Introduction to Queueing Networks

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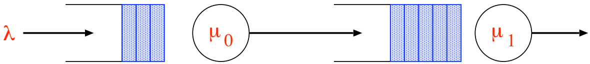
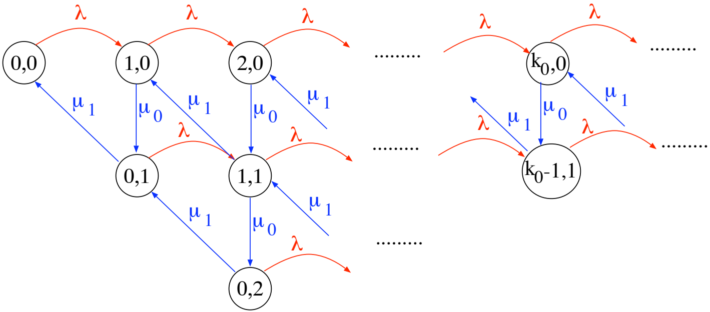
Outline

Example: Network of Queues





- **for $k_0 > 0, k_1 > 0$:**

$$(\mu_0+\mu_1+\lambda)p(k_0,k_1) = \mu_0 p(k_0+1,k_1-1)+\mu_1 p(k_0,k_1+1)+\lambda p(k_0-1,k_1)$$

- **for $k_0 > 0, k_1 = 0$:**

$$(\mu_0 + \lambda)p(k_0, 0) = \mu_1 p(k_0, 1) + \lambda p(k_0 - 1, 0)$$

- **for $k_0 = 0, k_1 > 0$:**

$$(\mu_1 + \lambda)p(0, k_1) = \mu_0 p(1, k_1 - 1) + \mu_1 p(0, k_1 + 1)$$

- **for $k_0 = 0, k_1 = 0$:**

$$\lambda p(0, 0) = \mu_1 p(0, 1)$$

Normalization :

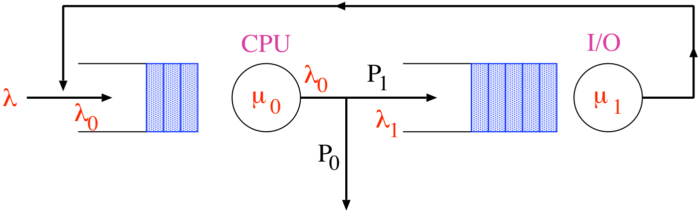
$$\sum_{k_0=0}^{\infty}\sum_{k_1=0}^{\infty} p(k_0,k_1) = 1$$

$$p(k_0,k_1) = (1-\rho_0)\rho_0^{k_0}(1-\rho_1)\rho_1^{k_1} \quad \text{for } k_0,k_1 = 0,1,\cdots$$

where $\rho_0 = \frac{\lambda}{\mu_0}$; $\rho_1 = \frac{\lambda}{\mu_1}$;
$P[N_0 = k_0] = (1-\rho_0)\rho_0^{k_0}$; $P[N_1 = k_1] = (1-\rho_1)\rho_1^{k_1}$

Open Queueing Network [Jackson 57]

It allows "feedback" and "product-form" can still be maintained.



$$
\begin{aligned}
p(k_0, k_1) &= (1-\rho_0)\rho_0^{k_0}(1-\rho_1)\rho_1^{k_1} \\
\lambda_0 &= \lambda + \lambda_1 \\
\lambda_1 &= \lambda_0 p_1
\end{aligned}
$$

Therefore,

$$\lambda_0 = \frac{\lambda}{1 - p_1}$$

$$\lambda_1 = \frac{\lambda p_1}{1 - p_1}$$

$$\rho_0 = \frac{\lambda_0}{\mu_0} = \frac{\lambda}{(1 - p_1)\mu_0} = \frac{\lambda}{p_0 \mu_0}$$

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{\lambda p_1}{p_0 \mu_1}$$

What is the average response time of a job?

By Little's formula, $T = \frac{\bar{N}}{\lambda}$ where $\bar{N}$ is the average no. of jobs in the "system".

$$\begin{aligned}
\bar{N} &= \bar{N_0} + \bar{N_1} = \sum_{k_0=0}^{\infty}\sum_{k_1=0}^{\infty}(k_0+k_1)p(k_0,k_1) \\
&= \sum_{k_0=0}^{\infty}\sum_{k_1=0}^{\infty}k_0 p(k_0,k_1) + \sum_{k_0=0}^{\infty}\sum_{k_1=0}^{\infty}k_1 p(k_0,k_1) \\
&= \sum_{k_0=0}^{\infty}k_0(1-\rho_0)\rho_0^{k_0}\sum_{k_1=0}^{\infty}(1-\rho_1)\rho_1^{k_1} \\
&\quad + \sum_{k_0=0}^{\infty}(1-\rho_0)\rho_0^{k_0}\sum_{k_1=0}^{\infty}k_1(1-\rho_1)\rho_1^{k_1}
\end{aligned}$$

$$\bar{N} = \frac{\rho_0}{1-\rho_0} + \frac{\rho_1}{1-\rho_1} \; ; \; E[T] = \frac{\bar{N}}{\lambda} = \left[\frac{\rho_0}{1-\rho_0} + \frac{\rho_1}{1-\rho_1}\right]\left[\frac{1}{\lambda}\right]$$

- Types of service centers
  - FCFS and service time is exponentially distributed.
  - Processor sharing (PS)
  - Last come first serve pre-emptive resume (LCFS-PR)
  - Infinite server (IS) or delay nodes
- We also allow a state dependent service rate ($\mu_i(n) =$ service rate at the i^{th} node where there is n customer).
  - Single server fixed rate (SSFR) where $\mu_i(n) = u_i$
  - Infinite server (IS) , $\mu_i(n) = n\mu_i$
  - Queue length dependent (QLD) with service rate $\mu_i(n)$

Jackson Network

- A queueing network with M nodes (labeled $i = 1, 2, \cdots, M$) s.t.
- Node i is QLD with rate $\mu_i(n)$ when it has n customers.
- A customer completing service at a node makes a probabilistic choice of either leaving the network or entering another node, independent of past history.
- The network is open and any external arrivals to node i is from a **Poisson** stream.

Jackson Network : Continue

$$\text{State space} \quad S = \{(n_1, n_2, \cdots, n_M) \mid n_i \geq 0\}$$

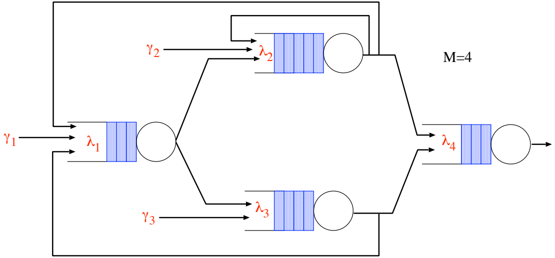
Routing probability matrix $Q = (q_{ij} \mid i, j = 1, 2, \cdots M)$

$$q_{i0} = 1 - \sum_{j=1}^{M} q_{ij}$$

- Let γ be the "TOTAL" external arrival rate to the open queueing network, the rate to node i is $\gamma_i = \gamma q_{0i}$ for $i = 1, 2, \cdots M$. So

$$\gamma = \sum_{i=1}^{M} \gamma_i \qquad (\sum q_{0i} = 1)$$

- All nodes in the Jackson network are QLD with exponential service time. Pictorially we can have:



- Let λ_i = mean arrival rate to node i,$(i = 1, 2, \cdots, M)$

$$\lambda_i = \gamma_i + \sum_{j=1}^{M} \lambda_j q_{ji}$$

there is **unique solution** to$\{\lambda_i\}$

- Example :

$$
\begin{aligned}
\lambda_1 &= \gamma_1 + \lambda_2 q_{21} + \lambda_3 q_{31} \\
\lambda_2 &= \gamma_2 + \lambda_1 q_{12} + \lambda_2 q_{22} \\
\lambda_3 &= \gamma_3 + \lambda_1 q_{13} \\
\lambda_4 &= \lambda_2 q_{24} + \lambda_3 q_{34}
\end{aligned}
$$

Jackson's Theorem: For a Jackson Network in steady state with arrival rate λ_i to node i,

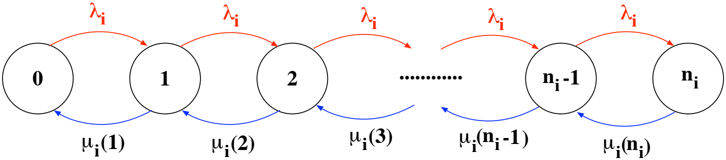
- The no. of customer at any node is independent of the number of customers at every other node.
- Node i behaves "stochastically" as if it were subjected to Poisson arrival rates λ_i
- Let $\pi(\vec{n}) = \text{Prob}[(n_1, n_2, \cdots n_M)]$ where $n_i \geq 0$, for **SSFR**:

$$\pi(\vec{n}) = \prod_{i=1}^{M} \left(1 - \frac{\lambda_i}{\mu_i}\right) \left(\frac{\lambda_i}{\mu_i}\right)^{n_i}$$

- For **QLD,** $M/M/c$ **queues**, define $u_i(r) = \mu_i \min(r, c_i)$ for $r \geq 0$, $i = 1, .., M$, and $\rho_i = \lambda_i/\mu_i$ for $i = 1, ..M$.

$$\pi(\vec{n}) = \prod_{i=1}^{M} C_i \left(\frac{\lambda_i^{n_i}}{\prod_{r=1}^{n_i} \mu_i(r)}\right) \quad ; \quad C_i = \left[\sum_{r=0}^{c_i-1} \frac{\rho_i^r}{r!} + \left(\frac{\rho_i^{c_i}}{c_i!}\right)\left(\frac{1}{1-\rho_i/c_i}\right)\right]^{-1}$$

The i^{th} node:



How about the normalization constant C_i ?

$$\pi(\vec{n}) = \pi(n_1, n_2, \cdots, n_M) = \prod_{i=1}^{M} \pi_i(n_i) = \prod_{i=1}^{M} C_i \left[\frac{\lambda_i^{n_i}}{\prod_{j=1}^{n_i} \mu_i(j)} \right]$$

where C_i can be found by $\sum_{n_i=0}^{\infty} C_i \left[\frac{\lambda_i^{n_i}}{\prod_{j=1}^{n_i} \mu_i(j)} \right] = 1$

So , what do we have?

- extremely powerful modeling tool to model a very large class of system.
- efficient solution
- we can compute mean queue length, utilization and throughout
- mean response time.

Comment

1. The arrival to each node, in general (unless it's only feed forward), is NOT a Poisson process.
2. How can we compute the $\bar{T}$ and each node utilization?
3. Optimal allocation : assume that the open network of SSFR nodes with arrival rate λ_i and μ_i for each node ($i = 1, 2, \cdots M$)

$$\begin{aligned}\text{Min } \bar{N} &= \sum_{i=1}^{M} \frac{\frac{\lambda_i}{\mu_i}}{1 - \frac{\lambda_i}{\mu_i}} \\ \text{s.t. } \sum_{i=1}^{M} \mu_i &= C = \text{constant}\end{aligned}$$

4. Application : Network, distributed system

Example 1

Consider a switching facility that transmits messages to a required destination. A NACK is sent by the destination when a packet has not been properly received. If so, the packet in error is retransmitted as soon as the NACK is received.

Assume the time to send a message and the time to receive a NACK are both exponentially distributed with parameter μ. Assume that packets arrive at the switch according to a Poisson process with rate λ^0. Let p, $0 < p \leq 1$, be the probability that a message is received correctly. Derive mean response time T.

We can model it as a Jackson network of one node with feedback, where $c_i = 1$ (SSFR), $p_{10} = p$ and $p_{11} = 1 - p$. Let $\pi(n)$, the probability of having n packets, is:

$$\begin{aligned} \lambda &= \lambda^0 + \lambda(1-p) \\ \pi(n) &= \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \quad n \geq 0 \end{aligned}$$

We have $\lambda = \lambda^0 / p$ and

$$\pi(n) = \left(1 - \frac{\lambda^0}{p\mu}\right)\left(\frac{\lambda^0}{p\mu}\right)^n \quad n \geq 0$$

Let N and T be the mean number of packet and mean response time.

$$\begin{aligned} N &= \frac{\lambda^0}{p\mu - \lambda^0} \\ T &= \frac{1}{p\mu - \lambda^0} \end{aligned}$$

Example 2

Similar to last example but now the switching facility is composed of K nodes in series, each model as $M/M/1$ queue with switching rate μ. What is the response time T?

- We have $\lambda_i^0 = 0$ for $i = 2, ..., K$ (no external arrival to nodes $2, ..., K$), $\mu_i = \mu$ for $i = 1, 2, ..., K$, $p_{i,i+1} = 1$ for $i = 1, ..., K-1$, and $p_{K,0} = p$ and $p_{K,1} = 1 - p$.
- $\lambda_1 = \lambda^0 + (1-p)\lambda_K$, $\lambda_i = \lambda_{i-1}$ for $i = 2, ..., K$. So

$$\lambda_i = \lambda^0/p \qquad \forall i = 1, ..., K.$$

By the Jackson's theorem, we have

$$\pi(\vec{n}) = \left(\frac{p\mu - \lambda^0}{p\mu}\right)^K \left(\frac{\lambda^0}{p\mu}\right)^{n_1+\cdots+n_K} \qquad \forall \vec{n} = (n_1, n_2, \ldots, n_K) \in \mathbb{N}^K$$

provided that $\lambda^0 < p\mu$. Let $E[N_i]$ be the average number of packets in queue i:

$$E[N_i] = \frac{\lambda^0}{p\mu - \lambda^0} \quad i = 1, \ldots, K.$$

Let $E[T]$ be the average response time:

$$E[T] = \sum_{i=1}^{K} E[N_i] = K\left(\frac{1}{p\mu - \lambda^0}\right).$$

Example 3: open central server network

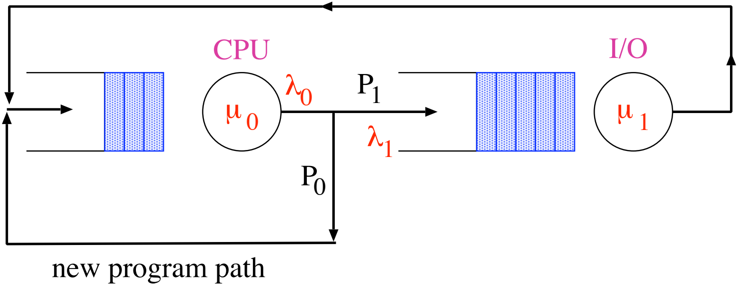
- A computer system with one CPU and two I/O devices. New jobs enter the system, wait for CPU resource, then possibly an I/O requests. When a job finishes its I/O, it may return for more CPU resource. Eventually a job completes and leave the system.
- It means that $K = 3$ (three nodes). $\lambda_i^0 = 0$ for $i = 2, 3$, $p_{2,1} = p_{3,1} = 1$, while $p_{1,0} > 0$.
- The traffic equations are: $\lambda_1 = \lambda_1^0 + \lambda_2 + \lambda_3$, $\lambda_2 = \lambda_1 p_{1,2}$, $\lambda_3 = \lambda_1 p_{1,3}$. Solving, we have: $\lambda_1 = \lambda_1^0 / p_{1,0}$, $\lambda_i = \lambda_1^0 p_{1,i} / p_{1,0}$ for $i = 2, 3$. Thus,

$$\pi(\vec{n}) = \left(1 - \frac{\lambda_1^0}{\mu_1 p_{1,0}}\right)\left(\frac{\lambda_1^0}{\mu_1 p_{1,0}}\right)^{n_1} \prod_{i=2}^{3}\left(1 - \frac{\lambda_1^0 p_{1,i}}{\mu_i p_{1,0}}\right)\left(\frac{\lambda_1^0 p_{1,i}}{\mu_i p_{1,0}}\right)^{n_i} \qquad \vec{n} \in \mathbb{N}^3$$

$$E[T] = \frac{1}{\mu_1 p_{1,0} - \lambda_1^0} + \sum_{i=2}^{3} \frac{p_{1,i}}{\mu_i p_{1,0} - \lambda_1^0 p_{1,i}}$$

Example of Queueing Network

- We fixed the total number of jobs be n in the system, where n is also called the "*degree of multiprogramming*".

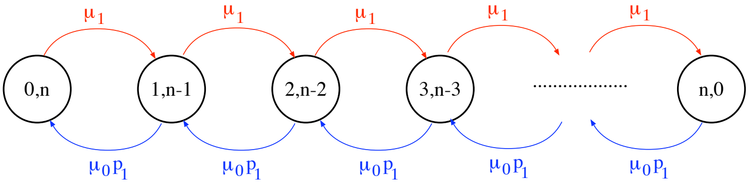


- State representation (k_0, k_1). Now with the constant that $k_0 + k_1 = n$

- Unlike the open queueing network, the state space is **finite**. The flow balance equations are:

$$
\begin{aligned}
(\mu_1 + \mu_0 p_1) p(k_0, k_1) &= \mu_0 p_1 p(k_0 + 1, k_1 - 1) + \mu_1 p(k_0 - 1, k_1 + 1) \\
\mu_1 p(0, n) &= \mu_0 p_1 p(1, n - 1) \\
\mu_0 p_1 p(n, 0) &= \mu_1 p(n - 1, 1)
\end{aligned}
$$

- If we draw the state transition diagram, we have:



- Using "traditional method", we know how to find the solution of $p(k_0, k_1)$
- New approach: Let $\rho_0 = \frac{a}{\mu_0}$; $\rho_1 = \frac{ap_1}{\mu_1}$, by substituting to the flow equations

$$p(k_0, k_1) = \frac{1}{C(n)} \rho_0^{k_0} \rho_1^{k_1} \qquad k_0, k_1 \geq 0$$

- The normalization factor $C(n)$ is chosen s.t

$$\sum_{k_0+k_1=n \,\&\, k_0,k_1\geq 0} p(k_0,k_1) = 1$$

- The choice of a (where $\rho_0 = \frac{a}{\mu_0}, \rho_1 = \frac{ap_1}{\mu_1}$) is very *arbitrary* in that the value of $p(k_0,k_1)$ will not change with a.
- If we define $\lambda_0 = a$ and $\lambda_1 = ap_1$, then (λ_0,λ_1) as the *relative throughput* of the corresponding nodes.
- We can choose, for example, $a = 1$ or $a = \mu_0$. Assume we choose $a = \mu_0$, then $\rho_0 = 1, \rho_1 = \frac{\mu_0 p}{\mu_1}$. Since

$$\begin{aligned} p(k_0,k_1) &= \frac{1}{C(n)}\rho_0^{k_0}\rho_1^{k_1} = \frac{1}{C(n)}\rho_1^{k_1} \\ 1 &= \frac{1}{C(n)}\sum_{k_1=0}^{n}\rho_1^{k_1} = \frac{1}{C(n)}\left[\frac{1-\rho_1^{n+1}}{1-\rho_1}\right] \end{aligned}$$

$$C(n) = \begin{cases} \frac{1-\rho_1^{n+1}}{1-\rho_1} & \text{where } \rho_1 \neq 1 \\ n+1 & \text{where } \rho_1 = 1 \text{(L'Hospital Rule)} \end{cases}$$

If we choose $a = 1$,then $\rho_0 = \frac{1}{\mu_0}$, $\rho_1 = \frac{p_1}{\mu_1}$

$$
\begin{aligned}
p(k_0, k_1) &= \frac{1}{C(n)}\rho_0^{k_0}\rho_1^{k_1} = \frac{1}{C(n)}\left(\frac{1}{\mu_0}\right)^{k_0}\left(\frac{p_1}{\mu_1}\right)^{k_1}, \text{ summing all } k_0, k_1 \\
1 &= \sum_{k_0=0}^{n}\sum_{k_1=n-k_0}^{n-k_0}\frac{1}{C(n)}\left(\frac{1}{\mu_0}\right)^{k_0}\left(\frac{p_1}{\mu_1}\right)^{k_1} \\
&= \sum_{k_0=0}^{n}\frac{1}{C(n)}\left(\frac{1}{\mu_0}\right)^{k_0}\left(\frac{p_1}{\mu_1}\right)^{n-k_0} \\
C(n) &= \sum_{k_0=0}^{n}\left(\frac{1}{\mu_0}\right)^{k_0}\left(\frac{p_1}{\mu_1}\right)^{n-k_0} = \left(\frac{p_1}{\mu_1}\right)^{n}\sum_{k_0=0}^{n}\left(\frac{1}{\mu_0}\right)^{k_0}\left(\frac{p_1}{\mu_1}\right)^{-k_0} \\
&= \left(\frac{p_1}{\mu_1}\right)^{n}\sum_{k_0=0}^{n}\left(\frac{\mu_1}{p_1\mu_0}\right)^{k_0} \\
C(n) &= \left(\frac{p_1}{\mu_1}\right)^{n}\left[\frac{1-\left(\frac{\mu_1}{p_1\mu_0}\right)^{n+1}}{1-\frac{\mu_1}{p_1\mu_0}}\right]
\end{aligned}
$$

$$
\begin{aligned}
\rightarrow p(k_0,k_1) &= \frac{1}{C(n)}\left(\frac{1}{\mu_0}\right)^{k_0}\left(\frac{p_1}{\mu_1}\right)^{k_1} \\
&= \frac{1}{C(n)}\left(\frac{1}{\mu_0}\right)^{k_0}\left(\frac{p_1}{\mu_1}\right)^{n-k_0} = \frac{1}{C(n)}\left(\frac{p_1}{\mu_0}\right)^{n}\left(\frac{\mu_1}{p_1\mu_0}\right)^{k_0} \\
p(k_0,k_1) &= \left(\frac{\mu_1}{p_1}\right)^{n}\left[\frac{1-\frac{\mu_1}{p_1\mu_0}}{1-(\frac{\mu_1}{p_1\mu_0})^{n+1}}\right]\left(\frac{p_1}{\mu_1}\right)^{n}\left(\frac{\mu_1}{p_1\mu_0}\right)^{k_0} \\
&= (p_1)^{-k_0}\left[\frac{1-p_1^{-1}}{1-p_1^{-(n+1)}}\right] \\
&= (p_1)^{-(n-k_1)}\left[p_1^{n}-\frac{1-p_1}{1-p_1^{n+1}}\right] \\
&= (p_1)^{k_1}\left(\frac{1-p_1}{1-p_1^{n+1}}\right)
\end{aligned}
$$

- The CPU utilization is

$$\begin{aligned} \text{CPU utilization} &= \text{Prob[CPU is busy]} \\ &= 1 - P(0,n) = 1 - \frac{\rho_1^n}{C(n)} \\ &= \begin{cases} \frac{\rho_1 - \rho_1^{n+1}}{1-\rho_1^{n+1}} & \rho_1 \neq 1 \\ \frac{n}{n+1} & \rho_1 = 1 \end{cases} \end{aligned}$$

- Average throughput is

$$E[T] = P[\text{CPU is busy}]\mu_0 p_0$$

Gordon - Newell network (1967)

- A Gordon-Newell network has M nodes ($i = 1, 2, \cdots M$) s.t.
  - Node i is QLD with rate $\mu_i(n)$ when there is n customers.
  - a customer completing service at a node chooses a node to enter next probabilistically, independent of past history
  - The network is **CLOSED** and has a fixed population K
- State space: $S = \{(n_1, n_2, \cdots n_M) \mid n_i \geq \phi, \sum_{i=1}^{M} n_i = K\}$

$$|S| = \binom{K+M-1}{M-1} \rightarrow \text{VERY LARGE NUMBER}$$

If $M = 5$, $K = 10$, $|S| = 1,001$.
If $M = 10$, $K = 35$, $|S| = 52,451,256$.

- Since there is **no** external arrival, $\gamma = \phi$.
  - Routing probabilities q_{ij} satisfy:

$$\sum_{j=1}^{M} q_{ij} = 1$$

  - traffic equations are:

$$\lambda_i = \sum_{j=1}^{M} \lambda_j q_{ji} \qquad i = 1, 2, \cdots, M$$

- The number of solutions $\{\lambda_i\}$ that satisfy the traffic equations is **infinite**.
- All solutions differ by a multiplicative factor C
- Let $(e_1, e_2, \cdots e_M)$ be any non-zero solution, that is $e_i = C\lambda_i$ (visit rate). Define: $x_i = \frac{e_i}{\mu_i}$
- $(e_1, e_2, \cdots e_M)$ is chosen by fixing one component to a convenient value, such as $e_1 = 1$

Gordon Newell Theorem

$$\pi(n_1, n_2, \cdots n_M) = \frac{1}{G} \prod_{i=1}^{M} x_i(n_i)$$

where $x_i(n_i) = \left[\frac{e_i^{n_i}}{\prod_{j=1}^{n_i} \mu_i(j)} \right]$ and $\sum_{i=1}^{M} n_i = K$

Therefore,

$$G = \sum_{\vec{n} \in S} \prod_{i=1}^{M} x_i(n_i)$$

Computation of G (assume SSFR)

Define $S(m,n) = \{(n_1, \cdots n_m) \mid n_i \geq 0, \sum_{i=1}^{m} n_i = n\}$

$$G(m,n) = \sum_{\vec{n} \in S(m,n)} \prod_{i=1}^{m} x_i(n_i) \quad \text{where} \quad x_i = \frac{e_i}{\mu_i}$$

$$\begin{aligned} G(m,n) &= \sum_{\substack{\vec{n} \in S(m,n);\\ n_m = 0}} \prod_{i=1}^{m} x_i^{n_i} + \sum_{\substack{\vec{n} \in S(m,n);\\ n_m > \phi}} \prod_{i=1}^{m} x_i^{n_i} \\ &= \sum_{\vec{n} \in S(m-1,n)} \prod_{i=1}^{m-1} x_i(n_i) + x_m \sum_{\substack{\vec{n} \in S(m,n);\\ k_i = n_i (i \neq m);\\ k_m = n_m - 1}} \prod_{i=1}^{m} x_i(k_i) \end{aligned}$$

$$G(m,n) = G(m-1,n) + x_m G(m,n-1) \quad m,n > \phi$$

In summary, for SSFR queueing network, we have

$$
\begin{aligned}
G(m,n) &= G(m-1,n) + \frac{e_m}{\mu_m} G(m,n-1) \quad m,n>0 \\
G(m,0) &= 1 \qquad\qquad m>0 \\
G(0,n) &= 0 \qquad\qquad n \geq 0
\end{aligned}
$$

n \ m	0	1	2	3		i		M-1	M
0	0	1	1	1				1	1
1	0	$\frac{e_1}{\mu_1}$	$\frac{e_1}{\mu_1}+\frac{e_2}{\mu_2}$						
2	0	$(\frac{e_1}{\mu_1})^2$							
	⋮	⋮			⋯⋯	↓ + (x) $\frac{e_i}{\mu_i}$ → + ↓	⋯⋯		
K-1	0	$(\frac{e_1}{\mu_1})^{K-1}$							↓
K	0	$(\frac{e_1}{\mu_1})^K$							→

Performance Measures

- Idle Probability for node i:

$$P(N_M = 0) = \frac{1}{G(M,K)} \sum_{\substack{\vec{n} \in S(M,K); \\ n_M = 0}} \prod_{i=1}^{M-1} \left(\frac{e_i}{\mu_i} \right)^{n_i} = \frac{G(M-1,K)}{G(M,K)}$$

- Another way to express it:

$$\begin{aligned} P(N_i = 0) &= \frac{1}{G(M,K)} \sum_{\substack{\vec{n} \in S(\mu,k); \\ n_i = 0}} \prod_{j=1}^{i-1} \left(\frac{e_i}{\mu_j} \right)^{n_j} \prod_{k=i+1}^{M} \left(\frac{e_k}{\mu_k} \right)^{n_k} \\ &= \frac{G(M \backslash i, K)}{G(M,K)} \end{aligned}$$

- Utilization of node i or (U_i):

$$U_i = 1 - P[N_i = 0] = 1 - \frac{G(M\backslash i, K)}{G(M, K)}$$

- How about $P[N_i \geq k]$:

$$P[N_i \geq k] = \frac{1}{G(M,K)} \sum_{\substack{\vec{n} \in S(M,K);\\ n_i \geq k}} \prod_{j=1}^{M} \left(\frac{e_j}{\mu_j}\right)^{n_j}$$

$$= \frac{1}{G(M,K)} \left(\frac{e_i}{\mu_i}\right)^k \sum_{\substack{m_j = n_j (j \neq i);\\ m_i = n_i - k;\\ \vec{n} \in S(M,K);\\ n_i \geq k}} \prod_{j=1}^{M} \left(\frac{e_j}{\mu_j}\right)^{m_j} = \left(\frac{e_i}{\mu_i}\right)^k \frac{G(M, K-k)}{G(M,K)}$$

- For $P[N_i \geq 1]$:

$$P[N_i \geq 1] = \mu_i = \left(\frac{e_i}{\mu_i}\right) \frac{G(M, K-1)}{G(M,K)}$$

(more useful)

The **throughout** of node i is

$$\lambda_i = \mu_i U_i = \mu_i \left(\frac{e_i}{\mu_i}\right) \frac{G(M,K-1)}{G(M,K)} = e_i \frac{G(M,K-1)}{G(M,K)}$$

$$\begin{aligned}
\text{Prob}[n_i = k] &= \pi_i(k) = P[N_i \geq k] - P[N_i \geq k+1] \\
\pi_i(k) &= \frac{1}{G(M,K)} \sum_{\substack{\vec{n}\in S(M,K);\\ n_i=k}} \prod_{j=1}^{M} \left(\frac{e_j}{\mu_j}\right)^{n_j} \\
&= \frac{\left(\frac{e_i}{\mu_i}\right)^k}{G(M,K)} \sum_{\substack{\vec{n}\in S(M,K-k);\\ n_i=0}} \prod_{j=1}^{M} \left(\frac{e_j}{\mu_j}\right)^{n_j} = \left(\frac{e_i}{\mu_i}\right)^k \frac{G(M\backslash i, K-k)}{G(M,K)} \\
\pi_i(k) &= P[N_i \geq k] - P[N_i \geq k+1] \\
&= \left(\frac{e_i}{\mu_i}\right)^k \left[\frac{G(M,K-k) - \frac{e_i}{\mu_i} G(M,K-k-1)}{G(M,K)}\right]
\end{aligned}$$

Expected no. of customer in node $i = L_i(K)$

$$
\begin{aligned}
L_i(K) &= \sum_{j=1}^{K} jP[N_i = j] = \sum_{j=1}^{K} P[N_i = j] \sum_{k=1}^{K} I_{(k \leq j)} \\
&= \sum_{k=1}^{K} \sum_{j=k}^{K} P[N_i = j] = \sum_{k=1}^{K} P[N_i \geq k] \\
L_i(K) &= \frac{1}{G(M,K)} \sum_{k=1}^{K} \left(\frac{e_i}{\mu_i}\right)^k G(M, K-k) \quad i = 1, 2, \cdots, M
\end{aligned}
$$

Derivation of above expression:

$$
\begin{aligned}
L_I(K) = \; & P[N_i = 1] + \\
& P[N_i = 2] + P[N_i = 2] + \\
& P[N_i = 3] + P[N_i = 3] + P[N_i = 3] + \cdots + \\
& P[N_i = K] + P[N_i = K] + P[N_I = K] + \cdots + P[N_i = K]
\end{aligned}
$$

Garden Newell Convolution Algorithm

- Assume M QLD nodes and K customers
- Define

$$
\begin{aligned}
f_i(Z) &= \sum_{k=0}^{\infty} X_i(k)Z^k = 1 + X_i(1)Z + X_i(2)Z^2 + X_i(3)Z^3 + \cdots \\
&= 1 + \left[\frac{e_i}{\mu_i(1)}\right] Z + \left[\frac{e_i^2}{\mu_i(1)\mu_i(2)}\right] Z^2 + \left[\frac{e_i^3}{\mu_i(1)\mu_i(2)\mu_i(3)}\right] Z^3 + \cdots \\
f(Z) &= f_1(Z)f_2(Z)\ldots f_M(Z)
\end{aligned}
$$

- The coefficient of Z^k in $f(Z) \rightarrow$ the sum of products of the form $X_1(n_1)X_2(n_2)\ldots X_M(n_M)$ such that $\sum_i n_i = k$

- Thus

$$f(Z) = 1 + G(M,1)Z + G(M,2)Z^2 + G(M,3)Z^3 + \cdots + G(M,k)Z^k + \cdots$$

- *IDEA: build up $f(Z)$ from partial products $g_i(Z)$ so that $G(i,k)$ is the coefficient of Z^k in $g_i(Z)$

$$\begin{aligned} g_1(Z) &= f_1(Z) \\ g_i(Z) &= g_{i-1}(Z) f_i(Z) \quad i = 2, 3, \cdots, M \end{aligned}$$

Therefore,

$$\begin{aligned} G(1,k) &= \text{coefficient of the } Z^k \text{ term in } g_1(Z) \\ G(1,k) &= X_1(k) = \frac{e_1^k}{\prod_{i=1}^{k}\mu_1(i)} \\ G(i,k) &= \sum_{j=0}^{k} G(i-1,j)x_i(k-j) \rightarrow \text{(convolution!)} \end{aligned}$$

Example :

$$\begin{aligned} G(2,k) &= \sum_{j=0}^{k} G(1,j)x_2(k-j) \\ &= G(1,0)x_2(k) + G(1,1)x_2(k-1) + \cdots + G(1,k)x_2(\phi) \\ &= \frac{e_2^k}{\prod_{j=1}^{k} x_2(j)} + \frac{e_1}{\mu_1(1)}\frac{e_2^{k-1}}{\prod_{j=1}^{k-1}\mu_2(j)} + \cdots \end{aligned}$$

$k \backslash \mu$	1	2	$\cdots$
0	1	1	$\cdots$
1	$\frac{e_1}{\mu_1(1)}$		$\cdots$
2	$\frac{e_1^2}{\mu_1(1)\mu_1(2)}$		$\cdots$
3	$\frac{e_1^3}{\mu_1(1)\mu_1(2)\mu_1(3)}$		$\cdots$
$\vdots$	$\vdots$		$\cdots$
K	$\frac{e_1^k}{\mu_1(1)...\mu_1(k)}$		$\cdots$

Performance Measure

- Prob[node i has k customers] $= \pi_i(k)$:

$$\pi_i(k) = \frac{1}{G(M,K)} \sum_{\substack{\vec{n} \in S(\mu,k); \\ n_i = k}} X_1(n_1) X_2(n_2) \ldots X_M(n_M)$$

$$\pi_i(k) = \frac{x_i(k)}{G(M,K)} G(M \backslash i, K-k)$$

- Expected number of customers in node i

$$E[L_i] = \sum_{k=0}^{K} k \pi_i(k) \Longrightarrow \text{very involve}$$

- Prob[node $i \geq 1$ customer] = ?

- Utilization of a node, in general, is very involved.

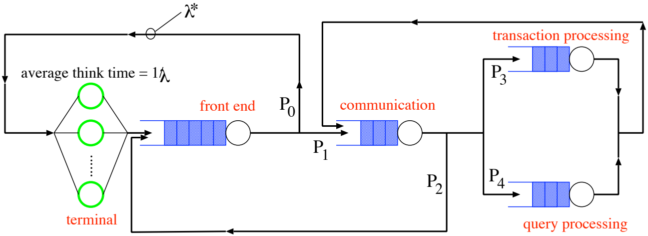
$$U_i(K) = Prob[n_i \geq 1] = \sum_{j=1}^{K} Prob[n_i = j]$$

- For a node that is SSFR, we have

$$U_i(K) = \left(\frac{e_i}{\mu_i}\right) \frac{G(M, K-1)}{G(M, K)}$$

- This holds even the other nodes are QLD servers!!

Example :



$$e_T = e_F p_0; \quad e_F = e_T + e_C p_2; \quad e_C = e_F p_1 + e_D + e_P;$$

$$e_D = e_C p_3; \quad e_P = e_C p_4$$

$$e_T = 1; \quad e_F = \frac{1}{p_0}; \quad e_C = \frac{1-p_0}{p_0 p_2}; \quad e_D = \frac{(1-p_0)p_3}{p_0 p_2}; \quad e_P = \frac{(1-p_4)p_4}{p_0 p_2}$$

$$\rho_T = \frac{e_T}{\lambda} = \frac{1}{\lambda}; \quad \rho_F = \frac{1}{p_0 \mu_F}; \quad \rho_C = \frac{(1-p_0)}{p_0 p_1 \mu_c}; \quad \rho_D = \frac{(1-p_0)p_3}{p_0 p_2 \mu_0} \quad \rho_P = \frac{(1-p_0)p_4}{p_0 p_2 \mu_p}$$

$$\pi(n_T, n_F, n_C, n_D, n_p) = \frac{1}{G} \frac{(\rho_T)^{n_T}}{n_T!} (\rho_F)^{n_F} (\rho_C)^{n_C} (\rho_D)^{n_D} (\rho_P)^{n_P}$$

U_F = ?

- Since U_F is a SSFR, we have

$$U_F = \rho_F \frac{G(5, K-1)}{G(5, K)} = \frac{1}{p_0 \mu_F} \frac{G(5, K-1)}{G(5, K)}$$

- Average throughput or rate of request completion is:

$$\lambda^* = \mu_F p_0 U_F = \frac{G(5, k-1)}{G(5, k)}$$

- but Little's Result ($N = \lambda T$)
  $T =$ the expected response time = $\frac{K}{\lambda^*} = \frac{KG(5,k)}{G(5,K-1)}$
  T = average think + average processing time = $\frac{KG(5,k)}{G(5,K-1)}$
  $T = \frac{1}{\lambda}$ + average processing time = $\frac{KG(5,k)}{G(5,K-1)}$
  Therefore, average processing time = $\frac{KG(5,k)}{G(5,K-1)} - \frac{1}{\lambda}$

Multiclass Open/Closed/Mixed Jackson Networks

Setting

- We have K customers and M nodes with μ_i exponential service rate for $i = 1, .., M$.
- R, an arbitrary but finite number of classes of customers.
- Let $p_{i,r;j,s}$ be the probability that a customer of class r completes service at node i will become class s in node j.
- The pairs (i, r) and (j, s) belong to the same **subchain** if the **same** customer can visit node i in class r and node j in class s.
- Let m be the number of subchains, let $E_1, \ldots, E_m$ be the set of states in each subchains.

Setting: continue

- Let n_{ir} be the number of customers of class r at node i. A "*closed*" system is characterized by

$$\sum_{(i,r)\in E_j} = \text{constant.} \quad \forall j = 1, \ldots, m.$$

- For an "*open*" system, a Poisson process with λ^0_{ir} is the external arrival rate of class r to node i. Customer may leave the system with $p_{i,r;0}$ so that $\sum_{j,s} p_{i,r;j,s} + p_{i,r;0} = 1$.
- Define $\boldsymbol{Q}(t) = (\boldsymbol{Q}_1(t), \ldots, \boldsymbol{Q}_M(t))$ where $\boldsymbol{Q}_i(t) = (Q_{i1}(t), \ldots, Q_{iR}(t))$ with $Q_{ir}(t)$ being the number of class r customers at node i. Note that $\boldsymbol{Q}(t)$ is **NOT** a CTMC because the class of a customer leaving a node is not known.
- Instead, we define $\boldsymbol{X}_i(t) = (I_{i1}(t), \ldots, I_{i,|Q_i(t)|}(t))$ where $I_{ij}(t) \in \{1, 2, ..., R\}$ is the class of the customer in position j in node i at time t. Then $(\boldsymbol{X}_1(t), ..., \boldsymbol{X}_M(t)), t \geq 0)$ is a CTMC.

Multiclass Open/Closed/Mixed Jackson Networks: For $k \in \{1, ..., m\}$ such that E_k is an open subchain, let $(\lambda_{ir})_{(i,r)\in E_k}$ be the "unique" strictly positive solution of the traffic equations

$$\lambda_{ir} = \lambda_{ir}^0 + \sum_{(j,s)\in E_k} \lambda_{js} p_{j,s;i,r} \quad \forall (i,r) \in E_k.$$

For $k \in \{1, ..., m\}$ such that E_k is a closed subchain, let $(\lambda_{ir})_{(i,r)\in E_k}$ be any non-zero solution of

$$\lambda_{ir} = \sum_{(j,s)\in E_k} \lambda_{js} p_{j,s;i,r} \quad \forall (i,r) \in E_k.$$

If $\sum_{r:(i,r)}$belongs to an open subchain $\lambda_{ir} < \mu_i$ for all $i = 1, 2, ..., M$, then

$$\pi(\vec{n}) = \frac{1}{G} \prod_{i=1}^{M} \left[n_i! \prod_{r=1}^{R} \frac{1}{n_{ir}!} \left(\frac{\lambda_{ir}}{\mu_i} \right)^{n_{ir}} \right]$$

for all $\vec{n} = (\vec{(}n_1), ..., \vec{(}n_M))$ in the state space, where $\vec{(}n_i) = (n_{i1}, ..., n_{iR}) \in \mathbb{N}^R$ and $n_i = \sum_{r=1}^{R} n_{ir}$. Here, G is the normalization constant.

Example

- There are $M = 2$ nodes and $R = 2$ classes of customers. There is no external arrival to node 2. External customers enter node 1 in class 1 with rate λ. Upon completion at node 1, a customer of class 1 is routed to node 2 with the probability 1. Upon completion at node 2, a customer of class a leaves with probability 1.
- There are always K customers of class 2 in the system. Upon service completion at node 1 (resp. node 2), customer of class 2 is routed back to node 2 (resp. node 1) in class 2 with probability 1. Let μ_i be the service rate at node $i = 1, 2$.

Example: continue

The state space S is

$$S = \{(n_{11}, n_{12}, n_{21}, n_{22}) \in \mathbb{N}^4 : n_{11} \geq 0, n_{21} \geq 0, n_{12} + n_{22} = K\}.$$

There are two subchains: E_1 (open), and E_2 (closed), with $E_1 = \{(1,1),(2,1)\}$ and $E_2 = \{(1,2),(2,2)\}$.

We find $\lambda_{11} = \lambda_{2,1} = \lambda$ and $\lambda_{12} = \lambda_{22}$. Take $\lambda_{12} = \lambda_{22} = 1$ for instance. We have

$$\pi(\vec{n}) = \frac{1}{G} \binom{n_1}{n_{11}} \binom{n_2}{n_{22}} \left(\frac{\lambda}{\mu_1}\right)^{n_{11}} \left(\frac{\lambda}{\mu_2}\right)^{n_{21}} \left(\frac{1}{\mu_1}\right)^{n_{12}} \left(\frac{1}{\mu_2}\right)^{n_{22}}$$

with $\lambda < \mu_i$, $i = 1,2$ (stability condition).

Example: continue (COMPUTING G)

$$
\begin{aligned}
G &= \sum_{\substack{n_{11}\geq 0; n_{21}\geq 0\\ n_{12}+n_{22}=K}} \left(\frac{\lambda}{\mu_1}\right)^{n_{11}} \left(\frac{\lambda}{\mu_2}\right)^{n_{21}} \left(\frac{1}{\mu_1}\right)^{n_{12}} \left(\frac{1}{\mu_2}\right)^{n_{22}} \\
&= \left(\sum_{n_{11}\geq 0} \frac{\lambda}{\mu_1}\right)^{n_{11}} \left(\sum_{n_{21}\geq 0} \frac{\lambda}{\mu_2}\right)^{n_{21}} \sum_{n_{12}+n_{22}=K} \left(\frac{1}{\mu_1}\right)^{n_{12}} \left(\frac{1}{\mu_2}\right)^{n_{22}} \\
&= \left(\prod_{i=1}^{2} \frac{\mu_i}{\mu_i - \lambda}\right) \left(\frac{1}{\mu_1}\right)^{K} \sum_{i=1}^{K} \left(\frac{\mu_1}{\mu_2}\right)^{i} \\
G &= \left(\prod_{i=1}^{2} \frac{\mu_i}{\mu_i - \lambda}\right) \left(\frac{1}{\mu_1}\right)^{K} \frac{1-(\mu_1/\mu_2)^{K+1}}{1-(\mu_1/\mu_2)} \quad \text{if } \mu_1 \neq \mu_2, \\
G &= \frac{K+1}{\mu^K} \left(\frac{\mu}{\mu - \lambda}\right)^2 \qquad \text{if } \mu_1 = \mu_2 = \mu.
\end{aligned}
$$

Extension to $M/M/c$

Let $c_i \geq 1$ be the number of servers at node i and define $\alpha_i(j) = \min(c_i, j)$ for $i = 1, ..., M$. Hence $\mu_i \alpha_i(j)$ is the service rate at node i when there are j customers. We have

$$\pi(\vec{n}) = \frac{1}{G} \prod_{i=1}^{M} \left[\left(\prod_{j=1}^{n_i} \frac{1}{\alpha_i(j)} \right) n_i! \left(\prod_{r=1}^{R} \frac{1}{n_{ir}!} \left(\frac{\lambda_{ir}}{\mu_i} \right)^{n_{ir}} \right) \right].$$

Extension to $M/M/c$ with state-depending routing

Let the total number of customer be $M(\vec{n}) = \sum_{i=1}^{M} n_i$. Let the external arrival rate of class r customer at node i be $\lambda_{ir}^{0}\gamma(M(\vec{n}))$, where γ is an arbitrary function from $\mathbb{N}$ into $[0, \infty)$. We have

$$\pi(\vec{n}) = \frac{d(\vec{n})}{G} \prod_{i=1}^{M} \left[\left(\prod_{j=1}^{n_i} \frac{1}{\alpha_i(j)} \right) n_i! \left(\prod_{r=1}^{R} \frac{1}{n_{ir}!} \left(\frac{\lambda_{ir}}{\mu_i} \right)^{n_{ir}} \right) \right].$$

where

$$d(\vec{n}) = \prod_{j=0}^{M(\vec{n})-1} \gamma(j).$$

and $d(\vec{n}) = 1$ if the network is closed.

A **classic piece** by F. Baskett, K.M. Chandy, R.R. Muntz and F.G. Palacios on "*Open, Closed, and Mixed Networks of Queues with Different Classes of Customers*, JACM, 22(2), 1975.

Terminology

- FCFS: First come first serve $M/M/c$ queue.
- PS: Processor sharing queue.
- LCFS: Last come first serve single server queue.
- IF: Infinite server queue

Characterization

- If node i is of type FCFS, then $\rho_{ir} = \lambda_{ir}/\mu_i$ for $r = 1, \ldots, R$, where R is the number of classes of customer, and μ_i is the exponential service times in node i.
- If node i is of type PS, LCFS, or IS, then $\rho_{ir} = \lambda_{ir}/\mu_{ir}$ for $r = 1, \ldots, R$, and μ_{ir} is the mean service time for customer of type r in node i.

  For nodes of types PS, LCFS, or IS, the service time distribution is **arbitrary**.

λ_{ir} is the solution of the traffic equations.

Theorem

For a BCMP network with M nodes and R classes of customer, which is open, closed, or mixed in which each node is of type FCFS, PS, LCFS, or IS, the steady state probabilities are:

$$\pi(\vec{n}) = \frac{d(\vec{n})}{G} \prod_{i=1}^{M} f_i(\vec{n}_i).$$

where $\vec{n} = (\vec{n}_1, \cdots, \vec{n}_M)$ *in the state space S with* $\vec{n}_i = (n_{i1}, n_{i2}, \cdots, n_{iR})$ *where* n_{ir} *is the number of jobs of class r at node i. Moreover,* $|\vec{n}_i| = \sum_{r=1}^{R} n_{ir}$ *for* $i = 1, 2, \ldots, M$.

$G < \infty$ *is the normalization constant such that* $\sum_{\vec{n} \in S} \pi(\vec{n}) = 1$, $d(\vec{n}) = \prod_{j=0}^{M(\vec{n})-1} \gamma(j)$ *if the arrivals depend on the total number of customers* $M(\vec{n}) = \sum_{i=1}^{M} |\vec{n}_i|$, *and* $d(\vec{n}) = 1$ *if the network is closed.*

$f_i(\vec{n}_i)$ for different types of nodes

- If node i is of type FCFS:

$$f_i(\vec{n}_i) = |\vec{n}_i|! \prod_{j=1}^{|\vec{n}_i|} \frac{1}{\alpha_i(j)} \prod_{r=1}^{R} \frac{\rho_{ir}^{n_{ir}}}{n_{ir}!}$$

- If node i is of type PS or LCFS:

$$f_i(\vec{n}_i) = |\vec{n}_i|! \prod_{r=1}^{R} \frac{\rho_{ir}^{n_{ir}}}{n_{ir}!}$$

- If node i is of type IS:

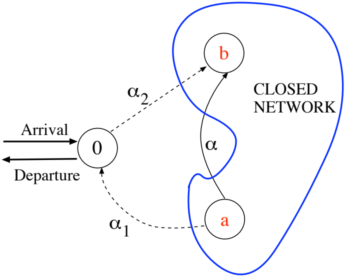
$$f_i(\vec{n}_i) = \prod_{r=1}^{R} \frac{\rho_{ir}^{n_{ir}}}{n_{ir}!}$$

Comment

- Solve the traffic equations λ_{ir} for $i = 1, ..., M$ and $r = 1, ..., R$.
- Use queueing network package to solve for G.
- When using queueing network package, all we need to enter are the topology of the network:
  - K, the number and the types of nodes,
  - R classes, and
  - routing matrix $[p_{i,r;j,s}]$.
  - external arrival rates
  - service rates, e.g., $\mu_i \alpha_i(j)$, for $j = 0, 1, ...$ for a node that is FCFS.
  - service rates μ_{ir} for customers of class $r = 1, ...R$ for a node that is PS, LCFS, or IS.

Mean value analysis [S. Lavenberg & M. Reiser]

- Derive **expected** performance measures while *avoiding derivation of steady state probability*



- Replace an arc α by $\alpha_1 \rightarrow$ node $\phi \rightarrow \alpha_2$

- Whenever a customer arrives at node ϕ (via along α_1), it departs from the network and is replaced by a "stochastically" identical customer who has the same routing probability along arc α_2
- This open network behaves exactly as the closed one (except node ϕ)

Mean Value analysis for state-dependent service rates

- The system throughput is still $T(K) = \frac{K}{\sum_{i=1}^{M} v_i W_i(k)}$
- Let $\pi_i(j|K)$ =Prob[node i has j customers where the network has K customers]

$$W_i(K) = \sum_{j=1}^{K} \pi_i(j-1|K-1)\frac{j}{\mu_i(j)}$$

Example:

$$\pi_i(0|K-1)\frac{1}{\mu_i(1)} + \pi_i(1|K-1)\frac{2}{\mu_i(2)} + \cdots + \pi_i(K-1|K-1)\frac{K}{\mu_i(K)}$$

- The mean queue length is:

$$L_i(K) = \sum_{j=1}^{K} j\pi_i(j|K)$$

- By definition, $\pi_i(0|0) = 1$ and

$$\pi_i(j|K) = \begin{cases} \frac{v_i T(K)}{\mu_i(j)}\pi_i(j-1|K-1) & j = 1, 2, \ldots, k \\ 1 - \sum_{k=1}^{K} \pi_i(k|K) & j = \phi \end{cases}$$

- For the α we chosen, let us define the **network throughput** T as the average rate customers pass along arc α in steady state.
- Now we can view T as the external arrival rate (γ) in the open network.
- Suppose that we have M nodes in the network. Define

$$\begin{aligned} v_i &= \text{average number of visit to node i by a customer} \\ &= (\textit{visitation rate}) \\ \lambda_i &= \text{average arrival rate of customer to node i} \end{aligned}$$

$$\lambda_i = T v_i \quad , \quad v_0 = v_a q_{ab}$$

- But since customers visit node ϕ EXACTLY ONCE, $v_0 = 1$, therefore:

$$v_a = \frac{1}{q_{ab}}; \quad v_j = \sum_{i=1}^{\mu} v_i q_{ij}$$

- Once one v_i is found,we can find other v_i's.

- Looking at node i, let L_i= average no. of customers, we have:

$$L_i = \lambda_i w_i \quad , \quad L_i = T v_i w_i$$

- But since $\sum_{i=1}^{M} L_i = K = T \sum_{i=1}^{M} v_i w_i$, therefore the system throughput

$$T = \frac{K}{\sum_{i=1}^{M} v_i w_i}$$

- If node i is infinite server,then

$$w_i = \frac{1}{\mu_i}$$

- If node i is single server fixed rate (SSFR),

$$w_i = \frac{1}{\mu_i}[Y_i + 1]$$

where Y_i is the mean number of customers seen by an arrival to node i.

Example:

$$\pi_i(j|K) = \begin{cases} \frac{v_i T(K)}{\mu_i(j)} \pi_i(j-1|K-1) & j = 1, 2, \ldots, k \\ 1 - \sum_{k=1}^{K} \pi_i(k|K) & j = \phi \end{cases}$$

- Assume it is node 1 (that is , i=1)

$$\begin{aligned}
\pi_1(1|1) &= \frac{v_1 T(1)}{\mu_1(1)} \pi_1(\phi|\phi) = \frac{v_1 T(1)}{\mu_1(1)} \\
\pi_1(\phi|1) &= 1 - \pi_1(1|1) \\
\pi_1(1|2) &= \frac{v_1 T(2)}{\mu_1(1)} \pi_1(\phi|1) \\
\pi_1(2|2) &= \frac{v_1 T(2)}{\mu_2(2)} \pi_1(1|1) \\
\pi_1(\phi|2) &= 1 - \pi(1|2) - \pi(2|2)
\end{aligned}$$