Introduction to Game Theory: Cooperative Games (2)

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Outline

Introduction

- Consider two games ν and μ in characteristic function form. Suppose the number of players is the same for both games. The question: *when can we say that ν and μ are essentially the same ?*
- For example, if we simply change the unit of payoff from one game to the other, the game is the same and we are simply multiplying the characteristic function by a positive constant.
- Another modification is that each player P_i received a fixed amount c_i independently on how he plays.
- Since the player can do nothing to change the c_i's, they would play as if these fixed amounts where not present.

Combining the above modifications, we have:

Definition

Let ν and μ be the two games in characteristic function form with the same number N of players. Then μ is strategically equivalent to ν if there exist constants $k > 0$, and $c_1, \ldots, c_N$, such that, for every coalition $\mathcal{S}$

$$\mu(\mathcal{S}) = k\nu(\mathcal{S}) + \sum_{P_i \in \mathcal{S}} c_i. \tag{1}$$

Note that ν and μ play symmetric roles, we can also express

$$\nu(\mathcal{S}) = (1/k)\mu(\mathcal{S}) + \sum_{P_i \in \mathcal{S}} (-c_i/k).$$

Example

- The game whose normal form appear in Table 1 has characteristic function:

$$\nu(\mathcal{P}) = 1,\ \nu(\emptyset) = 0,$$
$$\nu(\{P_1, P_2\}) = 1,\ \nu(\{P_1, P_3\}) = 4/3,\ \nu(\{P_2, P_3\}) = 3/4,$$
$$\nu(\{P_1\}) = 1/4,\ \nu(\{P2\}) = -1/3,\ \nu(\{P_3\}) = 0.$$

- Let $k = 2$ and c_1, c_2, and c_3 be -1, 0, -1 respectively, we have μ, which is strategically equivalent to ν with

$$\mu(\mathcal{P}) = (2)1 + (-1 + 0 - 1) = 0,\ \mu(\emptyset) = (2)0 = 0,$$
$$\mu(\{P_1, P_2\}) = (2)1 + (-1 + 0) = 1,$$
$$\mu(\{P_1, P_3\}) = (2)(4/3) + (-1 - 1) = 2/3,\ \mu(\{P_2, P_3\}) = 1/2,$$
$$\mu(\{P_1\}) = -1/2,\ \mu(\{P2\}) = -2/3,\ \mu(\{P_3\}) = -1.$$

In this example, μ is zero-sum.

Theorem

If ν and μ are strategically equivalent, and ν is inessential, then so is μ. Thus if ν is essential, so is μ.

Proof

Assuming if ν is inessential, compute

$$
\begin{aligned}
\sum_{i=1}^{N} \nu(\{P_i\}) &= \sum_{i=1}^{N} (k\nu(\{P_i\}) + c_i) \\
&= k \sum_{i=1}^{N} \nu(\{P_i\}) + \sum_{i=1}^{N} c_i = k\nu(\mathcal{P}) + \sum_{i=1}^{N} c_i = \mu(\mathcal{P}).
\end{aligned}
$$

This shows that μ is inessential. By symmetry, if μ is inessential, so is ν. This if ν is essential, so is μ.

Relationship between "Equivalence" and "Imputations"

Theorem

Let ν and μ be strategically equivalent N−person games. Then we have

- *An N−tuple $\mathbf{x}$ is an imputation for ν if and only if $k\mathbf{x} + \mathbf{c}$ is an imputation for μ.*
- *An imputation $\mathbf{x}$ dominates an imputation $\mathbf{y}$ through a coalition $\mathcal{S}$ with respect to ν if and only if $k\mathbf{x} + \mathbf{c}$ dominates $k\mathbf{y} + \mathbf{c}$ with respect to μ through the same coalition.*
- *An N−tuple $\mathbf{x}$ is in the core of ν if and only if $k\mathbf{x} + \mathbf{c}$ is in the core of μ.*

Proof

- Suppose that $\boldsymbol{x}$ is an imputation for ν. Then, for $1 \leq i \leq N$,

$$\mu(\{P_i\}) = k\nu(\{P_i\}) + c_i \leq kx_i + c_i,$$

which verifies individual rationality since $kx_i + c_i$ is the i^{th} component of $k\boldsymbol{x} + \boldsymbol{c}$.

- For collective rationality, we have

$$\mu(\mathcal{P}) \quad = \quad k\nu(\mathcal{P}) + \sum_{i=1}^{N} c_i = k\sum_{i=1}^{N} x_i + \sum_{i=1}^{N} c_i = \sum_{i=1}^{N} (kx_i + c_i)\,.$$

Therefore, $k\boldsymbol{x} + \boldsymbol{c}$ is an imputation for μ.

- The converse of this statement is true because of the symmetry of ν and μ.
- The other two statements of the theorem are proved in a similar way.

Remark

- The previous theorem tells us that if we are studying a game in characteristic function form, then we are simultaneously studying **all** games which are strategically equivalent to it.
- In case ν and μ are strategically equivalent, then we use the phrase "ν and μ are the same up to strategic equivalence".
- Another implication is that we can replace a game by another one whose characteristic function is particularly easy to work with.

Definition

A characteristic function μ is in $(0,1)-$reduced form if both the following hold:

- $\mu(\{P\}) = 0$ for every player P.
- $\mu(\mathcal{P}) = 1$.

Observation

- A game in $(0,1)-$reduced form is obviously essential.
- Conversely, it is also true that, up to strategic equivalence, every essential game is in $(0,1)-$reduced form.

Theorem

If ν is an essential game, then ν is strategically equivalent to a game μ in $(0,1)-$reduced form.

Proof

- Define
$$k = \frac{1}{\nu(\mathcal{P}) - \sum_{i=1}^{N} \nu(\{P_i\})} \geq 0,$$
and for $i \leq i \leq N$, define
$$c_i = -k\nu(\{P_i\}).$$
- Then μ is defined by Equation (1).
- It is easy to verify that μ is in $(0,1)-$reduced form.

Example

Let us consider the game whose normal form is given in Table 1.

- From the previous theorem, we have:

$$k = \frac{1}{1-(-1/12)} = \frac{12}{13}, \text{ and}$$

$$c_1 = -(12/13)(1/4) = -3/13, \; c_2 = 4/13, \; c_3 = 0.$$

- Then μ is given by:

$$\mu(\mathcal{P}) = 1, \mu(\emptyset) = 0,$$
$$\mu(\text{any one-player coalition}) = 0,$$
$$\mu(\{P_1, P_2\}) = (12/13)(1) - 3/13 + 4/13 = 1,$$
$$\mu(\{P_1, P_3\}) = \mu(\{P_2, P_3\}) = 1.$$

Example: continue

For this game μ, we immediately observe three things:

- All three two-person coalitions are *equally good*.
- If a two-person coalition forms, the players will probably divide the payoff equally (since the players have completely symmetric roles).
- There is no advantage to a two-player coalition in bringing in the third party to form a grand coalition.
- Conclusion:
  - One of the two-player coalitions will form, the player will split the payoff, the third player is left out in the cold.
  - Prevailing imputations will be either $(1/2, 1/2, 0)$, $(1/2, 0, 1/2)$ or $(0, 1/2, 1/2)$.
  - Our analysis *cannot* predict which coalitions will actually form.

Example: continue

If we transform these conclusions back into terms of ν, then

- one of the two-player coalitions will form.
- The prevailing imputation will be one of the three possibilities which can be computed using the relationship between ν and μ (will be shown later)

Another example

For the 3-player game $\mathcal{G}$, the $(0,1)-$reduced form μ is

- $\mu(\{P_1, P_2\}) = 3/8$, $\mu(\{P_1, P_3\}) = \mu(\{P_2, P_3\}) = 1/2$.
- In this game, the nice symmetry is lost and we can safely say that the *grand coalition* is likely to form.
- To make a guess about what the final imputation might be hazardous since we discussed that the core is large.

Theorem

Suppose μ and ν are $N-$person games in $(0,1)-$reduced form. If they are strategically equivalent, then they are all equal.

Proof

By definition of strategic equivalence, there exist constant $k > 0$, and $c_1, \ldots, C_N$ such that

$$\mu(\mathcal{S}) = k\nu(\mathcal{S}) + \sum_{P_i \in \mathcal{S}} c_i$$

for every coalition $\mathcal{S}$.

To prove that ν and μ are equal, we need to show that $k = 1$ and $c_i = 0$ for all i. Since both characteristic functions are zero for all one-player coalitions, we see that $c_i = 0, \forall i$. Since both characteristic functions are 1 for the grand coalition, we see that $k = 1$.

Classification of Small Games

Up to strategic equivalence, the number of games with two or three players is limited, as shown by the following three theorems.

Theorem

A two-player game in characteristic function form is either inessential or strategically equivalent to ν, where

$$\nu(\textit{the grand coalition}) = 1, \ \nu(\emptyset) = 0,$$
$$\nu(\textit{either one-player coalition}) = 0.$$

Classification of Small Games: continue

In the case of constant-sum games with three players, we have

Theorem

Every three-player constant-sum game in characteristic function form is either inessential or strategically equivalent to ν, where

$$\nu(\textit{the grand coalition}) = 1, \ \nu(\emptyset) = 0,$$
$$\nu(\textit{any two-player coalition}) = 1,$$
$$\nu(\textit{any one-player coalition}) = 0.$$

Remark: It basically says that every essential constant-sum game with three players is strategically equivalent to three-player, constant-sum, essential game in (0,1)-reduced form (or we called THREE). For example, the game given in Table 1 is strategically equivalent to THREE.

Classification of Small Games: continue

Up to strategic equivalence, the three-player games which are not necessarily constant-sum form a three-parameter family. We have

Theorem

Every three-player game in characteristic function form is either inessential or there exist constants a, b, c satisfying:

$$0 \leq a \leq 1, \ 0 \leq b \leq 1, \ 0 \leq c \leq 1,$$

such that the game is strategically equivalent to ν, where

$$\nu(\textit{the grand coalition}) = 1, \ \nu(\emptyset) = 0,$$

and

$$\nu(\textit{any one-player coalition}) = 0,$$
$$\nu(\{P_1, P_2\}) = a, \ \nu(\{P_1, P_3\}) = b, \ \nu(\{P_2, P_3\}) = c.$$

Two approaches to find solution

As a solution concept for cooperative game, core has a problem since it is either a) *empty*, or b) there are so many imputations in the core and we have no reasonable way to decide which ones are actually likely to occur.

Two approaches are proposed, the are:

- Stable sets of Imputations.
- Shapley Values

Definition

Let X be a set of imputations for a game in characteristic function form. Then we say that X is stable if the following two conditions hold:

- **(Internal Stability):** No imputation in X dominates any other imputation in X through any coalition.
- **(External Stability):** In $\boldsymbol{y}$ is any imputation outside X, then it is dominated through some coalition by some imputation inside X.

The idea of *stable sets of imputations* was introduced by von Neumann and Morgenstern in 1944, and they argued that a stable set is a solution of the game.

Comments on Stable Sets

- Note that an imputation inside a stable set may be *dominated* by some imputation outside. Of course, that outside imputation is, in turn, dominated by some other imputations inside (via external stability). So *transitive property* does not hold for imputation dominance.
- Since all imputations inside X are equal, one which actually prevails would be chosen in some way, say, via pure chance, custom,..etc. But there is a problem since there may be an imputation outside X which dominates a given imputation inside. And if a coalition is formed based on this outside imputation, X is not *stable* anymore.
- Someone argue that this chaotic series of formation and dissolutions of coalitions of different stable sets is ok, since real life often looks that way.

Example

For the three-person game in Table 1, we have showed (via (0,1)-reduced form), that it has a stable set. Let

$$X = \{(0, 1/2, 1/2), (1/2, 0, 1/2), (1/2, 1/2, 0)\}.$$

We have the following result

Theorem

The set X defined above is a stable set for THREE.

Proof

- We denote the characteristic function for THREE by μ.
- We first verify the internal stability.
  - By symmetry it is enough to show that the imputation $(0, 1/2, 1/2)$ does not dominate $(1/2, 0, 1/2)$ through any coalition.
  - The only possible coalition through which this domination could occur is $\{P_2\}$, but $\mu(\{P_2\}) = 0 < 1/2$, and this violates the feasibility condition in the definition of dominance.
- To verify about external stability, let $\boldsymbol{y}$ be an outside imputation:
  - We must show that one of the members of X dominates it through some coalition.
  - Note that there are at least two values of i for which $y_i < 1/2$. If this were not true, we would have $y_i \geq 1/2$ for two values of i.
  - Since each y_i is nonnegative, and $\sum_i y_i = 1$, this implies $\boldsymbol{y}$ is one of the imputation in X. This contradicts the assumption that $\boldsymbol{y} \notin X$.
  - By symmetry, we may assume that y_1 and y_2 are both less than $1/2$, but then (1/2,1/2,0) dominates $\boldsymbol{y}$ through coalition $\{P_1, P_2\}$.

Comment

- Note that there are imputations outside X which dominate members of X.
- Consider, for example, (2/3,1/3,0) dominates (1/2,0,1/2) through coalition $\{P_1, P_2\}$.
- On the other hand, (0, 1/2, 1/2) (a member of X) dominates (2/3, 1/3, 0) through $\{P_2, P_3\}$.

Stable Set for any games $\in$ THREE

- Since every essential constant-sum game with three players is strategically equivalent to THREE, one can use X to obtain the stable set for any such game.
- Let ν be the game whose normal form is shown in Table 1. Then

$$\mu(\mathcal{S}) = k\nu(CS) + \sum_{P_i \in \mathcal{S}} c_i,$$

  for every coalition $\mathcal{S}$, where $k = 12/13$, $c_1 = -3/13$, $c_2 = 4/13$ and $c_3 = 0$.
- Thus $\nu(\mathcal{S}) = (1/k)\mu(\mathcal{S}) + \sum_{P_i \in \mathcal{S}}(-c_i/k)$.
- Replacing each imputation $\boldsymbol{x}$ in X by $(1/k)\boldsymbol{x} - (1/k)\boldsymbol{c}$, gives us a stable set for ν, namely

$$\{(19/24, 5/24, 0), (19/24, -1/3, 13/24), (1/4, 5/24, 13/24)\}.$$

Shapley Values

- Proposed by L.S. Shapley in 1953, an interesting attempt to define, in a fair way, an imputation which embodies what the players' final payoffs "should" be.
- It takes into account a player's contribution to the success of the coalition she belongs to.
- If the characteristic function of the game is ν, and if $\mathcal{S}$ is the coalition to which player P_i belongs, then

$$\delta(P_i, \mathcal{S}) = \nu(\mathcal{S}) - \nu(\mathcal{S} - \{P_i\})$$

  is a measure of the amount that P_i has contributed to $\mathcal{S}$ by joining it.

Comment on $\delta(P_i, \mathcal{S})$

- Note that $\delta(P_i, \mathcal{S})$, by themselves, are not very revealing.
- Consider the game of THREE:
  - We have $\delta(P_i, \mathcal{P}) = 0$ for any P_i: no one contributed anything.
  - If $\mathcal{S}$ is any two-player coalition, then $\delta(P_i, \mathcal{S}) = 1$ for each player in $\mathcal{S}$. That is, the sum of contributions is greater than $\nu(S)$.

Intuition

- Notice that once the players have collectively agreed on an imputation, it might as well be assumed that it is the grand coalition which forms.
- Why? Because the condition of collective rationality ensures that the total of all payments (via the imputation) is $\nu(\mathcal{P})$.
- We concentrate on the *process* by which the grand coalition comes into being: it starts with the first player, then the 2nd player, ...etc. Or the process is characterized by an ordered list of players.

Example: four-person game

Suppose that its characteristic function is

$$
\begin{gathered}
\nu(\mathcal{P}) = 100,\ \nu(\emptyset) = 0, \\
\nu(\{P_1\}) = 0,\ \nu(\{P_2\}) = -10,\ \nu(\{P_3\}) = 10,\ \nu(\{P_4\}) = 0, \\
\nu(\{P_1, P_2\}) = 25,\ \nu(\{P_1, P_3\}) = 30,\ \nu(\{P_1, P_4\}) = 10, \\
\nu(\{P_2, P_3\}) = 10,\ \nu(\{P_2, P_4\}) = 10,\ \nu(\{P_3, P_4\}) = 30, \\
\nu(\{P_1, P_2, P_3\}) = 50,\ \nu(\{P_1, P_2, P_4\}) = 30, \\
\nu(\{P_1, P_3, P_4\}) = 50,\ \nu(\{P_2, P_3, P_4\}) = 40.
\end{gathered}
$$

One ordering of the players through which the grand coalition could form:

$$P_3, P_2, P_1, P_4.$$

The total number of ordering for the grand coalition to form: 4!. In general, if there are N players, we have $N!$ possibilities. Each ordering occurs with probability $1/N!$.

Example: four-person game (continue)

- Given that the grand coalition forms according to the given ordering:

$$\delta(P_1, \{P_3, P_2, P_1\}) = \nu(\{P_3, P_2, P_1\}) - \nu(\{P_3, P_2\}) = 50 - 10 = 40.$$

  This is a measure of the contribution of P_1 makes as she enters the growing coalition.
- The Shapley value, ϕ_i, is this:
  - Make the same sort of calculation for each of the $N!$ possible orderings of the players
  - Weight each one by the probability of $1/N!$ of that ordering to occur.
  - Add the results.
- We will show how to derive the Shapley value ϕ_i:
  - so that the computation of ϕ_i is somewhat easier.
  - show that $\phi = (\phi_1, \ldots, \phi_N)$ is an imputation.

Shapley value

- Note that out of the $N!$ ordering, there are many duplications.
- Suppose P_i occurs at position k. Denote the $\mathcal{S}$ be the set of k players up to and including P_i in this ordering. If we permute the part of the ordering coming before P_i, and permute the part coming after P_i, we obtain a new ordering in which P_i again is in the k^{th} position. In any of these permuted ordering, we have

$$\delta(P_i, \mathcal{S}) = \nu(\mathcal{S}) - \nu(\mathcal{S} - \{P_i\}).$$

- There are $(k-1)!$ permutations of the players coming before P_i and $(N-k)!$ permutations of players coming after P_i, the term $\delta(P_i, \mathcal{S})$ occurs $(k-1)!(N-k)!$ times.
- Finally, the Shapley value for P_i, or ϕ_i, is:

$$\phi_i = \sum_{P_i \in \mathcal{S}} \frac{(N - |\mathcal{S}|)!(|\mathcal{S}| - 1)!}{N!} \delta(P_i, \mathcal{S}). \quad (2)$$

Example

- Consider the game whose normal form is given in Table 1 with the given characteristic function.
- To find ϕ_1, there are four coalition containing P_1:

$$\{P_1\}, \{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_2, P_3\}.$$

  So Eq (2) has four terms in this case.
- We compute

$$\delta(P_1, \{P_1\}) = 1/4 - 0 = 1/4,\ \delta(P_1, \{P_1, P_2\}) = 1 - (-1/3) = 4/3,$$
$$\delta(P_1, \{P_1, P_3\}) = 4/3 - 0 = 4/3,\ \delta(P_1, \{P_1, P_2, P_3\}) = 1 - (3/4) = 1/4.$$

- Then

$$\phi_1 = \frac{2!0!}{3!}\frac{1}{4} + \frac{1!1!}{3!}\frac{4}{3} + \frac{1!1!}{3!}\frac{4}{3} + \frac{0!2!}{3!}\frac{1}{4} = \frac{11}{18}.$$

  Similarly, $\phi_2 = \frac{1}{36}$, $\phi_3 = \frac{13}{36}$. So $\phi = \left(\frac{11}{18}, \frac{1}{36}, \frac{13}{36}\right)$ is an imputation.

Interpretation

- Note that $\phi = (\frac{11}{18}, \frac{1}{36}, \frac{13}{36})$ is an imputation.
- P_1's Shapley value is largest of the three, indicating that P_1 is the strongest.
- P_2's Shapley value is very small.
- P_3 is in the middle.
- A glance at the characteristic function supports this "value"

More examples

- The 3-player game $\mathcal{G}$, we can compute the Shapley value and it is

$$(1/8, 5/8, 1/4).$$

  This numbers seem to reasonably reflect the advantage that player P_2 has in the game.
- For the Used Car game, the Shapley values are:

$$\phi_N = 433.333...,\ \phi_A = 83.333, ...\ \phi_M = 183.33....$$

  Thus, Mitchel gets the car for $433.33, but has to pay Agnew $83.33 as a bride for not bidding against him. And the Shapley vector indicates that Nixon is in the most powerful position.

Theorem

Let ν be a game in characteristic function form. Then the Shapley vector for ν is an imputations.

Proof:

- We prove "*individual rationality*", we must show that $\phi_i \geq \nu(\{P_i\})$.
- By super-additivity, if $P_i \in \mathcal{S}$, $\delta(P_i, \mathcal{S}) = \nu(\mathcal{S}) - \nu(\mathcal{S} - \{P_i\}) \geq \nu(\{P_i\})$. Thus,

$$\phi_i \geq \left(\sum_{P_i \in \mathcal{S}} \frac{(N - |\mathcal{S}|)!(|\mathcal{S}| - 1)!}{N!} \right) \nu(\{P_i\}).$$

- The sum in this inequality is the sum of the probabilities of the different orderings of the players, and it must equal 1, so $\phi_i \geq \nu(\{P_i\})$.

Proof: continue

- To prove "*collective rationality*", consider

$$\sum_{i=1}^{N} \phi_i = \sum_{i=1}^{N} \sum_{P_i \in \mathcal{S}} \frac{(N-|\mathcal{S}|)!(|\mathcal{S}|-1)!}{N!} \delta(P_i, \mathcal{S}).$$

- In the double sum, fix our attention on the terms involving $\nu(\mathcal{T})$, where $\mathcal{T}$ is a fixed nonempty coalition which is not equal to $\mathcal{P}$.
- There are two kinds of terms involving $\nu(\mathcal{T})$, those with positive coefficient (when $\mathcal{T} = \mathcal{S}$):

$$\frac{(N-|\mathcal{T}|)!(|\mathcal{T}|-1)!}{N!},$$

and those with a negative coefficient (when $\mathcal{T} = \mathcal{S} - \{P_i\}$):

$$-\frac{(N-1-|\mathcal{T}|)!|\mathcal{T}|!}{N!}.$$

Proof: continue

- The first occurs $|\mathcal{T}|$ times (one for each member of $\mathcal{T}$), and the second kind occurs $N - |\mathcal{T}|$ times (once for each player outside $\mathcal{T}$).
- The coefficient of the double sum is:

$$\frac{|\mathcal{T}|(N-|\mathcal{T}|)!(|\mathcal{T}|-1)!}{N!} - \frac{(N-|\mathcal{T}|)(N-1-|\mathcal{T}|)!|\mathcal{T}|!}{N!} =$$
$$\frac{(N-|\mathcal{T}|)!|\mathcal{T}|!}{N!} - \frac{N-|\mathcal{T}|)!|\mathcal{T}|!}{N!} = 0.$$

- Therefore, the only term left in the double sum are those involving the grand coalition, and those involving the empty coalition. Since $\nu(\emptyset) = 0$, we have

$$\sum_{i=1}^{N} \phi_i = \frac{N(0!)(N-1)!}{N!}\nu(\mathcal{P}) = \nu(\mathcal{P}).$$

Shapley value for The Lake Wobegon Game

- Continue