Introduction to Game Theory: Cooperative Games

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Outline

Introduction

- Under cooperative games, players can coordinate their strategies and share the payoff.
- In particular, sets of players, called coalitions, can
  - make binding agreements about joint strategies,
  - pool their individual agreements and,
  - redistribute the total in a specified way.
- Cooperative game theory applies both to zero-sum and non-zero-sum games.

Formal definition

- A coalition is simply a subset of the set of players which forms in order to coordinate strategies and to agree on how the total payoff is to be divided among the members.
- Let $\mathcal{P}$ be the set of players and there are N players in the system.
- A coalition is denoted by an uppercase script letters: $\mathcal{S}, \mathcal{T}, \mathcal{U}$,..etc.
- Given a coalition $\mathcal{S} \subseteq \mathcal{P}$, the *counter-coalition* to $\mathcal{S}$ is $\mathcal{S}^c = \mathcal{P} - \mathcal{S} = \{P \in \mathcal{P} : P \not\subseteq \mathcal{S}\}$.

Continue

strategy	payoff vectors
(1,1,1)	(-2,1,2)
(1,1,2)	(1,1,-1)
(1,2,1)	(0,-1,2)
(1,2,2)	(-1,2,0)
(2,1,1)	(1,-1,1)
(2,1,2)	(0,0,1)
(2,2,1)	(1,0,0)
(2,2,2)	(1,2,-2)

Table: Consider a 3−player game

- In this game, $\mathcal{P} = \{P_1, P_2, P_3\}$. There are eight coalitions:
  - 3 one-player coalitions: $\{P_1\}$, $\{P_2\}$, $\{P_3\}$.
  - 3 two-player coalitions: $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_2, P_3\}$.
  - Grand coalition: $\mathcal{P}$ itself and the empty coalition: $\emptyset$.
- In general, in a game with N players, there are 2^N coalitions.

Characteristic Function

- One simple way to view about cooperative game is a competition (non-cooperative) between two "players": coalition $\mathcal{S}$ and the counter coalition $\mathcal{S}^c$.
- Consider an $N-$player game $\mathcal{P} = \{P_1, \ldots, P_N\}$, and X_i is the strategy set for player P_i.
- The system has an non-empty coalition $\mathcal{S} \subseteq \mathcal{P}$ and $\mathcal{S}^c$.
- Pure joint strategies available to members of $\mathcal{S}$ (or $\mathcal{S}^c$) are the Cartesian product of those X_i's for which $P_i \in \mathcal{S}$ ($P_i \in \mathcal{S}^c$).
- We have a bi-matrix with rows (columns) correspond to the pure joint strategies of players in $\mathcal{S}$ ($\mathcal{S}^c$). An entry in the bi-matrix is a pair of number, with the first (second) being the sum of the payoffs to players in $\mathcal{S}$ ($\mathcal{S}^c$)

Example

- Consider a coalition $\mathcal{S} = \{P_1, P_3\}$, then $\mathcal{S}^c = \{P_2\}$.
- Coalition $\mathcal{S}$ has four pure joint strategies: $\{(1,1),(1,2),(2,1),(2,2)\}$. For $\mathcal{S}^c$, the strategies are 1 and 2. The bi-matrix is:

	1	2
(1,1)	(0,1)	(2,-1)
(1,2)	(0,1)	(-1,2)
(2,1)	(2,-1)	(1,0)
(2,2)	(1,0)	(-1,2)

- The *maximum value* for the coalition is called the characteristic function of $\mathcal{S}$ and it is denoted as $\nu(\mathcal{S})$. In other words, members of $\mathcal{S}$ are guaranteed to gain a total payoff of at least $\nu(\mathcal{S})$.

Example: continue

- Let us consider the previous game. For $\mathcal{S}$, pure joint strategy (1,2) is dominated by (1,1), pure joint strategy (2,2) is dominated by (2,1). We have

	1	2
(1,1)	(0,1)	(2,-1)
(2,1)	(2,-1)	(1,0)

- We solve the above non-cooperative game, we have $\nu(\mathcal{S}) = 4/3$ and $\nu(\mathcal{S}^c) = -1/3$.
- Computing in a similar way, we have $\nu(\{P_1, P_2\}) = 1$, $\nu(\{P_3\}) = 0$, $\nu(\{P_2, P_3\}) = 3/4$, $\nu(\{P_1\}) = 1/4$.
- The characteristic function for the grand coalition is simply the largest total payoff which the set of all players can achieve, it is easily seen that $\nu(\mathcal{P}) = 1$.
- Finally, by definition, the characteristic function of empty coalition is $\nu(\emptyset) = 0$.

Interpretation of the cooperative game

By examining the characteristic function, we can speculate which coalitions are likely to form.

- Since P_1 does better playing on his own than P_2 or P_3 playing on their own, P_2 and P_3 would bid against each other to try to *entice* P_1 into a coalition.
- In exchange, P_1 would demand a larger share of the total payoff to the coalition he joins and he would ask for more than $1/4$ since he get that much on his own.
- On the other hand, if P_1 demands too much, P_2 and P_3 might join together, excluding P_1 and gain a total of $3/4$.

The following theorem states that, *"in union, there is strength."*

Theorem (Super-additivity)

Let $\mathcal{S}$ and $\mathcal{T}$ be disjoint coalitions, then $\nu(\mathcal{S} \cup \mathcal{T}) \geq \nu(\mathcal{S}) + \nu(\mathcal{T})$.

Proof: Since each player uses the maximin solution method, a coalition guarantees that each player gain *at least as much* as if they do not form a coalition.
Using the previous example, we have

$$\nu(\{P_1, P_3\}) = 4/3 > \nu(\{P_1\}) + \nu(\{P_3\}) = 1/4 + 0 = 1/4.$$

Corollary

If $\mathcal{S}_1, \ldots, \mathcal{S}_k$ are pairwise disjoint coalitions, then

$$\nu(\cup_{i=1}^{k} \mathcal{S}_i) \geq \sum_{i=1}^{k} \nu(\mathcal{S}_i).$$

Corollary

For any N−person game, $\nu(\mathcal{P}) \geq \sum_{i=1}^{N} \nu(\{P_i\})$.

Definition

A game in characteristic function form consists of a set $\mathcal{P} = \{P_1, \ldots, P_N\}$ of players, together with a function ν, defined for all subsets of $\mathcal{P}$, such that

$$\nu(\emptyset) = 0,$$

and such that the super-additivity holds, that is:

$$\nu(\mathcal{S} \cup \mathcal{T}) \geq \nu(\mathcal{S}) + \nu(\mathcal{T}),$$

whenever $\mathcal{S}$ and $\mathcal{T}$ are disjoint coalitions of the players.

Definition

An $N-$person game ν in characteristic function form is said to be inessential if

$$\nu(\mathcal{P}) = \sum_{i=1}^{N} \nu(\{P_i\}).$$

In other words, in it is an inessential game, there is no point to form a coalition.

Theorem

Let $\mathcal{S}$ be any coalition of an inessential game, then $\nu(\mathcal{S}) = \sum_{P \in \mathcal{S}} \nu(\{P\})$.

Proof: Suppose not, then it must be $\nu(\mathcal{S}) > \sum_{P \in \mathcal{S}} \nu(\{P\})$. By the super-additivity property, we have

$$\nu(\mathcal{P}) \geq \nu(\mathcal{S}) + \nu(\mathcal{S}^c) > \sum_{i=1}^{N} \nu(\{P_i\}),$$

which contradicts the definition of an inessential game.

An inessential game does not make it unimportant. To illustrate:

Theorem

A two-person game which is zero-sum in its normal form is inessential in its characteristic function form.

Proof: For a zero-sum game, we can use the minimax theorem so that $\nu(\{P_1\})$ and $\nu(\{P_2\})$ are negative of each other, thus the sum is zero. In addition, we have $\nu(\mathcal{P}) = 0$. Thus $\nu(\mathcal{P}) = \nu(\{P_1\}) + \nu(\{P_2\})$.

Exercise

The 3-person game of Couples is played as follows. Each player chooses one of the other two players; these choices are made simultaneously. If a couple forms (e.g., if P_2 chooses P_3, and P_3 chooses P_2), then each member of that couple receives a payoff of $1/2$, while the person not in the couple receives -1. If no couple forms (e.g., if P_1 chooses P_2, P_2 chooses P_3 and P_3 chooses P_1), then each receives a payoff of zero.

Show that this game is a zero-sum and essential.

Introduction

- Suppose a coalition forms in an $N-$person game. We want to study the final distribution of the payoffs.
- This is important because players want to know how much they gain if they form a coalition.
- The amount going to the players form an $N-$tuple $\boldsymbol{x}$ of numbers.
- The $N-$tuple vector $\boldsymbol{x}$ must satisfy two conditions: individual rationality and collective rationality for coalition to occur.
- An $N-$tuple of payments to the players which satisfies both these conditions is call an imputation.

Definition

Let ν be an $N-$person game in characteristic function form with players $\mathcal{P} = \{P_1, \ldots, P_N\}$. An $N-$tuple $\boldsymbol{x}$ of real numbers is said to be an imputation if both the following conditions hold

- (*Individual Rationality*) For all players P_i, $x_i \geq \nu(\{P_i\})$.
- (*Collective Rationality*) We have $\sum_{i=1}^{N} x_i = \nu(\mathcal{P})$.

Remark: Individual rationality is reasonable. If $x_i < \nu(\{P_i\})$, then no coalition given P_i only the amount of x_i would ever form and P_i would do better going on his own.

To show the "Collective Rationality": $\sum_i^N x_i = \nu(\mathcal{P})$

- Let us first show $\sum_{i=1}^N x_i \geq \nu(\mathcal{P})$.
  - Assume this inequality is false, then we would have $\beta = \nu(\mathcal{P}) - \sum_i^N x_i > 0$.
  - Thus, the players could form a grand coalition and distribute the total payoff $\nu(\mathcal{P})$: $x_i^{'} = x_i + \beta/N$, giving every player more.
  - Hence, if $\boldsymbol{x}$ is to have a chance, the inequality should be "$\geq$".
- We then argue that $\sum_{i=1}^N x_i \leq \nu(\mathcal{P})$.
  - Suppose $\boldsymbol{x}$ occurs and that $\mathcal{S}$ is the coalition, members in $\mathcal{S}$ and $\mathcal{S}^c$ agree to $\boldsymbol{x}$ as their payoffs.
  - Using super-additivity:

$$\sum_{i=1}^N x_i = \sum_{P_i \in \mathcal{S}} x_i + \sum_{P_i \in \mathcal{S}^c} x_i = \nu(\mathcal{S}) + \nu(\mathcal{S}^c) \leq \nu(\mathcal{P}).$$

- Combining both conditions, we must have $\sum_{i=1}^N x_i = \nu(\mathcal{P})$.

Example

- Consider the game in Table 1, any 3-tuple $\boldsymbol{x}$ which satisfies the conditions:

$$x_1 + x_2 + x_3 = 1; x_1 \geq 1/4; x_2 \geq -1/3; x_3 \geq 0.$$

  is a valid imputation.
- It is easy to see that there are **infinite** many 3-tuples which satisfy these conditions, e.g.,
  - $(1/3, 1/3, 1/3)$,
  - $(1/4, 3/8, 3/8)$,
  - $(1, 0, 0)$.

Theorem

Let ν be an $N-$person game in characteristic function form. If ν is inessential, then it has only one imputation, namely,

$$\boldsymbol{x} = (\nu(\{P_1\}), \ldots, \nu(\{P_N\})).$$

If ν is essential, then it has infinitely many imputations.

Proof

- Suppose first that ν is inessential, and $\boldsymbol{x}$ is an imputation.
  - If, for some j, $x_j > \nu(\{P_j\})$,
  - then $\sum_{i=1}^{N} x_i > \sum_{i=1}^{N} \nu(\{P_l\}) = \nu(\mathcal{P})$. This is a contradiction to collective rationality.
- Suppose ν is essential
  - Let $\beta = \nu(\mathcal{P}) - \sum_{i=1}^{N} \nu(\{P_i\}) > 0$.
  - For any $N-$tuple $\boldsymbol{\alpha}$ of nonnegative number summing to β, we have

$$x_i = \nu(\{P_i\}) + \alpha_i,$$

    which defines an imputation.
  - Obviously there are infinitely many choices of $\boldsymbol{\alpha}$, so there are infinitely many imputations for essential game.

Remark

- For an essential game, the issue is to find out which imputations deserve to be called "solutions".
- For the game in Table 1, none of the three imputations listed earlier seems likely to occur.
- Imputation $(1/4, 3/8, 3/8)$, it is unstable because P_1 and P_2 could form a coalition and gain a total payoff of at least 1.

Dominance of Imputations

Some imputations are more "preferable".

Definition

Let ν be a game in characteristic function form, let $\mathcal{S}$ be a coalition, and let $\boldsymbol{x}$, $\boldsymbol{y}$ be imputations. We say that $\boldsymbol{x}$ dominates $\boldsymbol{y}$ through coalition $\mathcal{S}$, or $\boldsymbol{x} \succ_{\mathcal{S}} \boldsymbol{y}$, if the following conditions hold:

- $x_i > y_i$ for all $P_i \in \mathcal{S}$.
- $\sum_{P_i \in \mathcal{S}} x_i \leq \nu(\mathcal{S})$.

Remark: the second condition of the definition says that $\boldsymbol{x}$ is feasible, that the players in $\mathcal{S}$ attain enough payoff so that $\boldsymbol{x}_i$ can be paid to $P_i \in \mathcal{S}$.

Example: (a) (1/3,1/3,1/3) dominates (1,0,0) through coalition $\{P_2, P_3\}$ since $\nu(\{P_2, P_3\}) = 3/4$. **(b)** (1/4,3/8,3/8) dominates (1/3,1/3,1/3) through $\{P_2, P_3\}$. **(c)** (1/2,1/2,0) dominates (1/3,1/3,1/3) through $\{P_1, P_2\}$ since $\nu(\{P_1, P_2\}) = 1$.

The Core

Observation: an imputation which is dominated through some coalition would never become permanently established and there is a tendency for this coalition to break up and be replaced by one which gives its members a larger share.

Definition

Let ν be a game in characteristic form. The core of ν consists of all imputations which are not dominated by any other imputations through any coalition.

If an imputation $\boldsymbol{x}$ is in the core, there is no group of players which has a reason to form a coalition and replace $\boldsymbol{x}$. Therefore, the core is the "solution concept" of $N-$person cooperative games. As we will soon seen, this solution concept is ok as long as the core is not empty.

How to determine whether *x* is in the core?

Theorem

*Let ν be a game in characteristic function form with N players, and **x** be an imputation. Then **x** is in the core of ν if and only if*

$$\sum_{P_i \in \mathcal{S}} x_i \geq \nu(\mathcal{S}),$$

for every coalition $\mathcal{S}$.

Corollary

*Let ν be a game in characteristic function form with N players and **x** be an N−tuple of numbers. Then **x** is an imputation in the core if and only if the following two conditions hold:*

- $\sum_{i=1}^{N} x_i = \nu(\mathcal{P})$.
- $\sum_{P_i \in \mathcal{S}} x_i \geq \nu(\mathcal{S})$ *for every coalition $\mathcal{S}$.*

Let us find the core of the game in Table 1. By the corollary, (x_1, x_2, x_3) is an imputation in the core iff:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, & (1)\\ x_1 &\geq 1/4, & (2)\\ x_2 &\geq -1/3, & (3)\\ x_3 &\geq 0, & (4)\\ x_1 + x_2 &\geq 1, & (5)\\ x_1 + x_3 &\geq 4/3, & (6)\\ x_2 + x_3 &\geq 3/4. & (7) \end{aligned}$$

Analysis: From Eq. (1),(4) and (5), we have $x_3 = 0$, $x_1 + x_2 = 1$. From Eq. (6)-(7), we have $x_1 \geq 4/3$, $x_2 \geq 3/4$. Adding these, we have $x_1 + x_2 \geq 25/12 > 1$. This is a contradiction. So the core of this game is **empty**.

Another 3-player game $\mathcal{G}$ whose characteristic function is given by

$$\begin{aligned}
\nu(\{P_1\}) &= -1/2, && (8)\\
\nu(\{P_2\}) &= 0, && (9)\\
\nu(\{P_3\}) &= -1/2, && (10)\\
\nu(\{P_1, P_2\}) &= 1/4, && (11)\\
\nu(\{P_1, P_3\}) &= 0, && (12)\\
\nu(\{P_2, P_3\}) &= 1/2, && (13)\\
\nu(\{P_1, P_2, P_3\}) &= 1. && (14)
\end{aligned}$$

Note that (a) super-additivity holds for this example.

A 3-tuple $\boldsymbol{x}$ is an imputation in the core of this game if and only if:

$$\begin{aligned}
x_1 + x_2 + x_3 &= 1, \\
x_1 &\geq -1/2, \\
x_2 &\geq 0, \\
x_3 &\geq -1/2, \\
x_1 + x_2 &\geq 1/4, \\
x_1 + x_3 &\geq 0, \\
x_2 + x_3 &\geq 1/2.
\end{aligned}$$

The system has many solutions, For example, (1/3,1/3,1/3) is in the core.

Example (The Used Car Game)

Nixon has an old car and he wants to sell. The car is worth nothing to Nixon unless he can sell it. Two persons: Agnew and Mitchell, want the car. Agnew values the car at $500 while Mitchell values it at $700. The game consists of each of the prospective buyers bidding on the car, and Nixon either accepting one of the higher bids, or rejecting both of them.

Abbreviate the names by N, A and M, the characteristic function form of the game is:

$$\nu(\{N\}) = \nu(\{A\}) = \nu(\{M\}) = 0$$
$$\nu(\{N, A\}) = 500, \nu(\{N, M\}) = 700, \nu(\{A, M\}) = 0,$$
$$\nu(\{N, A, M\}) = 700.$$

Justification

- Consider $\nu(\{N\})$ and its counter-coalition $\{A, M\}$. N has two pure strategies: (a) accept the higher bid, or (b) reject both if the higher bid is less than some lower bound. There is a joint strategy for $\{A, M\}$ in which both bid for zero. By definition of maximin value, $\nu(\{N\}) = 0$.
- $\nu(\{A\}) = \nu(\{M\}) = 0$ because the counter-coalition can always reject that player's bid.
- Coalition $\{N, A\}$ has many joint strategies which result in a payoff to it of \$500, e.g., A pays \$500 to N, payoff to N is \$500 and payoff to A is zero (value of car minus the money). Note that they cannot get more than \$500 without the cooperation of M. Similarly, $\{N, M\} = 700$.
- Finally, the grand coalition has the value of \$700 since it is the large possible sum of payoffs, e.g., if M pays \$700 for the car.

Imputation of the game

- An imputation $\boldsymbol{x}$ is in the core iff

$$x_N, x_A, x_M \geq 0$$
$$x_N + x_A + x_M = 700$$
$$x_N + x_A \geq 500; \qquad x_N + x_M \geq 700; \qquad x_A + x_M \geq 0.$$

- We solve these and give: $500 \leq x_N \leq 700$; $x_M = 700 - x_N$; $x_A = 0$.
- **Interpretation:** M gets the car with a bid between \$500 and \$700 (x_N is the bid). Agnew does not get the car, but his presence forces the prices up over \$500.

Additional observations

- Since the game is cooperative, it is possible that A and M to conspire: A bids for zero, M bids for \$300. N gets \$300, and M pays A \$200. The imputation (x_N, x_A, x_M) is $(300, 200, 200)$. However, it is **NOT** in the core because it is dominated by previous imputation via $\{N, M\}$, e.g., (500, 0, 200).
- Another possibility is that Agnew and Mitchell play as above, but Nixon rejects the bid. He keeps the car and the 3-tuple payoff is (0,0,0). Note that this is **NOT** an imputation, because eventhough *individual rationality* holds, but *collective rationality* does not because $x_N + x_A + x_M = 0 < \nu(\{N, A, M\}) = 700$.

Let us consider games whose normal forms are zero-sum.

Definition

Let ν be a game in characteristic function form. We say that ν is constant-sum if, for every coalition $\mathcal{S}$, we have

$$\nu(\mathcal{S}) + \nu(\mathcal{S}^c) = \nu(\mathcal{P}).$$

Further, ν is zero-sum if it is constant sum and if, in addition $\nu(\mathcal{P}) = 0$.

Example: The game in Table 1 is constant-sum, while the Used Car Game is not since

$$\nu(\{N, A\}) + \nu(\{M\}) = 500 + 0 \neq 700 = \nu(\{N, A, M\}).$$

(*) Note that a game which is constant-sum in its normal form is different from a game which is constant-sum in its characteristic function form.

Definition

Let π be an $N-$person game in normal form. Then we say that π is constant-sum if there is a constant c such that

$$\sum_{i=1}^{N} \pi_i(x_1, \ldots, x_N) = c,$$

for all choices of strategies $x_1, \ldots, x_N$ for players $P_1, \ldots, P_N$ respectively. If $c = 0$, this reduces to zero-sum.

Theorem

If an $N-$person game π is constant-sum in its normal form, then its characteristic function is also constant-sum.

Proof: Let c be the constant value of π. Define a new game τ by subtracting c/N from every payoff in π. Thus

$$\tau_i(x_1, \ldots, x_N) = \pi_i(x_1, \ldots, x_N) - c/N$$

for all choice of i and all choices of strategies. Thus τ is zero-sum. We can show (homework ??) that the characteristic function μ of τ is zero-sum. Now it is easy to see that the characteristic function ν of π is related to μ by

$$\nu(\mathcal{S}) = \mu(\mathcal{S}) + kc/N,$$

where k is the number of players in $\mathcal{S}$. From this, we can see ν is constant-sum.

Theorem

If ν is both essential and constant-sum, then its core is empty.

Proof: Suppose ν has N players: $\mathcal{P} = \{P_1, \ldots, P_N\}$. Assume ν is esstential and there is an imputation $\boldsymbol{x}$ in the core, then we show that ν is inessential. For any player P_j, by individual rationality, we have

$$x_j \geq \nu(\{P_j\}).$$

Since $\boldsymbol{x}$ is in the core, we have

$$\sum_{i \neq j} x_i \geq \nu(\{P_j\}^c). \tag{15}$$

Adding these inequalities, and using collective rationality $\nu(\mathcal{P}) = \sum_{i=1}^{N} x_i \geq \nu(\{P_j\}) + \nu(\{P_j\}^c) = \nu(\mathcal{P})$, by the constant-sum property. It follows that Eq (15) is acutually an equality. Since it holds for every j, we have $\nu(\mathcal{P}) = \sum_{i=1}^{N} \nu(\{P_i\})$, which says that ν is inessential.

The theory of cooperative game has been applied to problems like (a) distribution of power in UN Security Council, (b) to understand the Electoral College method of electing US presidents.

Example of a voting game

- The municipal government of Lake Wobegon, Minnesota, is run by a City Council and a Mayor.
- The Council consists of six Aldermen and a Chairman.
- A bill can become a law in two ways:
  - A majority of the Council (with the Chairman voting only in case of a tie among the Aldermen) approves it and the Mayor signs it.
  - The Council passes it, the Mayor vetoes it, but at least six of the seven members of the Council then vote to override the veto (in this case, the Chairman always votes).
- The game consists of all eight people involved signing approval or disapproval of the given bill.

Example: continue

- The payoffs would be in units of "power" gained by being on the *winning* side.
- Define a *winning coalition* if it can pass a bill into law, e.g., a coalition consisting of any three Aldermen, the Chairman and the mayor. We define $\nu(\mathcal{S}) = 1$ if $\mathcal{S}$ is a winning coalition.
- Define a coalition which is not willing a *losing coalition*, e.g., the coalition consisting of four Aldermen is a losing since they do not have the votes to override the mayor's veto. We define $\nu(\mathcal{S}) = 0$ if $\mathcal{S}$ is a losing coalition.
- Note, every "one" player coalition is losing, the grand coalition is winning.

Example: continue

- An 8−tuple $(x_M, x_C, x_1, \ldots, x_6)$ is an imputation if and only if

$$x_M, x_C, x_1, \ldots, x_6 \geq 0;$$
$$x_M + x_C + x_1 + \cdots + x_6 = 1.$$

Theorem

The Lake Wobegon game has an empty core.

Proof

- Suppose on the contrary, $(x_M, x_C, x_1, \ldots, x_6)$ is in the core.
- Now any coalition consisting of at least six members of the Council is winning. Thus

$$x_C + x_1 + \cdots + x_6 \geq 1,$$

  and the same inequality holds if any one of the terms in it is dropped.
- Since all x's are nonnegative, and the sum of all eight is 1. This implies that all the x's in the inequality above are zero. This is a contradiction.

Example

Definition

A game ν in characteristic form is call simple if all the following holds:

- $\nu(\mathcal{S})$ is either 0 or 1, for every coalition $\mathcal{S}$.
- $\nu(\text{the grand coalition}) = 1$.
- $\nu(\text{any one-player coalition}) = 0$.

In a simple game, a coalition $\mathcal{S}$ with $\nu(\mathcal{S}) = 1$ is called a winning coalition, and one with $\nu(\mathcal{S})$ is called losing.

A four-person game is given in characteristic function form as follows:

$$\nu(\{P_1\}) = -1, \nu(\{P_2\}) = 0, \nu(\{P_3\}) = -1, \nu(\{P_4\}) = 0,$$
$$\nu(\{P_1, P_2\}) = 0, \nu(\{P_1, P_3\}) = -1, \nu(\{P_1, P_4\}) = 1,$$
$$\nu(\{P_2, P_3\}) = 0, \nu(\{P_2, P_4\}) = 1, \nu(\{P_3, P_4\}) = 0,$$
$$\nu(\{P_1, P_2, P_3\}) = 1, \nu(\{P_1, P_2, P_4\}) = 2,$$
$$\nu(\{P_1, P_3, P_4\}) = 0, \nu(\{P_2, P_3, P_4\}) = 1,$$
$$\nu(\{P_1, P_2, P_3, P_4\}) = 2, \nu(\{\emptyset\}) = 0.$$

Verify that ν is a characteristic function. Is the core of this came nonempty?

Solution

- To verify ν is a characteristic function, we have to check that super-additivity,

$$\nu(\mathcal{S} \cup \mathcal{T}) \geq \nu(\mathcal{S}) + \nu(\mathcal{T}),$$

holds whenever $\mathcal{S}$ and $\mathcal{T}$ are disjoint coalitions.

- This is easily check, for example

$$\nu(\{P_1, P_2, P_4\}) = 2 \geq -1 + 1 = \nu(\{P_1\}) + \nu(\{P_2, P_4\}).$$

- By the previous corollary, (x_1, x_2, x_3, x_4) is in the core if and only if both the following hold:

$$x_1 + x_2 + x_3 + x_4 = \nu(\{P_1, P_2, P_3, P_4\}) = 2,$$
$$\sum_{P_i \in \mathcal{S}} x_i \geq \nu(\mathcal{S}).$$

It is easy to check, for example (1,0,0,1) and (0,1,0,1) satisfy these conditions. Thus the core is not empty.