Introduction to Game Theory: Games with Continuous Strategy Sets

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Introduction

- So far, we have considered players choose action from a discrete set.
- It is possible for the pure strategy set to be subsets of the real line, or infinite dimension.
- For example, the pure strategy (action) sets are a subset of real number \([a, b]\).
- A pure strategy is a choice \(x \in [a, b]\).
- A mixed strategy is defined by giving a function \(p(x)\) such that the probability that the choice lies between \(x\) and \(x + dx\) is \(p(x)dx\).
- The existence of NE for games with continuous pure-strategy sets was proved independently by Debreu, Glicksburg and Fan.
- Let us study some classical games with continuous strategy sets.
Cournot Duopoly

Consider two firms competing for a market by producing some infinitely divisible product (e.g., petroleum).

We allow firms to choose how much they produce, e.g., firm $i$ decides on $q_i$, the quantity to produce, in which $q_i \in [0, \infty)$.

Each unit production cost is $c$.

Let $Q = q_1 + q_2$, which is the total quantity produced by both firms.

The market price depends on $Q$, which is

$$P(Q) = \begin{cases} 
  P_0(1 - \frac{Q}{Q_0}) & \text{if } Q < Q_0, \\
  0 & \text{if } Q \geq Q_0.
\end{cases}$$

Payoff for firm $i$ is

$$\pi_i(q_1, q_2) = q_iP(Q) - cq_i$$

for $i = 1, 2$.

Obviously, $q_i \in [0, Q_0]$. 
Solution for firm 1

Consider firm 1 against every possible choice of firm 2, the best response is to find $\hat{q}_1$ that maximizes $\pi_1(q_1, q_2)$, or $\frac{\partial \pi_1}{\partial q_1}(\hat{q}_1, q_2) = 0$.

Solving $\hat{q}_1 = \frac{Q_0}{2} \left( 1 - \frac{q_2}{Q_0} - \frac{c}{P_0} \right)$

We need to check it is the "best", not "worst" response by

$$\frac{\partial^2 \pi_1}{\partial q_1^2}(\hat{q}_1, q_2) = -2 \left( \frac{P_0}{Q_0} \right) < 0.$$ 

Need to check $\hat{q}_1 + q_2 \leq Q_0$, or

$$\hat{q}_1 + q_2 = \frac{Q_0}{2} \left( 1 - \frac{q_2}{Q_0} - \frac{c}{P_0} \right) + q_2 = \frac{Q_0}{2} + \frac{q_2}{2} - \frac{cQ_0}{2P_0}$$

$$< \frac{Q_0}{2} + \frac{Q_0}{2} - \frac{cQ_0}{2P_0} = Q_0 \left( 1 - \frac{c}{2P_0} \right) < Q_0.$$
The Cournot Duopoly Model

Overall solution

- Similarly, \( \hat{q}_2 = \frac{Q_0}{2} \left( 1 - \frac{q_1}{Q_0} - \frac{c}{P_0} \right) \).
- A pure strategy NE is \((q_1^*, q_2^*)\), each is a best response to the other. So we need to solve:

\[
q_1^* = \frac{Q_0}{2} \left( 1 - \frac{q_2^*}{Q_0} - \frac{c}{P_0} \right); \quad q_2^* = \frac{Q_0}{2} \left( 1 - \frac{q_1^*}{Q_0} - \frac{c}{P_0} \right);
\]

- The solution is: \( q_1^* = q_2^* = \frac{Q_0}{3} \left( 1 - \frac{c}{P_0} \right) \equiv q_c^* \).
- Payoff of each firm:

\[
\pi_1(q_c^*, q_c^*) = \pi_2(q_c^*, q_c^*) = q_c^* P(2q_c^*) - cq_c^* = \frac{Q_0 P_0}{9} \left( 1 - \frac{c}{P_0} \right)^2.
\]
Comparison with monopoly

- Under monopoly, the payoff is
  \[ \pi_m(q) = qP(q) - cq. \]

- Solving, we have
  \[ q_m^* = \frac{Q_0}{2} \left( 1 - \frac{c}{P_0} \right). \]

- Since \( q_m < 2q_c^* \), the price for unit good is higher in the monopoly market than the competitive market. This implies competition can benefit consumer.
Comparison with cartel

Suppose both firms form a cartel and agree to produce at $q_1 = q_2 = q_m^*/2$, and the payoff is

$$\pi_1(q_m^*/2, q_m^*/2) = \pi_2(q_m^*/2, q_m^*/2) = \frac{1}{2} q_m^* P(q_m^*) - \frac{1}{2} c q_m^*$$

$$= \frac{Q_0 P_0}{8} \left(1 - \frac{c}{P_0}\right)^2,$$

which is higher than the Cournot payoff and the price for customer is the same as the monopoly market.

This conclusion is **unstable** because the best response to cartel:

$$\hat{q} = \frac{Q_0}{2} \left(1 - \frac{q_m^*}{2 Q_0} - \frac{c}{P_0}\right) = \frac{3}{4} q_m^* > \frac{1}{2} q_m^*.$$

We are not saying cartel is not possible, this only says cartel will not occur in the situations described by the Cournot model.
Exercise 1

Consider the "asymmetric Cournot duopoly game" where the marginal cost for firm 1 is $c_1$ and the marginal cost for firm 2 is $c_2$.

- If $0 < c_i < P_0/2$, $\forall i$, what is the Nash equilibrium?
- If $c_1 < c_2 < P_0$ but $2c_2 > P_0 + c_1$, what is the Nash equilibrium?


Solution to Exercise 1

- Payoffs of firms: $\pi_i(q_1, q_2) = q_i \left[ P_0 \left( 1 - \frac{q_1 + q_2}{Q_0} \right) - c_i \right]$. 
- The best response is $\hat{q}_1 = \frac{Q_0}{2} \left( 1 - \frac{q_2}{Q_0} - \frac{c_1}{P_0} \right)$ and $\hat{q}_2 = \frac{Q_0}{2} \left( 1 - \frac{q_1}{Q_0} - \frac{c_2}{P_0} \right)$.
- NE strategies are found by solving the above simultaneous equations, we have: $q_1^* = \frac{Q_0}{3} \left( 1 - \frac{2c_1 - c_2}{P_0} \right)$, $q_2^* = \frac{Q_0}{3} \left( 1 - \frac{2c_2 - c_1}{P_0} \right)$.
- For this to be NE, we need $q_1^* > 0$ and $q_2^* > 0$, which implies $2c_1 - c_2 < P_0$ and $2c_2 - c_1 < P_0$.
- If $0 < c_1, c_2 < P_0/2$, the above conditions satisfied so they are the NE strategies.
- If $2c_2 > P_0 + c_1$, then $q_2^* < 0$ so the above cannot be NE. In this case, the NE is: $q_1^* = \frac{Q_0}{2} \left( 1 - \frac{c_1}{P_0} \right)$, $q_2^* = 0$. 

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Exercise 2

Consider the $n$–player Cournot game. We have $n$ identical firms (i.e., same production cost) produce quantities $q_1, q_2, \ldots, q_n$. The market price is given by $P(Q) = P_0(1 - Q/Q_0)$ where $Q = \sum_{i=1}^{n} q_i$. Find the symmetric Nash equilibrium (i.e., $q_i^* = q^* \forall i$). What happens to each firm’s profit as $n \to \infty$?
Assume all other firms except firm 1 are producing quantity $q$ and firm 1 is producing (possibly different) quantity $q_1$, then

$$\pi_1(q_1, q, \ldots, q) = q_1 \left[ P_0 \left( 1 - \frac{q_1 + (n-1)q}{Q_0} \right) - c \right].$$

The best response for firm 1 is:

$$\hat{q}_1 = \frac{Q_0}{2} \left( 1 - (n-1) \frac{q}{Q_0} - \frac{c}{P_0} \right).$$

The symmetric Nash equilibrium $q^*$ is:

$$q^* = \frac{Q_0}{2} \left( 1 - (n-1) \frac{q^*}{Q_0} - \frac{c}{P_0} \right) = \frac{Q_0}{n+1} \left( 1 - \frac{c}{P_0} \right).$$

This gives a profit to each firm of

$$\pi_i(q^*, \ldots, q^*) = q^* \left[ P_0 \left( 1 - \frac{nq^*}{Q_0} \right) - c \right] = \frac{Q_0 P_0}{(n+1)^2} \left( 1 - \frac{c}{P_0} \right)^2.$$

So $\lim_{n \to \infty} \pi_i = 0$. 
Bertrand Model of Duopoly

- Consider a case of *differentiated* products with two firms 1 and 2 choose prices $p_1$ and $p_2$ respectively.
- The quantity that consumers demand from firm $i$ is
  \[ q_i(p_i, p_j) = a - p_i + bp_j, \quad b > 0. \]
- Assume no fixed costs of production and marginal costs are constant at $c$, where $c < a$.
- Both firms act simultaneously.
- Each firm’s strategy space is $S_i = [0, \infty)$. 
- A typical strategy $s_i$ is now a price choice, $p_i \geq 0$. 

Bertrand Model of Duopoly

- Profit function of firm $i$:
  \[ \pi_i(p_i, p_j) = q_i(p_i, p_j)(p_i - c) = [a - p_i + bp_j][p_i - c] \]

- ($p_1^*, p_2^*$) is a NE if for each firm $i$, $p_i^*$ solves:
  \[ \max_{0 \leq p_i < \infty} \pi_i(p_i, p_j^*) = \max_{0 \leq p_i < \infty} [a - p_i + bp_j^*][p_i - c] \]

- The solution to firm $i$’s optimization is
  \[ p_i^* = \frac{1}{2}(a + bp_j^* + c), \quad i = 1, 2. \]

- Solving these two equations, we have
  \[ p_1^* = p_2^* = \frac{a + c}{2 - b}. \]
Final-Offer Arbitration

- Parties in dispute of wages: a firm and a union.
- Firm and union make offer \textit{simultaneously}: \(w_f\) and \(w_u\).
- Arbitrator chooses one of the offer as the settlement.
- Arbitrator has an ideal settlement of \(x\). She simply chooses the offer that is closer to \(x\) (provided \(w_f < w_u\)).
- Formally: choose \(w_f\) if \(x < (w_f + w_u)/2\); choose \(w_u\) if \(x > (w_f + w_u)/2\). \textit{(If tie, toss a coin to choose either} \(w_f\) \textit{or} \(w_u\)).
- Arbitrator knows \(x\) but parties do not. The parties believe \(x\) is randomly distributed according to PDF \(F(x)\) or pdf \(f(x)\).
## Final-Offer Arbitration

- \( \text{Prob}(w_f \text{ chosen}) = \text{Prob}\{ x < \frac{w_f + w_u}{2} \} = F\left\{ \frac{w_f + w_u}{2} \right\}. \)
- \( \text{Prob}(w_u \text{ chosen}) = \text{Prob}\{ x > \frac{w_f + w_u}{2} \} = 1 - F\left\{ \frac{w_f + w_u}{2} \right\}. \)

The expected wage settlement is:

\[
\begin{align*}
& w_f \text{Prob}(w_f \text{ chosen}) + w_u \text{Prob}(w_u \text{ chosen}) \\
= & \quad w_f F\left( \frac{w_f + w_u}{2} \right) + w_u \left[ 1 - F\left( \frac{w_f + w_u}{2} \right) \right].
\end{align*}
\]

- Firm (union) wants to minimize (maximize) the expected wage settlement.
Final-Offer Arbitration

**Optimization at Nash equilibrium** \((w_f^*, w_u^*)\)

**Firm:** \(\min_{w_f} w_f F \left( \frac{w_f + w_u^*}{2} \right) + w_u^* \left[ 1 - F \left( \frac{w_f + w_u^*}{2} \right) \right]\)

**Union:** \(\max_{w_u} w_f^* F \left( \frac{w_f^* + w_u}{2} \right) + w_u \left[ 1 - F \left( \frac{w_f^* + w_u}{2} \right) \right]\)

Solving:

\[
(w_u^* - w_f^*) \frac{1}{2} f \left( \frac{w_f^* + w_u^*}{2} \right) = F \left( \frac{w_f^* + w_u^*}{2} \right)
\]

\[
(w_u^* - w_f^*) \frac{1}{2} f \left( \frac{w_f^* + w_u^*}{2} \right) = \left[ 1 - F \left( \frac{w_f^* + w_u^*}{2} \right) \right]
\]

This implies that \(F \left( \frac{w_f^* + w_u^*}{2} \right) = \frac{1}{2}\). The average of the offer must equal to the median of the arbitrator’s preferred settlement.
Final-Offer Arbitration

If $F$ is a normal distribution with mean $m$ and variance $\sigma^2$, then

$$\frac{w_f^* + w_u^*}{2} = m \quad \text{and} \quad w_u^* - w_f^* = \frac{1}{f(m)} = \sqrt{2\pi\sigma^2}.$$ 

At the Nash equilibrium, we have

$$w_u^* = m + \sqrt{\pi\sigma^2}/2 ; \quad w_f^* = m - \sqrt{\pi\sigma^2}/2.$$ 

- Parties’ offers are centered around the expectation of the arbitrator’s preferred settlement (i.e., $m$).
- The gap between the offers increase with the parties’ uncertainty about the arbitrator’s preferred settlement (i.e., $\sigma^2$).
The Stackelberg Duopoly

Similar to the Cournot model, we have two firms, each needs to determine the amount of production, and the same market price $P(Q) = P_0(1 - Q/Q_0)$ where $Q = q_1 + q_2$.

However, we have **sequential decision**: Firm 1 (or market leader) decides first and then firm 2 decides. We assume each firm wants to maximize its profit, and $P_0 > c$.

Determine $q_1^*$, $q_2^*$, payoffs $\pi_1(q_1^*, q_2^*)$ and $\pi_2(q_1^*, q_2^*)$. 
Solution to the Stackelberg Duopoly Model

- We first use **backward induction** to find the subgame perfect NE by finding the best response of firm 2, \( \hat{q}_2(q_1) \), for every possible value of \( q_1 \).

- Given that firm 1 knows firm 2’s best response, we find the best response of firm 1, \( \hat{q}_1(\hat{q}_2) \), so as to find the NE for this game.
Solution to the Stackelberg Duopoly Model: continue

- Firm 2’s profit: \( \pi_2(q_1, q_2) = q_2[P(Q) - c] \) and the best response to a choice of \( q_1 \) is found by solving: \( \frac{\partial \pi_2}{\partial q_2}(q_1, q_2) = 0 \), which gives
  \[
  \hat{q}_2(q_1) = \frac{Q_0}{2} \left(1 - \frac{q_1}{Q_0} - \frac{c}{P_0}\right).
  \]

- Firm 1 chooses \( q_1 \) based on the best response of \( \hat{q}_2(q_1) \), firm 1’s payoff:
  \[
  \pi_1(q_1, \hat{q}_2(q_1)) = q_1 \left[P_0 \left(1 - \frac{q_1 + \hat{q}_2(q_1)}{Q_0}\right) - c\right] = q_1 \left(\frac{P_0}{2}\right) \left(1 - \frac{q_1}{Q_0} - \frac{c}{P_0}\right).
  \]

- Firm 1 maximizes its profit at: \( \hat{q}_1 = \frac{Q_0}{2} \left(1 - \frac{c}{P_0}\right) \).
The Stackelberg Duopoly Model

Solution to the Stackelberg Duopoly Model: continue

- By evaluation \( \frac{\partial \pi_1(q_1, \hat{q}_2)}{\partial q_1} = 0 \), one can find that firm 1 maximizes its profit at: \( \hat{q}_1 = \frac{Q_0}{2} \left(1 - \frac{c}{P_0}\right) \).

- The Nash equilibrium is:

\[
q_1^* = \frac{Q_0}{2} \left(1 - \frac{c}{P_0}\right) ; \quad q_2^* = \hat{q}_2(q_1^*) = \frac{Q_0}{4} \left(1 - \frac{c}{P_0}\right).
\]

- Some interesting note:

1. Leader’s advantage: since \( q_1^* > q_2^* \), this implies \( \pi_1(q_1^*, q_2^*) > \pi_2(q_1^*, q_2^*) \).

2. The price of the good is cheaper under the Stackelberg duopoly than Cournot duopoly.

- HW: Exercise 6.5.
3-period Bargaining Game: 1 unit of resource

- In the first period, Player 1 proposes to take $s_1$ of the resource, leaving $1 - s_1$ to Player 2.
- Player 2 either **accepts** (and the game ends with payoffs $s_1$ to Player 1 and $1 - s_1$ to Player 2), or **reject** (the game continues).
- In the second period, Player 2 proposes that Player 1 to take $s_2$ of the resource, leaving $1 - s_2$ to Player 2.
- Player 1 either **accepts** (and the game ends with payoffs $s_2$ to Player 1 and $1 - s_2$ to Player 2), or **reject** (the game continues).
- In the third period, Player 1 receives $s$ of the resource, player 2 receives $1 - s$ of the resource, where $0 < s < 1$.

There is a **discount** factor $\delta$ per period, $0 < \delta < 1$. 
Consider Player 2’s optimal offer if the 2nd period is reached.

Player 1 is facing a choice, choose \( s_2 \) or receive \( \delta s \). Player 1 will accept the offer iff

\[ s_2 \geq \delta s. \]

Player 2’s 2nd-period decision:

1. receiving \( 1 - \delta s \) (by offering \( s_2 = \delta s \) to Player 1), or
2. receiving \( \delta (1 - s) \) in the third period.

Since \( 1 - \delta s > \delta (1 - s) \), Player 2’s optimal 2nd-round choice is \( s_2^* = \delta s \) and Player 1 will accept.
Solution: continue

- Player 1 is facing a choice in the 1st-period.
- Player 2 will only accept the offer in the 1st-period iff
  1. $1 - s_1 \geq \delta(1 - s_2^*)$, or
  2. $s_1 \leq 1 - \delta(1 - s_2^*)$.
- Player 1’s 1st-period decision:
  1. receiving $1 - \delta(1 - s_2^*) = 1 - \delta(1 - \delta s)$ (making that bid), or
  2. receiving $\delta s_2^* = \delta^2 s$.
- Since $1 - \delta(1 - \delta s) > \delta^2 s$, so Player 1’s optimal 1st-period offer is $s_1^* = 1 - \delta(1 - \delta s)$.
- The solution of the game should end in the 1st-period with $(s_1^*, 1 - s_1^*)$, where $s_1^* = 1 - \delta(1 - \delta s)$. 
Extension to infinite rounds

- What about if we have *infinite* number of rounds?
- Truncate the infinite-horizon game and apply the logic from the finite-horizon case.
- The game in the 3rd period, should it be reached, is identical to the game beginning in the 1st period.
- Let $S_H$ be the highest payoff player 1 can achieve in any backwards-induction outcome of the game as a whole.
Extension to infinite rounds: continue

- Using $S_H$ as the 3rd period payoff to player 1.
- Player 1’s first-period payoff is $f(S_H)$ where
  \[ f(s) = 1 - \delta + \delta^2 s. \]
- But $S_H$ is also the highest possible 1st-period payoff, so $f(S_H) = S_H$.
- The only value of $s$ that satisfy $f(s) = s$ is
  \[ s^* = 1/(1 + \delta). \]
- Solution is, in the first round, player 1 offers
  \((s^*, 1 - s^*) = (1/(1 + \delta), \delta/(1 + \delta))\) to player 2, who will accept.
Bank Runs

- Two investors each deposited $D$ with a bank.
- The bank invested in a project. If it’s forced to liquidate before the project matures, a return of $2r$, where $D > r > D/2$. If the project matures, a return of $2R$, where $R > D$.
- Investors can withdraw on date 1 (before the project matures) or date 2 (after the project matures).
- The game is:
  1. If both investors make withdrawals at date 1, each receives $r$, game ends.
  2. If only one makes withdrawal at date 1, that investor receives $D$, other receives $2r - D$, game ends.
  3. If both withdraw at date 2, each receives $R$, game ends.
  4. If only one withdraws at date 2, that investor receives $2R - D$, other receives $D$, game ends.
  5. If neither makes withdrawal at date 2, banks returns $R$ to each investor, game ends.
"Normal-Form" of the game

For two dates

<table>
<thead>
<tr>
<th>Date 1</th>
<th>Investor 2 (Withdraw)</th>
<th>Investor 2 (Don’t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investor 1</td>
<td>r,r</td>
<td>D, 2r-D</td>
</tr>
<tr>
<td>(Withdraw)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Investor 1</td>
<td>2r-D, D</td>
<td>next stage</td>
</tr>
<tr>
<td>(Don’t)</td>
<td></td>
<td></td>
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<table>
<thead>
<tr>
<th>Date 2</th>
<th>Investor 2 (Withdraw)</th>
<th>Investor 2 (Don’t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investor 1</td>
<td>R,R</td>
<td>2R-D,D</td>
</tr>
<tr>
<td>(Withdraw)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Investor 1</td>
<td>D, 2R-D</td>
<td>R,R</td>
</tr>
<tr>
<td>(Don’t)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Consider date 2, since $R > D$ (and so $2R - D > R$), “withdraw” strictly dominates “don’t”, we have a unique Nash equilibrium.

For date 1, we have:

<table>
<thead>
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<th>Investor 2 (Don’t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investor 1 (Withdraw)</td>
<td>$r, r$</td>
<td>$D, 2r-D$</td>
</tr>
<tr>
<td>Investor 1 (Don’t)</td>
<td>$2r-D, D$</td>
<td>$R, R$</td>
</tr>
</tbody>
</table>

Since $r < D$ (and so $2r - D < r$), we have two pure-strategy Nash Equilibrium, (a) both withdraw, (b) both don’t withdraw, with the 2nd NE being efficient.
Tariffs and Imperfect Competition

- Consider two countries, denoted by \( i = 1, 2 \), each setting a tariff rate \( t_i \) per unit of product.
- A firm produces output, both for home consumption and export.
- Consumer can buy from a local firm or foreign firm.
- The market clearing price for country \( i \) is \( P(Q_i) = a - Q_i \), where \( Q_i \) is the quantity on the market in country \( i \).
- A firm in \( i \) produces \( h_i(e_i) \) units for local (foreign) market, i.e., \( Q_i = h_i + e_j \).
- The production cost of firm \( i \) is \( C_i(h_i, e_i) = c(h_i + e_i) \) and it pays \( t_j e_i \) to country \( j \).
Tariffs and Imperfect Competition Game

First, the government *simultaneously* choose tariff rates $t_1$ and $t_2$.
Second, the firms observe the tariff rates, decide $(h_1, e_1)$ and $(h_2, e_2)$ *simultaneously*.
Third, payoffs for both firms and governments:
(1) Profit for firm $i$:

$$
\pi_i(t_i, t_j, h_i, e_i, h_j, e_j) = \left[ a - (h_i + e_j) \right] h_i + \left[ a - (e_i + h_j) \right] e_i - c(h_i + e_i) - t_j e_i
$$

(2) Welfare for government $i$:

$$
W_i(t_i, t_j, h_i, e_i, h_j, e_j) = \frac{1}{2} Q_i^2 + \pi_i(t_i, t_j, h_i, e_i, h_j, e_j) + t_i e_j
$$
Suppose the governments have chosen \( t_1 \) and \( t_2 \).

If \((h_1^*, e_1^*, h_2^*, e_2^*)\) is a NE for firm 1 and 2, firm \( i \) needs to solve \( \max_{h_i, e_i \geq 0} \pi_i(t_i, t_j, h_i, e_i, h_j^*, e_j^*) \). After re-arrangement, it becomes two separable optimizations:

\[
\begin{align*}
\max_{h_i \geq 0} h_i[a - (h_i + e_j^*) - c]; \quad & \max_{e_i \geq 0} e_i[a - (e_i + h_j^*) - c] - t_i e_i.
\end{align*}
\]

Assuming \( e_j^* \leq a - c \) and \( h_j^* \leq a - c - t_j \), we have

\[
\begin{align*}
h_i^* &= \frac{1}{2} \left( a - e_j^* - c \right); \quad e_i^* = \frac{1}{2} \left( a - h_j^* - c - t_j \right), \quad i = 1, 2.
\end{align*}
\]

Solving, we have

\[
\begin{align*}
h_i^* &= \frac{a - c - t_i}{3}; \quad e_i^* = \frac{a - c - 2t_j}{3}, \quad i = 1, 2.
\end{align*}
\]
Tariffs and Imperfect Competition Game: 1st stage

- In the first stage, government $i$ payoff is:

$$W_i(t_i, t_j, h^*_i, e^*_1, h^*_2, e^*_2) = W_i(t_i, t_j)$$

since $h^*_i (e^*_i)$ is a function of $t_i (t_j)$.

- If $(t^*_1, t^*_2)$ is a NE, each government solves:

$$\max_{t_i \geq 0} W_i(t_i, t^*_j).$$

Solving the optimization, we have $t^*_i = \frac{a-c}{3}$, for $i = 1, 2$. which is a dominant strategy for each government.

- Substitute $t^*_i$, we have

$$h^*_i = \frac{4(a-c)}{9}; \quad e^*_i = \frac{a-c}{9}, \text{ for } i = 1, 2.$$
Comment on Tariffs and Imperfect Competition Game

- In the subgame-perfect outcome, the aggregate quantity on each market is \( 5(a - c)/9 \).
- But if two governments cooperate, they seek socially optimal point and they solve the following optimization problem:

\[
\max_{t_1, t_2 \geq 0} W_1(t_1, t_2) + W_2(t_1, t_2)
\]

The solution is \( t_1^* = t_2^* = 0 \) (no tariff) and the aggregate quantity is \( 2(a - c)/3 \).
- Therefore, for the above game, we have a unique NE, and it is socially inefficient.
Model for War of Attrition

- Two players compete for a resource of value $v$, i.e., two companies engaged in a price war.
- The strategy for each player is a choice of a persistence time, $t_i$, where $t_i \in [0, \infty)$. 
- Three assumptions
  - The cost of the contest is related only to its duration.
  - The player that persists the longest gets all of the resource.
  - The cost paid by each player is proportional to the shortest persistence time chosen, or no cost is incurred after one player quits and the contest ends.

What are the pure NE strategies?
What are the mixed NE strategies?
Solution to the War of Attrition

- Payoffs for the two players are:

\[
\pi_1(t_1, t_2) = \begin{cases} 
  v - ct_2 & \text{if } t_1 > t_2, \\
  -ct_1 & \text{if } t_1 \leq t_2.
\end{cases}
\]

\[
\pi_2(t_1, t_2) = \begin{cases} 
  v - ct_1 & \text{if } t_2 > t_1, \\
  -ct_2 & \text{if } t_2 \leq t_1.
\end{cases}
\]

- One pure strategy NE is: \( t_1^* = \frac{v}{c} \) and \( t_2^* = 0 \), giving

\[
\pi_1\left(\frac{v}{c}, 0\right) = v \quad \text{and} \quad \pi_2\left(\frac{v}{c}, 0\right) = 0.
\]
Why NE?

- It is a NE because for player 1:
  \[ \pi_1(t_1, 0) = v, \forall t_i > 0 \quad \text{and} \quad \pi_1(0, 0) = 0, \]
  which gives
  \[ \pi_1(t_1, t_2^*) \leq \pi_1(t_1^*, t_2^*) \quad \forall t_1. \]

- For player 2, we have
  \[ \pi_2(v/c, t_2) = -ct_2 < 0, \forall t_2 \leq v/c \quad \text{and} \quad \pi_2(v/c, t_2) = 0, \forall t_2 > v/c. \]
  Hence
  \[ \pi_2(t_1^*, t_2) \leq \pi_2(t_1^*, t_2^*) \quad \forall t_2. \]
Other NE

The second pure strategy NE is: \( t_1^* = 0 \) and \( t_2^* = \frac{v}{c} \). Giving \( \pi_1(0, v/c) = 0 \) and \( \pi_2(0, v/c) = v \). Analysis is similar to previous argument.

There is a mixed-strategy Nash equilibrium. For detail, refer to the textbook.