pi(W, \boldsymbol{x}_\epsilon) - \pi(L, \boldsymbol{x}_\epsilon)\right) \\
&= (p^* - p)(4p^* - 3 - 4\epsilon(p^* - p))
\end{aligned}
$$

Solution: continue

Taking each candidate ESSs in turn:

- σ_W: $p^* = 1$, so

  $$\delta_\pi = (1-p)(1-4\epsilon(1-p)) > 0, \; \forall p \neq 1, \; \text{and } \epsilon < \bar{\epsilon} = 1/4.$$

  So it is an ESS.

- σ_L: $p^* = 0$, so

  $$\delta\pi = p(3-4\epsilon p) > 0, \; \forall p \neq 0, \; \text{and } \epsilon < \bar{\epsilon} = 3/4.$$

  So σ_L is an ESS.

- σ_m: $p^* = 3/4$, so

  $$\delta_\pi = -4\epsilon\left(\frac{3}{4} - p\right)^2 < 0, \forall p \neq \frac{3}{4} \text{ and } \epsilon > 0.$$

  So it is "**not**" an ESS.

Given an individual plays against an opponent that has been randomly chosen from the population, the payoff depends on what both individual do.

The Hawk-Dove Game

- Individuals can use one of two possible pure strategies: (a) H: be aggressive, (b) D: be non-aggressive.
- In general, individual can use a randomized strategy $\sigma = (p, 1-p)$ with probability p of using H.
- A population consists of individuals that are aggressive with probability x, i.e., $\boldsymbol{x} = (x, 1-x)$, this can arise because
  - a monomorphic population, everyone uses $\sigma = (x, 1-x)$, or
  - a polymorphic population, a fraction x of population use $\sigma_H = (1, 0)$ and a fraction $1-x$ use $\sigma_D = (0, 1)$. Let consider only monomorphic population.

The Hawk-Dove Game: continue

- There is a resource (e.g., food, breeding site,...etc) with value v. The outcome of a conflict depends on the types of two individuals that meet.
- Possible combinations:
  1. a hawk and a dove: hawk wins,
  2. a dove and a dove: they "share" the resource evenly,
  3. a hawk and a hawk: they fight with one winner gets the resource and the other loser pays a cost (i.e., injury) of c.
- What is the outcome of the game? What is the ESS?

Solution

- The payoff of an individual:

$$\pi(\sigma, \boldsymbol{x}) = px\frac{v-c}{2} + p(1-x)v + (1-p)(1-x)\frac{v}{2}.$$

- Assume $v < c$, there is no pure-strategy ESS. Why?
  - In a population of Doves ($x = 0$),

$$\pi(\sigma, \boldsymbol{x}_D) = pv + (1-p)\frac{v}{2} = (1+p)\frac{v}{2}.$$

    It is best to set $p = 1$ (play hawk). As a consequence, the proportion of more aggressive individual will increase.
  - In a population of Hawks ($x = 1$),

$$\pi(\sigma, \boldsymbol{x}_H) = p\frac{v-c}{2}.$$

    It is best to set $p = 0$ because $(v - c) < 0$. As a consequence, the proportion of less aggressive individual will increase.

Solution: continue

- Is there a mixed strategy ESS, $\sigma^* = (p^*, 1-p^*)$? For σ^* to be ESS, it must be a best response to the population $\boldsymbol{x}^* = (p^*, 1-p^*)$ that it generates.
- In the population $\boldsymbol{x}^*$, the payoff of an arbitrary $\sigma = (p, 1-p)$:

$$\begin{aligned}\pi(\sigma, \boldsymbol{x}^*) &= pp^*\frac{v-c}{2} + p(1-p^*)v + (1-p)(1-p^*)\frac{v}{2} \\ &= (1-p^*)\frac{v}{2} + \frac{pc}{2}\left[\frac{v}{c} - p^*\right].\end{aligned}$$

- If $p^* < v/c$, then the best response is $\hat{p} = 1$ (i.e., $\hat{p} \neq p^*$).
- If $p^* > v/c$, then the best response is $\hat{p} = 0$ (i.e., $\hat{p} \neq p^*$).
- If $p^* = v/c$, then any choice of p (including p^*) gives the same payoff, so we have

$$\sigma^* = \left(\frac{v}{c}, 1 - \frac{v}{c}\right),$$

as a candidate ESS when $v < c$.

- To confirm σ^* is an ESS, we must show that for $\sigma = (p, 1-p) \neq \sigma^*$, $\pi(\sigma^*, \boldsymbol{x}_\epsilon) > \pi(\sigma, \boldsymbol{x}_\epsilon)$, where

$$\begin{aligned}\boldsymbol{x}_\epsilon &= ((1-\epsilon)p^* + \epsilon p, ((1-\epsilon)(1-p^*) + \epsilon(1-p)) \\ &= (p^* + \epsilon(p-p^*), 1-p^* + \epsilon(p^*-p)).\end{aligned}$$

- We have

$$\begin{aligned}\pi(\sigma^*, \boldsymbol{x}_\epsilon) &= p^*(p^* + \epsilon(p-p^*))\frac{v-c}{2} + p^*(1-p^* + \epsilon(p^*-p))v + \\ &\quad (1-p^*)(1-p^* + \epsilon(p^*-p))\frac{v}{2}, \\ \pi(\sigma, \boldsymbol{x}_\epsilon) &= p(p^* + \epsilon(p-p^*))\frac{v-c}{2} + p(1-p^* + \epsilon(p^*-p))v + \\ &\quad (1-p)(1-p^* + \epsilon(p^*-p))\frac{v}{2}.\end{aligned}$$

- Substituting $p^* = v/c$, we have

$$\pi(\sigma^*, \boldsymbol{x}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) = \frac{\epsilon c}{2}(p^*-p)^2 > 0.$$

so $\sigma^* = (p^*, 1-p^*)$ is an ESS.

Homework

- Consider a Hawk-Dove game with $v \geq c$, show that playing H is an ESS.

Example: The evolution of money

- In an remote island, inhabitants have to decide to use either "beads" or "shells" as tokens of money in commerce.
- A transaction is only successful if both parties use the same form of token.
- Assume that a trader gets a utility increment of 1 if the transaction is successful and 0 if it fails.
- The general strategy to an individual is to use beads with p, i.e., $\sigma = (p, 1 - p)$. The population profile $\boldsymbol{x} = (x, 1 - x)$.
- What is an ESS ?

Solution

- An individual attempts to trade with a randomly selected member of the population, his payoff

$$\pi(\sigma, \boldsymbol{x}) = px + (1-p)(1-x) = (1-x) + p(2x-1).$$

We see that

$$x > \frac{1}{2} \longrightarrow \hat{p} = 1 \quad \text{and} \quad p = 1 \longrightarrow x = 1.$$

So $\sigma_b^* = (1, 0)$ is a potential ESS with $\boldsymbol{x} = (1, 0)$.

- The post-entry population is:

$$\boldsymbol{x}_\epsilon = (1-\epsilon)(1,0) + \epsilon(p, 1-p) = (1-\epsilon(1-p), \epsilon(1-p)).$$

Solution: continue

- In this population, the payoff for an arbitrary strategy is

$$\pi(\sigma, \boldsymbol{x}_\epsilon) = \epsilon(1-p) + p(1 - 2\epsilon(1-p)).$$

- The payoff for the candidate ESS is $\pi(\sigma_b^*, \boldsymbol{x}_\epsilon) = 1 - \epsilon(1-p)$, so

$$\begin{aligned} \pi(\sigma_b^*, \boldsymbol{x}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) &> 0, \\ \Longleftrightarrow (1-p)(1 - 2\epsilon(1-p)) &> 0. \end{aligned}$$

- Now, $\forall p \neq p^*$, we have $(1-p) > 0$, so σ_b^* is an ESS if and only iff $\epsilon(1-p) < \frac{1}{2}$. That is $\bar{\epsilon} = \frac{1}{2}$.

Solution: continue

- The strategy $\sigma_s^* = (0, 1)$ is another ESS because the post-entry population,

$$\boldsymbol{x}_\epsilon = (\epsilon p, 1 - \epsilon p),$$

the payoff for an arbitrary strategy is

$$\pi(\sigma, \boldsymbol{x}_\epsilon) = (1 - \epsilon p) - p(1 - 2\epsilon p),$$

and the payoff for the candidate ESS is

$$\pi(\sigma_b^*, \boldsymbol{x}_\epsilon) = 1 - \epsilon p.$$

- We have:

$$\pi(\sigma_b^*, \boldsymbol{x}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) > 0 \Longleftrightarrow p(1 - 2\epsilon p) > 0.$$

- Now, $\forall p \neq p^*$, we have $p > 0$, so σ_s^* is an ESS if and only if $\epsilon p < \frac{1}{2}$, i.e., $\bar{\epsilon} = \frac{1}{2}$.

Solution: continue

The final candidate for an ESS is $\sigma_m^* = (\frac{1}{2}, \frac{1}{2})$ because

$$x = \frac{1}{2} \Longrightarrow \hat{p} \in [0,1] \Longrightarrow x \in [0,1].$$

(including, of course, $x = 1/2$). Consider the post-entry population

$$\boldsymbol{x}_\epsilon \;=\; (1-\epsilon)\left(\frac{1}{2},\frac{1}{2}\right) + \epsilon(p, 1-p) = \left(\frac{1}{2} - \frac{1}{2}\epsilon(1-2p), \frac{1}{2} + \frac{1}{2}\epsilon(1-2p)\right).$$

The payoff for an arbitrary strategy is $\pi(\sigma, \boldsymbol{x}_\epsilon) = \frac{1}{2} + \frac{1}{2}\epsilon(1-2p)^2$, and the payoff for the candidate ESS is $\pi(\sigma_m^*, \boldsymbol{x}_\epsilon) = \frac{1}{2}$. So

$$\pi(\sigma_m^*, \boldsymbol{x}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) > 0 \Longleftrightarrow -\frac{1}{2}\epsilon(1-2p)^2 > 0.$$

Because $\epsilon > 0$ and $p \neq \frac{1}{2}$, this condition **cannot be satisfied**; so σ_m^* is **not** an ESS. So whether to use beads or shells depends on the initial condition.

Homework

Consider a Prisoners' Dilemma where the payoffs for an interaction between two individuals are given by:

	P_2 (C)	P_2 (D)
P_1 (C)	3, 3	0, 5
P_1 (D)	5, 0	1, 1

If a population of individuals play this pairwise contest, what is the ESS?

ESSs and Nash Equilibria

- In this section, we show that ESSs in a pairwise contest population game correspond to a (possibly empty) subset of the set of Nash equilibria for an associated two-player game.
- In a pairwise contest population game, the payoff to a focal individual using σ in a population with profile $\boldsymbol{x}$ is

$$\pi(\sigma, \boldsymbol{x}) = \sum_{s \in \boldsymbol{S}} \sum_{s' \in \boldsymbol{S}} p(s) x(s') \pi(s, s'). \quad (1)$$

- Note that the above payoff is the same as a two-player game against an opponent using a strategy $\sigma^{'}$ that assigns $p^{'}(s) = x(s) \, \forall s \in \boldsymbol{S}$. So there is an association between a two-player game with a population game involving pairwise contests.

Definition

In a pairwise contest population game has payoffs given by Eq. (1), then the associated two-player game is the game with the payoffs given by the numbers $\pi_1(s, s') = \pi(s, s') = \pi_2(s', s)$.

Comment

In a monomorphic population, if σ^* is an ESS, then $\boldsymbol{x}^* = \sigma^*$. So if there is a NE in the associated game corresponding to the ESS in the population game, then it must be of the form (σ^*, σ^*). That is, symmetric NE can be associated with an ESS but an asymmetric one cannot.

Theorem

Let σ^* be an ESS in a pairwise contest, then $\forall\sigma \neq \sigma^*$, either

1. $\pi(\sigma^*, \sigma^*) > \pi(\sigma, \sigma^*)$, *or*
2. $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ *and* $\pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$.

Conversely, if either (1) or (2) holds for each $\sigma \neq \sigma^*$ in a two-player game, then σ^* is an ESS in the corresponding population game.

Proof

If σ^* is an ESS, then by definition, for ϵ sufficiently small,

$$\pi(\sigma^*, \boldsymbol{x}_\epsilon) > \pi(\sigma, \boldsymbol{x}_\epsilon)$$

where $\boldsymbol{x}_\epsilon = (1-\epsilon)\sigma^* + \epsilon\sigma$. For pairwise contests, this condition can be rewritten as:

$$(1-\epsilon)\pi(\sigma^*, \sigma^*) + \epsilon\pi(\sigma^*, \sigma) \quad > \quad (1-\epsilon)\pi(\sigma, \sigma^*) + \epsilon\pi(\sigma, \sigma). \qquad (2)$$

Converse: If condition 1 holds, then Eq. (2) can be satisfied for some sufficiently small ϵ. If condition 2 holds, then Eq. (2) is satisfied for all $0 < \epsilon < 1$.

Direct: Suppose $\pi(\sigma^*, \sigma^*) < \pi(\sigma, \sigma^*)$, then $\exists\epsilon$ sufficiently small that the Eq. (2) is violated. So we have Eq. (2) $\Longrightarrow \pi(\sigma^*, \sigma^*) \geq \pi(\sigma, \sigma^*)$. If $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$, then Eq. (2) $\Longrightarrow \pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$.

Remark 1

- The Nash equilibrium condition is

$$\pi(\sigma^*, \sigma^*) \geq \pi(\sigma, \sigma^*) \quad \forall \sigma \neq \sigma^*$$

  So the condition $\pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$ in Eq. (2) is a supplementary requirement that eliminates some Nash equilibria from consideration.
- In other words, there may be a Nash equilibrium in the two-player game but no corresponding ESS in the population game.
- The supplementary condition is especially relevant in the case of mixed-strategy Nash equilibria.

Remark 2

Previous theorem provides an alternative for finding an ESS in a pairwise contest population game:

1. write down the associated two-player game;
2. find the symmetric Nash equilibria of this game;
3. test the Nash equilibria using condition (1) or (2) above.

Any NE strategy σ^* passes these tests is an ESS, leading to a population profile of $\boldsymbol{x}^* = \sigma^*$.

Example

- Consider the Hawk-Dove game again. The associated two-player game is:

	P_2 (H)	P_2 (D)
P_1 (H)	$\frac{v-c}{2}, \frac{v-c}{2}$	$v, 0$
P_1 (D)	$\frac{v-c}{2}, \frac{v-c}{2}$	$\frac{v}{2}, \frac{v}{2}$

- For $v < c$, there is no symmetric pure-strategy Nash equilibria.
- To find a mixed-strategy Nash equilibrium, we use the Equality of Payoffs theorem:

$$\begin{aligned} \pi_1(H, \sigma^*) &= \pi_1(D, \sigma^*) \\ q^* \frac{v-c}{2} + (1-q^*)v &= (1-q^*)\frac{v}{2} \\ q^* &= \frac{v}{c}. \end{aligned}$$

Continue:

- By the symmetry of the problem, we can deduce that player 1 also plays H with probability $p^* = v/c$.
- To show that $\sigma^* = (p^*, 1 - p^*)$ is an ESS, we should that either condition (1) or (2) holds for $\sigma \neq \sigma^*$.
- Since σ^* is a mixed strategy, the Equality of Payoffs theorem tells us that $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$. So condition (1) does not hold, we have to check whether $\pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$. Now

$$\begin{aligned}
\pi(\sigma^*, \sigma) &= p^* p \frac{v-c}{2} + p^*(1-p)v + (1-p^*)(1-p)\frac{v}{2}; \\
\pi(\sigma, \sigma) &= p^2 \frac{v-c}{2} + p(1-p)v + (1-p)^2 \frac{v}{2}.
\end{aligned}$$

- After some algebraic operations, we have $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) = \frac{c}{2}(p^* - p)^2 > 0 \quad \forall p \neq p^*$. So, σ^* is an ESS.

HW: Exercise 8.7.

Motivation

- So far, we have assumed the pairwise contests game is symmetric.
- In many situations, players engaged in a contest can be different:
  - There may be buyers vs. seller,
  - A firm holding a monopoly vs. a firm seeking to enter the market.
- Such differences between individual may lead to an asymmetric payoff and asymmetric available actions.
- In a population, an individual may find themselves playing a particular role in one game and playing another role in later encounter.
- Thus, the general strategy must specify behavior for **all roles:** use s in role r and s' in role r'.

Our study of Asymmetric Pairwise contests

- We want to determine the payoff if we know:
  - how often an individual assumes a particular role,
  - how often the individuals who meet are playing in which role.
- For the following study, we make the following assumptions:
  - We assume that there are only two roles of interest.
  - A player in one role *always* meets a player in the other role (or role asymmetry).
  - Each individual finds themselves playing each role with equal probability.

Example

- A variation of the hawk-dove game in which two individuals are contesting ownership of a territory that one of them currently controls.
- Value of the territory and the costs of contests are the same for both players.
- The difference with the standard hawk-dove game is that players can now *condition* their behavior on the role they occupy: "owner" or "intruder".
- Therefore, pure strategy is of the form "play Hawk if owner, play Dove if intruder", which is represented by HD.
- The full set of strategies is: $\{HH, HD, DH, DD\}$
- We assume that any contest involves one player in each role and each player has an equal chance of being an owner or an intruder.
- With these assumptions, what is the payoff?

Solution

- Payoff table:

	HH	HD	DH	DD
HH	$\frac{v-c}{2}, \frac{v-c}{2}$	$\frac{3v-c}{4}, \frac{v-c}{4}$	$\frac{3v-c}{4}, \frac{v-c}{4}$	$v, 0$
HD	$\frac{v-c}{4}, \frac{3v-c}{4}$	$\frac{v}{2}, \frac{v}{2}$	$\frac{2v-c}{4}, \frac{2v-c}{4}$	$\frac{3v}{4}, \frac{v}{4}$
DH	$\frac{v-c}{4}, \frac{3v-c}{4}$	$\frac{2v-c}{4}, \frac{2v-c}{4}$	$\frac{v}{2}, \frac{v}{2}$	$\frac{3v}{4}, \frac{v}{4}$
DD	$0, v$	$\frac{v}{4}, \frac{3v}{4}$	$\frac{v}{4}, \frac{3v}{4}$	$\frac{v}{2}, \frac{v}{2}$

- For example, when *HH* plays against *HD*, we have the possibilities of *H* vs. *H*, *H* vs. *D*, *H* vs. *H*, and *H* vs. *D*:

$$\frac{1}{4}\left(\frac{v-c}{2}\right) + \frac{1}{4}v + \frac{1}{4}\left(\frac{v-c}{2}\right) + \frac{1}{4}v = \frac{3v-c}{4}.$$

- Two symmetric pure-strategy NE: [*HD*, *HD*] and [*DH*, *DH*] because for $v < c$, we have: $\frac{v}{2} > \frac{3v-c}{4} > \frac{v}{4} > \frac{2v-c}{4}$.
- The strategies *HD* and *DH* are both ESSs. There is no mixed strategy ESS. HW: Exercise 8.8

Theorem

In a pairwise contest game that possesses role asymmetry, all evolutionary stable strategies are pure.