where we write \( \rho = \lambda / \mu \). For stability we require \( \lambda < m \mu \), or \( \rho < 1 \). We note that \( D(z) \) is the negative of the polynomial in Eq. (1.84) (with \( m \) replacing \( r \)), and so we may use the results of Problem 4.10. Thus \( D(z) \) has one root at \( z = 1 \), \( m - 1 \) roots in the range \( |z| < 1 \), and one root (say, \( z_m \)) in \( |z| > 1 \). By analyticity of \( Q(z) \) for \( |z| \leq 1 \), the \( m \) roots of \( D(z) \) satisfying \( |z| \leq 1 \) cancel with the \( m \) numerator roots leaving

\[
Q(z) = K \frac{1}{z_m - z}
\]

Now \( Q(1) = 1 \) implies \( K = z_m - 1 \). Thus

\[
Q(z) = \frac{z_m - 1}{z_m - z}
\]

\[\square\]

**PROBLEM 5.14**

Consider an M/G/1 system with bulk service. Whenever the server becomes free, he accepts two customers from the queue into service simultaneously, or, if only one is on queue, he accepts that one; in either case, the service time for the group (of size 1 or 2) is taken from \( B(x) \). Let \( q_n \) be the number of customers remaining after the \( n \)th service instant. Let \( v_n \) be the number of arrivals during the \( n \)th service. Define \( B^*(s) \), \( Q(z) \), and \( V(z) \) as transforms associated with the random variables \( \bar{x} \), \( \bar{q} \), and \( \bar{v} \) as usual. Let \( \rho = \lambda \bar{x} / 2 \).

(a) Using the method of imbedded Markov chains, find

\[
E(\bar{q}) = \lim_{n\to\infty} E(q_n)
\]

in terms of \( \rho \), \( \sigma_2 \), and \( P[\bar{q} = 0] \equiv p_0 \).

(b) Find \( Q(z) \) in terms of \( B^*(s) \), \( p_0 \), and \( p_1 \equiv P[\bar{q} = 1] \).

(c) Express \( p_1 \) in terms of \( p_0 \).

**SOLUTION**

(a) Clearly we may write

\[
q_{n+1} = \begin{cases} 
q_n - 2 + v_{n+1} & q_n \geq 2 \\
q_n - 1 + v_{n+1} & q_n = 1 \\
v_{n+1} & q_n = 0
\end{cases}
\]

Introducing the function

\[
\Delta_{2,k} \triangleq \begin{cases} 
2 & k \geq 2 \\
1 & 0 \leq k \leq 2
\end{cases}
\]
we have \( q_{n+1} = q_n - \Delta_{2,n} + v_{n+1} \). Letting \( n \to \infty \) and taking expectations

\[
\bar{q} = \bar{q} - E[\Delta_{2,n}] + \bar{v}
\]

But

\[
E[\Delta_{2,n}] = \sum_{k=0}^{\infty} \Delta_{2,n} P[\bar{q} = k] = P[\bar{q} = 1] + \sum_{k=2}^{\infty} 2k P[\bar{q} = k] = p_1 + 2(1 - p_0 - p_1)
\]

So

\[
\bar{v} = E[\Delta_{2,n}] = 2 - 2p_0 - p_1
\]

Recall that

\[
q_{n+1} = q_n - \Delta_{2,n} + v_{n+1}
\]

Squaring this equation gives

\[
q_{n+1}^2 = q_n^2 - 2q_n \Delta_{2,n} + \Delta_{2,n}^2 + 2v_{n+1} (q_n - \Delta_{2,n}) + v_{n+1}^2
\]

Let \( n \to \infty \) and take expectations

\[
\bar{q}^2 = \bar{q}^2 - 2E[\bar{q} \Delta_{2,n}] + E[\Delta_{2,n}^2] + 2 \bar{v}E[\bar{q} - \Delta_{2,n}] + \bar{v}^2
\]

since \( \bar{v} \) and \( \bar{q} \) are independent. So

\[
2E[\bar{q} \Delta_{2,n}] = E[\Delta_{2,n}^2] + \bar{v}^2 + 2 \bar{v} (E[\bar{q}] - E[\Delta_{2,n}])
\]

Now \( E[\Delta_{2,n}] = \bar{v} \) and

\[
E[\Delta_{2,n}^2] = \sum_{k=1}^{\infty} \Delta_{2,n}^2 P[\bar{q} = k] = P[\bar{q} = 1] + 4 \sum_{k=2}^{\infty} k^2 P[\bar{q} = k] = p_1 + 4(1 - p_0 - p_1)
\]

Also

\[
E[\bar{q} \Delta_{2,n}] = \sum_{k=1}^{\infty} k \Delta_{2,n} P[\bar{q} = k] = P[\bar{q} = 1] + \sum_{k=2}^{\infty} 2k P[\bar{q} = k] = p_1 + 2 \sum_{k=1}^{\infty} k P[\bar{q} = k] - 2p_1 = 2\bar{q} - p_1
\]

Thus

\[
2(2\bar{q} - p_1) = 4 - 4p_0 - 3p_1 + \bar{v}^2 + 2\bar{v}(\bar{q} - \bar{v})
\]

Therefore, using \( p_1 = 2 - 2p_0 - \bar{v} \), we find

\[
\bar{q} = \frac{2 - 2p_0 + \bar{v} + \bar{v}^2 - 2(\bar{v})^2}{4 - 2\bar{v}}
\]

From Problem 5.11, we know that \( V(q) = B^*(\lambda - \lambda_2) \). Then, by differentiation, we have

\[
\bar{v} = V'(\bar{q}) = \lambda \bar{x} = 2p
\]

and

\[
\bar{v}^2 - \bar{v} = V''(\bar{q}) = \lambda^2 \bar{x}^2
\]

Thus

\[
\bar{q} = \frac{2(1 - p_0) + 2p + \lambda^2 \bar{x}^2 + 2p - 2(4p^2)}{4 - 4p}
\]

\[
\bar{q} = \rho + \frac{2(1 - p_0) + \lambda^2 \bar{x}^2 - 4p^2}{4(1 - \rho)}
\]

(b) We have

\[
Q(z) = E[z^\bar{q}] = E[z^{\bar{q} - \Delta_{2,n} + v_{n+1}}] = E[z^{\bar{q}}] E[z^{\Delta_{2,n} + v_{n+1}}] = V(z) E[z^{\Delta_{2,n}}]
\]
more ready for service, they are both "taken into" service as above. If one or none is ready, then the server waits until a pair is ready, and so on. Let

\[ q_n = \text{number of customers left behind in the system immediately following the } n\text{th critical time} \]

\[ v_n = \text{number of customers arriving during the combined service time of the } n\text{th pair of customers} \]

\( a \) Derive a relationship between \( q_{n+1}, q_n, \) and \( v_{n+1} \).

\( b \) Find

\[ V(z) = \sum_{k=0}^{\infty} P[v_k = k]z^k \]

\( c \) Derive an expression for \( Q(z) = \lim_{n \to \infty} Q_n(z) \) in terms of \( p_0 = P[\bar{q} = 0] \), where

\[ Q_n(z) = \sum_{k=0}^{\infty} P[q_k = k]z^k \]

\( d \) How would you solve for \( p_0 \)?

\( e \) Describe (do not calculate) two methods for finding \( \bar{q} \).

**SOLUTION**

\( a \) Since the server refuses to serve any customers unless at least two are in the system, we see that

\[ q_{n+1} = \begin{cases} q_n - 2 + v_{n+1} & q_n \geq 2 \\ v_{n+1} & q_n \leq 1 \end{cases} \]

Introducing the function

\[ \Delta_{2k} \triangleq \begin{cases} 2 & k \geq 2 \\ 0 & 0 \leq k \leq 2 \end{cases} \]

we have

\[ q_{n+1} = q_n - \Delta_{2q_n} + v_{n+1} \]

\( b \) \( V(z) \) is the z-transform for the number of arrivals in an interval, which is the sum of two service times. The transform of the pdf for this interval is clearly \([B^*(z^2)]^2\). By analogy to the development in Problem 5.11, we see that

\[ V(z) = [B^*(\lambda - \lambda z)]^2 \]

**PROBLEM 5.15**

Consider an M/G/1 queuing system with the following variation. The server refuses to serve any customers unless at least two customers are ready for service, at which time both are "taken into" service. These two customers are served individually and independently, one after the other. The instant at which the second of these two is finished is called a "critical" time and we shall use these critical times as the points in an embedded Markov chain. Immediately following a critical time, if there are two
(c) From part (a) we have
\[ Q_{n+1}(z) = E[z^{u_{n+1}}] = E[z^{u_{n}+1+u_{n+1}}] = E[z^{u_{n}}]E[z^{u_{n+1}}] \]
by independence of \( q_n \) and \( u_{n+1} \). Letting \( n \to \infty \), we have
\[ Q(z) = E[z^q] = V(z)E[z^{\Delta Q(z)}] \]
But
\[ E[z^{\Delta Q(z)}] = \sum_{k=0}^{\infty} z^{k} p_{k} \]
\[ = p_{0} + p_{1} + \sum_{k=2}^{\infty} z^{k-1} p_{k} \]
\[ = p_{0} + p_{1} + \frac{1}{z} \{ Q(z) - p_{0} - p_{1}z \} \]
Thus
\[ Q(z) = \frac{V(z)}{z^2} \left[ Q(z) + (z^2 - 1)p_{0} + (z^2 - z)p_{1} \right] \]
\[ Q(z) = \frac{V(z)}{z^2} \frac{(1 - z^2)p_{0} + (z - z^2)p_{1}}{V(z) - z^2} \]
To eliminate \( p_{1} \) we proceed as follows:
\[ 1 = \frac{Q(1)}{V(1)} = \lim_{z \to 1} \frac{1 - z^2)p_{0} + (z - z^2)p_{1}}{V(z) - z^2} \]
Using L'Hospital's rule and \( V'(1) = \bar{v} \), we find
\[ \bar{v} = 2 - 2p_{0} - p_{1} \]
(This could also be obtained from the equation \( \bar{q} = \bar{q} + E[\Delta Q(z)] + \bar{v} \). But \( V(z) = [B/(\lambda - \lambda z)]^2 \), so that
\[ \bar{v} = V'(1) = 2\lambda \bar{x} \]
Thus
\[ 2\lambda \bar{x} = 2 - 2p_{0} - p_{1} \]
and
\[ p_{1} = 2 - 2p_{0} - 2\lambda \bar{x} \]
Substituting this expression for \( p_{1} \) into \( Q(z) \), we have
\[ Q(z) = [B/(\lambda - \lambda z)] ^2 \frac{(1 - z)(1 - p_{0}) + 2z(1 - \lambda \bar{x})}{[B/(\lambda - \lambda z)] ^2 - z^2} \]
(d) Equate roots of the denominator of \( Q(z) \) with that of the numerator for \( |z| < 1 \) using analyticity of \( Q(z) \).
(e) (1) Use the relation \( \bar{q} = Q'(1) \).
(2) Square the equation in part (a), let \( n \to \infty \), and take expectations. \square

PROBLEM 5.16

Consider an M/G/1 queueing system in which service is given as follows. Upon entry into service, a coin is tossed, which has probability \( p \) of giving Heads. If the result is Heads, then the service time for that customer is zero seconds. If Tails, his service time is drawn from the following exponential distribution:
\[ pe^{-px} \quad x \geq 0 \]
(a) Find the average service time \( \bar{x} \).
(b) Find the variance of service time \( \sigma_{x}^2 \).
(c) Find the expected waiting time \( W \).
(d) Find \( W'(x) \).
(e) From (a), find the expected waiting time \( W \).
(f) From (a), find \( W'(x) \).

SOLUTION

The service time density \( b(x) \) is given by
\[ b(x) = p\delta(x) + (1 - p)pe^{-px} \quad x \geq 0 \]
(a) The mean service time is
\[ \bar{x} = 0 \cdot p + \frac{1}{p} \cdot (1 - p) = \frac{1 - p}{p} \]
[Thus, for stability, we require \( p > \lambda/(\lambda + 1) \).]
(b) The second moment of service time is
\[ \bar{x}^2 = 0 \cdot p + \frac{2}{p^2} \cdot (1 - p) = \frac{2(1 - p)}{p^2} \]
Thus
\[ \sigma_{x}^2 = \bar{x}^2 - (\bar{x})^2 = \frac{2(1 - p)}{p^2} - \frac{(1 - p)^2}{p^2} \]
\[ \sigma_{x}^2 = \frac{1 - p^2}{p^2} \]
(c) For $M/G/1, W = \lambda \bar{X}^2 / 2(1 - \rho)$ by Eq. (1.101). Using $\rho = \lambda \mathbb{E} = \lambda(1 - \rho) / \rho$, we have

\[ W = \frac{2(1 - \rho)}{\rho^2} \frac{\lambda^2}{\lambda(1 - \rho)} \]
\[ W = \frac{\rho}{p(1 - \rho)} \]

(d) We first note that

\[ B^*(s) = p + (1 - p) \frac{p}{s + p} = \frac{p(s + 1)}{s + p} \]

Thus, by Eq. (1.105),

\[ W^*(s) = \frac{s(1 - \rho)}{s - \lambda + \lambda p(s + 1)} \frac{s + p}{s + p} \]
\[ W^*(s) = \frac{(1 - \rho)(s + p)}{s + p(1 - \rho)} \]

(e) Differentiating gives

\[ W^{(1)}(s) = (1 - \rho) \left[ \frac{[s + p(1 - \rho)] - (s + p)}{(s + p(1 - \rho))^2} \right] \]
\[ = (1 - \rho) \frac{-p}{(s + p(1 - \rho))^2} \]

Thus

\[ W = -W^{(1)}(0) = \frac{\rho}{p(1 - \rho)} \]

[same as part (c)].

(f) We have

\[ W^*(s) = \frac{(1 - \rho)(s + p)}{s + p(1 - \rho)} \]
\[ = (1 - \rho) + \frac{pp(1 - \rho)}{s + p(1 - \rho)} \]

Inverting we find the pdf as

\[ w(y) = (1 - \rho)\delta(y) + pp(1 - \rho)e^{-\rho(1 - \rho)y} \quad y \geq 0 \]

Thus the PDF is

\[ W(y) = 1 - \rho e^{-\rho(1 - \rho)y} \quad y \geq 0 \]