Introduction to Game Theory: Games with Continuous Strategy Sets

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Outline

Introduction

- So far, we have considered players choose action from a discrete set.
- it is possible for that the pure strategy set is from subsets of the real line, or infinite dimension.
- For example, the pure strategy (action) sets are a subset of real number $[a, b]$.
- A pure strategy is a choice $x \in [a, b]$.
- A mixed strategy is defined by giving a function $p(x)$ such that the probability that the choice lies between x and $x + dx$ is $p(x)dx$.
- The existence of NE for games with continuous pure-strategy sets was proved independently by Debreu, Glicksburg and Fan.
- Let us study some classical games with continuous strategy sets.

Cournot Duopoly

- Consider two firms competing for a market by producing some infinitely divisible product (e.g., petroleum).
- We allow firms to choose how much they produce, e.g., firm i decides on q_i, the quantity to produce, in which $q_i \in [0, \infty)$.
- Each unit production cost is c.
- Let $Q = q_1 + q_2$, which is the total quantity produced by both firms.
- The market price depends on Q, which is

$$P(Q) = \begin{cases} P_0(1 - \frac{Q}{Q_0}) & \text{if } Q < Q_0, \\ 0 & \text{if } Q \geq Q_0. \end{cases}$$

- Payoff for firm i is

$$\pi_i(q_1, q_2) = q_i P(Q) - cq_i \quad \text{for } i = 1, 2.$$

- Obviously, $q_i \in [0, Q_0]$.

Solution for firm 1

- Consider firm 1 against every possible choice of firm 2, the best response is to find $\hat{q}_1$ that maximizes $\pi_1(q_1, q_2)$, or $\frac{\partial \pi_1}{\partial q_1}(\hat{q}_1, q_2) = 0$.
- Solving $\hat{q}_1 = \frac{Q_0}{2}\left(1 - \frac{q_2}{Q_0} - \frac{c}{P_0}\right)$
- We need to check it is the "best", not "worst" response by

$$\frac{\partial^2 \pi_1}{\partial q_1^2}(\hat{q}_1, q_2) = -2\left(\frac{P_0}{Q_0}\right) < 0.$$

- Need to check $\hat{q}_1 + q_2 \leq Q_0$, or

$$\begin{aligned}
\hat{q}_1 + q_2 &= \frac{Q_0}{2}\left(1 - \frac{q_2}{Q_0} - \frac{c}{P_0}\right) + q_2 = \frac{Q_0}{2} + \frac{q_2}{2} - \frac{cQ_0}{2P_0} \\
&< \frac{Q_0}{2} + \frac{Q_0}{2} - \frac{cQ_0}{2P_0} = Q_0\left(1 - \frac{c}{2P_0}\right) < Q_0.
\end{aligned}$$

Overall solution

- Similarly, $\hat{q}_2 = \frac{Q_0}{2}\left(1 - \frac{q_1}{Q_0} - \frac{c}{P_0}\right)$.
- A pure strategy NE is (q_1^*, q_2^*), each is a best response to the other. So we need to solve:

$$q_1^* = \frac{Q_0}{2}\left(1 - \frac{q_2^*}{Q_0} - \frac{c}{P_0}\right); \quad q_2^* = \frac{Q_0}{2}\left(1 - \frac{q_1^*}{Q_0} - \frac{c}{P_0}\right);$$

- The solution is: $q_1^* = q_2^* = \frac{Q_0}{3}\left(1 - \frac{c}{P_0}\right) \equiv q_c^*$.
- Payoff of each firm:

$$\pi_1(q_c^*, q_c^*) = \pi_2(q_c^*, q_c^*) = q_c^* P(2q_c^*) - cq_c^* = \frac{Q_0 P_0}{9}\left(1 - \frac{c}{P_0}\right)^2.$$

Comparison with monopoly

- Under monopoly, the payoff is

$$\pi_m(q) = qP(q) - cq.$$

- Solving, we have

$$q_m^* = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right).$$

- Since $q_m < 2q_c^*$, the price for unit good is higher in the monopoly market than the competitive market. This implies competition can benefit consumer.

Comparison with cartel

- Suppose both firms form a cartel and agree to produce at $q_1 = q_2 = q_m^*/2$, and the payoff is

$$\begin{aligned}\pi_1(q_m^*/2, q_m^*/2) = \pi_2(q_m^*/2, q_m^*/2) &= \frac{1}{2}q_m^* P(q_m^*) - \frac{1}{2}cq_m^* \\ &= \frac{Q_0 P_0}{8}\left(1 - \frac{c}{P_0}\right)^2,\end{aligned}$$

  which is higher than the Cournot payoff and the price for customer is the same as the monopoly market.
- This conclusion is **unstable** because the best response to cartel:

$$\hat{q} = \frac{Q_0}{2}\left(1 - \frac{q_m^*}{2Q_0} - \frac{c}{P_0}\right) = \frac{3}{4}q_m^* > \frac{1}{2}q_m^*.$$

- We are not saying cartel is not possible, this only says cartel will not occur in the situations described by the Cournot model.

Exercise 1

- Consider the "*asymmetric Cournot duopoly game*" where the marginal cost for firm 1 is c_1 and the marginal cost for firm 2 is c_2.
- If $0 < c_i < P_0/2$, $\forall i$, what is the Nash equilibrium?
- If $c_1 < c_2 < P_0$ but $2c_2 > P_0 + c_1$, what is the Nash equilibrium?

Solution to Exercise 1

- Payoffs of firms: $\pi_i(q_1, q_2) = q_i \left[P_0\left(1 - \frac{q_1+q_2}{Q_0}\right) - c_i\right]$.
- The best response is $\hat{q}_1 = \frac{Q_0}{2}\left(1 - \frac{q_2}{Q_0} - \frac{c_1}{P_0}\right)$ and $\hat{q}_2 = \frac{Q_0}{2}\left(1 - \frac{q_1}{Q_0} - \frac{c_2}{P_0}\right)$
- NE strategies are found by solving the above simultaneous equations, we have: $q_1^* = \frac{Q_0}{3}\left(1 - \frac{2c_1-c_2}{P_0}\right)$, $q_2^* = \frac{Q_0}{3}\left(1 - \frac{2c_2-c_1}{P_0}\right)$.
- For this to be NE, we need $q_1^* > 0$ and $q_2^* > 0$, which implies $2c_1 - c_2 < P_0$ and $2c_2 - c_1 < P_0$.
- If $0 < c_1, c_2 < P_0/2$, the above conditions satisfied so they are the NE strategies.
- If $2c_2 > P_0 + c_1$, then $q_2^* < 0$ so the above cannot be NE. In this case, the NE is: $q_1^* = \frac{Q_0}{2}(1 - \frac{c_1}{P_0})$, $q_2^* = 0$.

Exercise 2

- Consider the $n-$player Cournot game. We have n identical firms (i.e., same production cost) produce quantities $q_1, q_2, \ldots, q_n$. The market price is given by $P(Q) = P_0(1 - Q/Q_0)$ where $Q = \sum_{i=1}^{n} q_i$. Find the symmetric Nash equilibrium (i.e., $q_i^* = q^* \, \forall i$). What happens to each firm's profit as $n \to \infty$?

- Assume all other firms except firm 1 are producing quantity q and firm 1 is producing (possibly different) quantity q_1, then

$$\pi_1(q_1, q, \ldots, q) = q_1 \left[P_0 \left(1 - \frac{q_1 + (n-1)q}{Q_0} \right) - c \right].$$

- The best response for firm 1 is:

$$\hat{q}_1 = \frac{Q_0}{2} \left(1 - (n-1)\frac{q}{Q_0} - \frac{c}{P_0} \right).$$

- The symmetric Nash equilibrium q^* is:

$$q^* = \frac{Q_0}{2} \left(1 - (n-1)\frac{q^*}{Q_0} - \frac{c}{P_0} \right) = \frac{Q_0}{n+1} \left(1 - \frac{c}{P_0} \right).$$

- This gives a profit to each firm of

$$\pi_i(q^*, \ldots, q^*) = q^* \left[P_0 \left(1 - \frac{nq^*}{Q_0} \right) - c \right] = \frac{Q_0 P_0}{(n+1)^2} \left(1 - \frac{c}{P_0} \right)^2.$$

So $\lim_{n\to\infty} \pi_i = 0$.

Bertrand Model of Duopoly

- Consider a case of *differentiated* products with two firms 1 and 2 choose prices p_1 and p_2 respectively.
- The quantity that consumers demand from firm i is

$$q_i(p_i, p_j) = a - p_i + bp_j, \;\; b > 0.$$

- Assume no fixed costs of production and marginal costs are constant at c, where $c < a$.
- Both firms act simultaneously.
- Each firm's strategy space is $S_i = [0, \infty)$.
- A typical strategy s_i is now a price choice, $p_i \geq 0$.

Bertrand Model of Duopoly

- Profit function of firm i:

$$\pi_i(p_i, p_j) = q_i(p_i, p_j)[p_i - c] = [a - p_i + bp_j][p_i - c]$$

- (p_1^*, p_2^*) is a NE if for each firm i, p_i^* solves:

$$\max_{0 \leq p_i < \infty} \pi_i(p_i, p_j^*) = \max_{0 \leq p_i < \infty} [a - p_i + bp_j^*][p_i - c]$$

- The solution to firm i's optimization is

$$p_i^* = \frac{1}{2}(a + bp_j^* + c), \quad i = 1, 2.$$

- Solving these two equations, we have

$$p_1^* = p_2^* = \frac{a + c}{2 - b}.$$

Final-Offer Arbitration

- Parties in dispute of wages: a firm and a union.
- Firm and union make offer *simultaneously*: w_f and w_u.
- Arbitrator chooses one of the offer as the settlement.
- Arbitrator has an ideal settlement of x. She simply chooses the offer that is closer to x (provided $w_f < w_u$).
- Formally: choose w_f if $x < (w_f + w_u)/2$; choose w_u if $x > (w_f + w_u)/2$. *(If tie, toss a coin to choose either w_f or w_u).*
- Arbitrator knows x but parties do not. The parties believe x is randomly distributed according to PDF $F(x)$ or pdf $f(x)$.

Final-Offer Arbitration

- $\text{Prob}(w_f \text{ chosen}) = \text{Prob}\{x < \frac{w_f + w_u}{2}\} = F\{\frac{w_f + w_u}{2}\}$.
- $\text{Prob}(w_u \text{ chosen}) = \text{Prob}\{x > \frac{w_f + w_u}{2}\} = 1 - F\{\frac{w_f + w_u}{2}\}$.
- The expected wage settlement is:

$$
\begin{aligned}
& w_f \text{Prob}(w_f \text{ chosen}) + w_u \text{Prob}(w_u \text{ chosen}) \\
& \qquad = w_f F\left(\frac{w_f + w_u}{2}\right) + w_u \left[1 - F\left(\frac{w_f + w_u}{2}\right)\right].
\end{aligned}
$$

- Firm (union) wants to *minimize* (*maximize*) the expected wage settlement.

Final-Offer Arbitration

Optimization at Nash equilibrium (w_f^*, w_u^*)

Firm: $$\min_{w_f} w_f F\left(\frac{w_f + w_u^*}{2}\right) + w_u^*\left[1 - F\left(\frac{w_f + w_u^*}{2}\right)\right]$$

Union: $$\max_{w_u} w_f^* F\left(\frac{w_f^* + w_u}{2}\right) + w_u\left[1 - F\left(\frac{w_f^* + w_u}{2}\right)\right]$$

Solving:

$$(w_u^* - w_f^*)\frac{1}{2} f\left(\frac{w_f^* + w_u^*}{2}\right) = F\left(\frac{w_f^* + w_u^*}{2}\right)$$

$$(w_u^* - w_f^*)\frac{1}{2} f\left(\frac{w_f^* + w_u^*}{2}\right) = \left[1 - F\left(\frac{w_f^* + w_u^*}{2}\right)\right]$$

This implies that $F\left(\frac{w_f^* + w_u^*}{2}\right) = \frac{1}{2}$. The average of the offer must equal to the median of the arbitrator's preferred settlement.

Final-Offer Arbitration

If F is a normal distribution with mean m and variance σ^2, then

$$\frac{w_f^* + w_u^*}{2} = m \quad \text{and} \quad w_u^* - w_f^* = \frac{1}{f(m)} = \sqrt{2\pi\sigma^2}.$$

At the Nash equilibrium, we have

$$w_u^* = m + \sqrt{\pi\sigma^2/2}\ ;\ w_f^* = m - \sqrt{\pi\sigma^2/2}.$$

- Parties' offers are centered around the expectation of the arbitrator's preferred settlement (i.e., m).
- The gap between the offers increase with the parties' uncertainty about the arbitrator's preferred settlement (i.e., σ^2).

The Stackelberg Duopoly

- Similar to the Cournot model, we have two firms, each needs to determine the amount of production, and the same market price $P(Q) = P_0(1 - Q/Q_0)$ where $Q = q_1 + q_2$.
- However, we have sequential decision: Firm 1 (or market leader) decides first and then firm 2 decides. We assume each firm wants to maximize its profit, and $P_0 > c$.
- Determine q_1^*, q_2^*, payoffs $\pi_1(q_1^*, q_2^*)$ and $\pi_2(q_1^*, q_2^*)$.

Solution to the Stackelberg Duopoly Model

- We first use backward induction to find the subgame perfect NE by finding the best response of firm 2, $\hat{q}_2(q_1)$, for every possible value of q_1.
- Given that firm 1 knows firm 2's best response, we find the best response of firm 1, $\hat{q}_1(\hat{q}_2)$, so as to find the NE for this game.

Solution to the Stackelberg Duopoly Model: continue

- Firm 2's profit: $\pi_2(q_1, q_2) = q_2[P(Q) - c]$ and the best response to a choice of q_1 is found by solving: $\frac{\partial \pi_2}{\partial q_2}(q_1, q_2) = 0$, which gives

$$\hat{q}_2(q_1) = \frac{Q_0}{2}\left(1 - \frac{q_1}{Q_0} - \frac{c}{P_0}\right).$$

- Firm 1 chooses q_1 based on the best response of $\hat{q}_2(q_1)$, firm 1's payoff:

$$\begin{aligned}\pi_1(q_1, \hat{q}_2(q_1)) &= q_1\left[P_0\left(1 - \frac{q_1 + \hat{q}_2(q_1)}{Q_0}\right) - c\right] \\ &= q_1\left(\frac{P_0}{2}\right)\left(1 - \frac{q_1}{Q_0} - \frac{c}{P_0}\right).\end{aligned}$$

- Firm 1 maximizes its profit at: $\hat{q}_1 = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right)$.

Solution to the Stackelberg Duopoly Model: continue

- By evaluation $\frac{\partial \pi_1(q_1, \hat{q}_2)}{\partial q_1} = 0$, one can find that firm 1 maximizes its profit at: $\hat{q}_1 = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right)$.
- The Nash equilibrium is:

$$q_1^* = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right); q_2^* = \hat{q}_2(q_1^*) = \frac{Q_0}{4}\left(1 - \frac{c}{P_0}\right).$$

- Some interesting note:
  1. Leader's advantage: since $q_1^* > q_2^*$, this implies $\pi_1(q_1^*, q_2^*) > \pi_2(q_1^*, q_2^*)$.
  2. The price of the good is cheaper under the Stackelberg duopoly than Cournot duopoly.
- HW: Exercise 6.5.

3-period Bargaining Game: 1 unit of resource

- In the first period, Player 1 proposes to take s_1 of the resource, leaving $1 - s_1$ to Player 2.
- Player 2 either **accepts** (and the game ends with payoffs s_1 to Player 1 and $1 - s_1$ to Player 2), or **reject** (the game continues).
- In the second period, Player 2 proposes that Player 1 to take s_2 of the resource, leaving $1 - s_2$ to Player 2.
- Player 1 either **accepts** (and the game ends with payoffs s_2 to Player 1 and $1 - s_2$ to Player 2), or **reject** (the game continues).
- In the third period, Player 1 receives s of the resource, player 2 receives $1 - s$ of the resource, where $0 < s < 1$.

There is a **discount** factor δ per period, $0 < \delta < 1$.

Solution

- Consider Player 2's optimal offer if the 2nd period is reached.
- Player 1 is facing a choice, choose s_2 or receive δs. Player 1 will accept the offer **iff**

$$s_2 \geq \delta s.$$

- Player 2's 2nd-period decision:
  1. receiving $1 - \delta s$ (by offering $s_2 = \delta s$ to Player 1), or
  2. receiving $\delta(1 - s)$ in the third period.
- Since $1 - \delta s > \delta(1 - s)$, Player 2's optimal 2nd-round choice is $s_2^* = \delta s$ and Player 1 will accept.

Solution: continue

- Player 1 is facing a choice in the 1st-period.
- Player 2 will only accept the offer in the 1st-period iff
  1. $1 - s_1 \geq \delta(1 - s_2^*)$, or
  2. $s_1 \leq 1 - \delta(1 - s_2^*)$.
- Player 1's 1st-period decision:
  1. receiving $1 - \delta(1 - s_2^*) = 1 - \delta(1 - \delta s)$ (making that bid), or
  2. receiving $\delta s_2^* = \delta^2 s$.
- Since $1 - \delta(1 - \delta s) > \delta^2 s$, so Player 1's optimal 1st-period offer is $s_1^* = 1 - \delta(1 - \delta s)$.
- The solution of the game should end in the 1st-period with $(s_1^*, 1 - s_1^*)$, where $s_1^* = 1 - \delta(1 - \delta s)$.

Extension to infinite rounds

- What about if we have *infinite* number of rounds?
- Truncate the infinite-horizon game and apply the logic from the finite-horizon case.
- The game in the 3rd period, should it be reached, is identical to the game beginning in the 1st period.
- Let S_H be the highest payoff player 1 can achieve in any backwards-induction outcome of the game as a whole.

Extension to infinite rounds: continue

- Using S_H as the 3rd period payoff to player 1.
- Player 1's first-period payoff is $f(S_H)$ where

$$f(s) = 1 - \delta + \delta^2 s.$$

- But S_H is also the highest possible 1st-period payoff, so $f(S_H) = S_H$.
- The only value of s that satisfy $f(s) = s$ is

$$s^* = 1/(1+\delta).$$

- Solution is, in the first round, player 1 offers $(s^*, 1-s^*) = (1/(1+\delta), \delta/(1+\delta))$ to player 2, who will accept.

Bank Runs

- Two investors each deposited D with a bank.
- The bank invested in a project. If it's forced to liquidate before the project matures, a return of $2r$, where $D > r > D/2$. If the project matures, a return of $2R$, where $R > D$.
- Investors can withdraw on date 1 (before the project matures) or date 2 (after the project matures).
- The game is:
  1. If both investors make withdrawals at date 1, each receives r, game ends.
  2. If only one makes withdrawal at date 1, that investor receives D, other receives $2r - D$, game ends.
  3. If both withdraw at date 2, each receives R, game ends.
  4. If only one withdraws at date 2, that investor receives $2R - D$, other receives D, game ends.
  5. If neither makes withdrawal at date 2, banks returns R to each investor, game ends.

"Normal-Form" of the game

For two dates

Date 1	Investor 2 **(Withdraw)**	Investor 2 **(Don't)**
Investor 1 **(Withdraw)**	r,r	D, 2r-D
Investor 1 **(Don't)**	2r-D, D	next stage

Date 2	Investor 2 **(Withdraw)**	Investor 2 **(Don't)**
Investor 1 **(Withdraw)**	R,R	2R-D,D
Investor 1 **(Don't)**	D, 2R-D	R,R

Analysis

- Consider date 2, since $R > D$ (and so $2R - D > R$), "**withdraw**" strictly dominates "**don't**", we have a unique Nash equilibrium.
- For date 1, we have:

Date 1	Investor 2 **(Withdraw)**	Investor 2 **(Don't)**
Investor 1 **(Withdraw)**	r,r	D, 2r-D
Investor 1 **(Don't)**	2r-D, D	R,R

- Since $r < D$ (and so $2r - D < r$), we have two pure-strategy Nash Equilibrium, (a) both withdraw, (b) both don't withdraw, with the 2nd NE being efficient.

Tariffs and Imperfect Competition

- Consider two countries, denoted by $i = 1, 2$, each setting a tariff rate t_i per unit of product.
- A firm produces output, both for home consumption and export.
- Consumer can buy from a local firm or foreign firm.
- The market clearing price for country i is $P(Q_i) = a - Q_i$, where Q_i is the quantity on the market in country i.
- A firm in i produces $h_i(e_i)$ units for local (foreign) market, i.e., $Q_i = h_i + e_j$.
- The production cost of firm i is $C_i(h_i, e_i) = c(h_i + e_i)$ and it pays $t_j e_i$ to country j.

Tariffs and Imperfect Competition Game

- First, the government *simultaneously* choose tariff rates t_1 and t_2.
- Second, the firms observe the tariff rates, decide (h_1, e_1) and (h_2, e_2) *simultaneously*.
- Third, payoffs for both firms and governments:
  (1) Profit for firm i:

$$\begin{aligned}\pi_i(t_i, t_j, h_i, e_i, h_j, e_j) &= [a - (h_i + e_j)]h_i + [a - (e_i + h_j)]e_i \\ &\quad - c(h_i + e_i) - t_j e_i\end{aligned}$$

  (2) Welfare for government i:

$$W_i(t_i, t_j, h_i, e_i, h_j, e_j) = \frac{1}{2}Q_i^2 + \pi_i(t_i, t_j, h_i, e_i, h_j, e_j) + t_i e_j$$

Tariffs and Imperfect Competition Game: 2nd stage

- Suppose the governments have chosen t_1 and t_2.
- If $(h_1^*, e_1^*, h_2^*, e_2^*)$ is a NE for firm 1 and 2, firm i needs to solve $\max_{h_i, e_i \geq 0} \pi_i(t_i, t_j, h_i, e_i, h_j^*, e_j^*)$. After re-arrangement, it becomes two separable optimizations:

$$\max_{h_i \geq 0} h_i[a - (h_i + e_j^*) - c]; \ \max_{e_i \geq 0} e_i[a - (e_i + h_j^*) - c] - t_j e_i.$$

- Assuming $e_j^* \leq a - c$ and $h_j^* \leq a - c - t_j$, we have

$$h_i^* = \frac{1}{2}\left(a - e_j^* - c\right) \ ; \ e_i^* = \frac{1}{2}\left(a - h_j^* - c - t_j\right), \ \ i = 1, 2.$$

- Solving, we have

$$h_i^* = \frac{a - c - t_i}{3} \ ; \ e_i^* = \frac{a - c - 2t_j}{3}, \ \ i = 1, 2.$$

Tariffs and Imperfect Competition Game: 1st stage

- In the first stage, government i payoff is:

$$W_i(t_i, t_j, h_1^*, e_1^*, h_2^*, e_2^*) = W_i(t_i, t_j)$$

  since h_i^* (e_i^*) is a function of t_i (t_j).
- If (t_1^*, t_2^*) is a NE, each government solves:

$$\max_{t_i \geq 0} W_i(t_i, t_j^*).$$

- Solving the optimization, we have $t_i^* = \frac{a-c}{3}$, for $i = 1, 2$. which is a *dominant strategy* for each government.
- Substitute t_i^*, we have

$$h_i^* = \frac{4(a-c)}{9} \; ; \; e_i^* = \frac{a-c}{9}, \text{ for } i = 1, 2.$$

Comment on Tariffs and Imperfect Competition Game

- In the subgame-perfect outcome, the aggregate quantity on each market is $5(a-c)/9$.
- But if two governments cooperate, they seek socially optimal point and they solve the following optimization problem :

$$\max_{t_1, t_2 \geq 0} W_1(t_1, t_2) + W_2(t_1, t_2)$$

  The solution is $t_1^* = t_2^* = 0$ (no tariff) and the aggregate quantity is $2(a-c)/3$.
- Therefore, for the above game, we have a unique NE, and it is socially inefficient.

Model for War of Attrition

- Two players compete for a resource of value v, i.e., two companies engaged in a price war.
- The strategy for each player is a choice of a *persistence time*, t_i, where $t_i \in [0, \infty)$.
- Three assumptions
  - The cost of the contest is related only to its duration.
  - The player that persists the longest gets *all* of the resource.
  - The cost paid by each player is proportional to the *shortest persistence* time chosen, or no cost is incurred after on player quits and the contest ends.
- What are the pure NE strategies?
- What are the mixed NE strategies?

Solution to the War of Attrition

- Payoffs for the two players are:

$$\pi_1(t_1, t_2) = \begin{cases} v - ct_2 & \text{if } t_1 > t_2, \\ -ct_1 & \text{if } t_1 \leq t_2. \end{cases}$$

$$\pi_2(t_1, t_2) = \begin{cases} v - ct_1 & \text{if } t_2 > t_1, \\ -ct_2 & \text{if } t_2 \leq t_1. \end{cases}$$

- One pure strategy NE is: $t_1^* = \frac{v}{c}$ and $t_2^* = 0$, giving

$$\pi_1(v/c, 0) = v \quad \text{and} \quad \pi_2(v/c, 0) = 0.$$

Why NE?

- It is a NE because for player 1:

$$\pi_1(t_1, 0) = v, \; \forall t_i > 0 \quad \text{and} \quad \pi_1(0, 0) = 0,$$

which gives

$$\pi_1(t_1, t_2^*) \leq \pi_1(t_1^*, t_2^*) \; \forall t_1.$$

- For player 2, we have

$$\pi_2(v/c, t_2) = -ct_2 < 0, \; \forall t_2 \leq v/c \text{ and } \pi_2(v/c, t_2) = 0, \; \forall t_2 > v/c.$$

Hence

$$\pi_2(t_1^*, t_2) \leq \pi_2(t_1^*, t_2^*) \quad \forall t_2.$$

Other NE

- The second pure strategy NE is: $t_1^* = 0$ and $t_2^* = \frac{v}{c}$. Giving $\pi_1(0, v/c) = 0$ and $\pi_2(0, v/c) = v$. Analysis is similar to previous argument.
- There is a mixed-strategy Nash equilibrium. For detail, refer to the textbook.