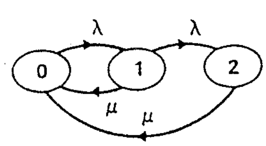
CHAPTER 4

MARKOVIAN QUEUES

PROBLEM 4.1

Consider the Markovian queueing system shown below. Branch labels are birth and death rates. Node labels give the number of customers in the system.



(a) Solve for p_k.
(b) Find the average number in the system.
(c) For $\lambda = \mu$, what values do we get for parts (a) and (b)? Try to interpret these results.
(d) Write down the transition rate matrix Q for this problem and give the matrix equation relating Q to the probabilities found in part (a).

SOLUTION

(a) Using the flow conservation law for states 0 and 2 and the conservation of probability, we get the following three independent equations:

$$\lambda p_0 = \mu p_1 + \mu p_2$$

$$\mu p_2 = \lambda p_1$$

$$p_0 + p_1 + p_2 = 1$$

Solving this gives

$$p_0 = \frac{\mu}{\lambda + \mu}$$

$$p_1 = \frac{\lambda\mu}{(\lambda + \mu)^2}$$

$$p_2 = \frac{\lambda^2}{(\lambda + \mu)^2}$$

(b) We have

$$\overline{N} = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 = \frac{\lambda\mu + 2\lambda^2}{(\lambda + \mu)^2}$$

$$\overline{N} = \frac{\lambda(2\lambda + \mu)}{(\lambda + \mu)^2}$$

(c) If $\lambda = \mu$, the results in parts (a) and (b) become

$$p_0 = \tfrac{1}{2},\ p_1 = p_2 = \tfrac{1}{4},\ \overline{N} = \tfrac{3}{4}$$

To interpret these results, consider a cycle from state 0 back to state 0. The rate out of state 0 is λ ($= \mu$), which puts the system into state 1. The rate out of state 1 is $\lambda + \mu = 2\mu$, and so the fraction of time spent in state 1 must be half that spent in state 0. From state 1 we arrive at state 2 with probability $\frac{1}{2}$ (or return directly to state 0 with probability $\frac{1}{2}$) and depart state 2 at rate μ; therefore we spend as much time, on the average, in state 2, (i.e., $\frac{1}{2} \cdot (1/\mu)$) as in state 1 (i.e., $1/2\mu$).

(d) Equation (1.53) implies that $-q_{ii}$ is the rate at which the system departs from state i, while q_{ij} ($i \neq j$) is the rate at which it moves from state i to state j. Thus

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\mu + \lambda) & \lambda \\ \mu & 0 & -\mu \end{bmatrix}$$

From Eq. (1.56) we have directly that

$$\boldsymbol{\pi}\mathbf{Q} = 0 \qquad (\boldsymbol{\pi} = \mathbf{p} = [p_0, p_1, p_2])$$ □

PROBLEM 4.2

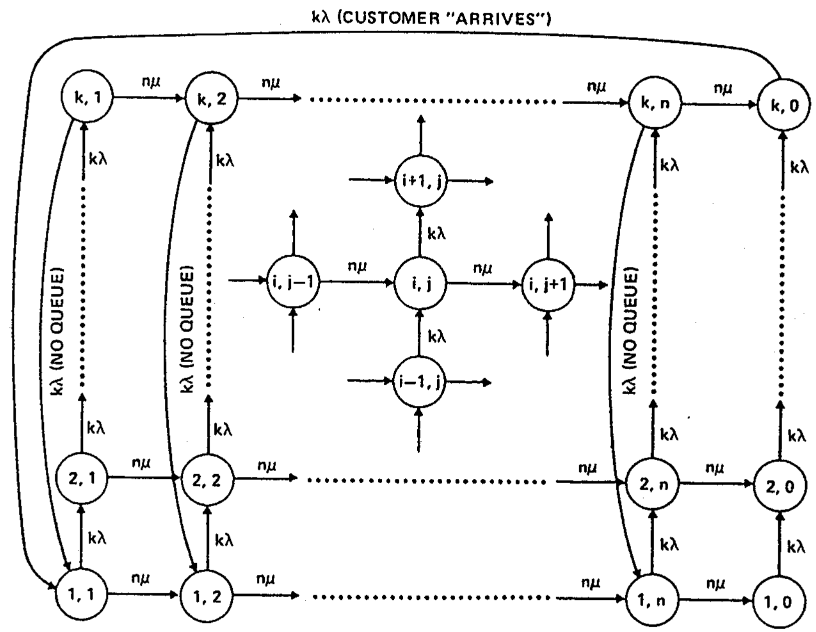
Consider an $E_k/E_n/1$ queueing system where *no* queue is permitted to form. A customer who arrives to find the service facility busy is "lost" (he departs with no service). Let (i, j) be the system state in which the "arriving" customer is in the ith arrival stage and the customer in service is in the jth service stage (note that there is always some customer in the arrival mechanism and that if there is no customer in the service facility, then we let $j = 0$). Let $1/k\lambda$ be the average time spent in any arrival stage and $1/n\mu$ be the average time spent in any service stage.

(a) Draw the state diagram showing all the transition *rates*.

(b) Write down the equilibrium equation for (i, j) where $1 < i < k$, $1 < j \leq n$.

SOLUTION

(a) The state-transition-rate diagram is



(b) Using Flow Out = Flow In, we obtain

$$(k\lambda + n\mu)p_{ij} = k\lambda p_{i-1,j} + n\mu p_{i,j-1} \qquad \text{for } 1 < i < k,\ 1 < j \leq n$$ □

PROBLEM 4.3

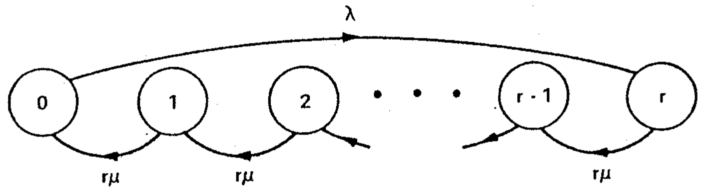
Consider an $M/E_r/1$ system in which *no* queue is allowed to form. Let j be the number of stages of service left in the system and let P_j be the equilibrium probability of being in state (i, j).

(a) Find P_j, $j = 0, 1, \ldots, r$.

(b) Find the probability of a busy system.

SOLUTION

The state-transition-rate diagram is



(a) The flow equations are

$$\lambda P_0 = r\mu P_1 \qquad j = 0$$
$$r\mu P_j = r\mu P_{j+1} \qquad 1 \le j \le r-1$$
$$r\mu P_r = \lambda P_0 \qquad j = r$$

Of these $r+1$ equations, one is redundant; using the first r we see that

$$\frac{\lambda}{r\mu}P_0 = P_1 = P_2 = \cdots = P_{r-1} = P_r$$

Also $\sum_{j=0}^{r} P_j = 1$ implies that

$$P_0 + \sum_{j=1}^{r} \frac{\lambda}{r\mu}P_0 = 1$$

Thus

$$P_0 = \frac{\mu}{\lambda + \mu}$$

and therefore

$$P_j = \frac{\lambda}{r(\lambda+\mu)} \qquad 1 \le j \le r$$

(b) We have

$$P[\text{busy system}] = 1 - P_0 = 1 - \frac{\mu}{\lambda+\mu}$$

$$P[\text{busy system}] = \frac{\lambda}{\lambda+\mu}$$

□

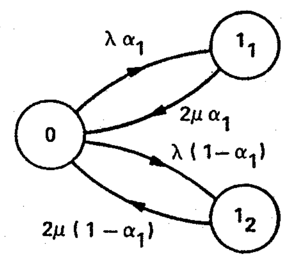
PROBLEM 4.4

Consider an $M/H_2/1$ system in which *no* queue is allowed to form. Service is of the hyperexponential type with $\mu_1 = 2\mu\alpha_1$ and $\mu_2 = 2\mu(1-\alpha_1)$.

(a) Solve for the equilibrium probability of an empty system.
(b) Find the probability that stage 1 is occupied.
(c) Find the probability of a busy system.

SOLUTION

Let 1_i represent the state when there is one customer in the system and that customer is in stage i. The state diagram for this system is as follows:



As usual, we have two independent flow equations and the conservation of probability:

$$\lambda p_0 = 2\mu\alpha_1 p_{1_1} + 2\mu(1-\alpha_1)p_{1_2}$$
$$\lambda\alpha_1 p_0 = 2\mu\alpha_1 p_{1_1}$$
$$p_0 + p_{1_1} + p_{1_2} = 1$$

Thus

$$p_0 = \frac{\mu}{\lambda+\mu}$$

$$p_{1_1} = p_{1_2} = \frac{\lambda}{2(\lambda+\mu)}$$

(a) The probability of an empty system is

$$P[\text{empty system}] = p_0 = \frac{\mu}{\lambda+\mu}$$

(b) The probability that stage 1 is busy is

$$P[\text{stage 1 busy}] = p_{1_1} = \frac{\lambda}{2(\lambda+\mu)}$$

(c) The probability of a busy system is

$$P[\text{busy system}] = 1 - p_0 = p_{1_1} + p_{1_2} = \frac{\lambda}{\lambda + \mu} \qquad \square$$

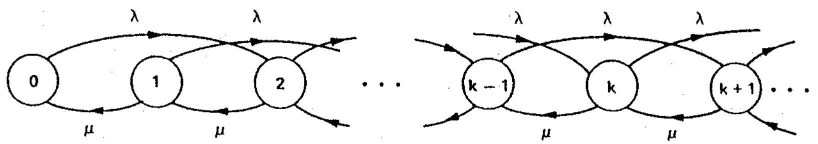
PROBLEM 4.5

Consider an M/M/1 system with parameters λ and μ in which exactly two customers arrive at each arrival instant.

(a) Draw the state-transition-rate diagram.
(b) By inspection, write down the equilibrium equations for p_k $(k = 0, 1, 2, \ldots)$.
(c) Let $\rho = 2\lambda/\mu$. Express $P(z)$ in terms of ρ and z.
(d) Find $P(z)$ by using the bulk arrival result given in Eq. (1.82).
(e) Find the mean and variance of the number of customers in the system from $P(z)$.
(f) Repeat parts (a)–(e) with exactly r customers arriving at each arrival instant (and $\rho = r\lambda/\mu$).

SOLUTION

(a) The state-transition-rate diagram is as follows:



(b) The equilibrium equations are

$$\lambda p_0 = \mu p_1 \qquad k = 0$$

$$(\lambda + \mu)p_1 = \mu p_2 \qquad k = 1$$

$$(\lambda + \mu)p_k = \lambda p_{k-2} + \mu p_{k+1} \qquad k \geq 2$$

(c) Multiply the kth equation by z^k and sum for $k \geq 0$. This gives

$$\lambda \sum_{k=0}^{\infty} p_k z^k + \mu \sum_{k=1}^{\infty} p_k z^k = \lambda \sum_{k=2}^{\infty} p_{k-2} z^k + \mu \sum_{k=0}^{\infty} p_{k+1} z^k$$

$$\lambda P(z) + \mu[P(z) - p_0] = \lambda z^2 P(z) + \frac{\mu}{z}[P(z) - p_0]$$

$$P(z) = \frac{\mu p_0 \left(1 - \frac{1}{z}\right)}{\lambda(1 - z^2) + \mu\left(1 - \frac{1}{z}\right)} = \frac{\mu p_0}{\mu - \lambda z(z + 1)}$$

(Note that the average arrival rate $\overline{\lambda} = 2\lambda$, and so $\rho = \overline{\lambda}\overline{x} = 2\lambda/\mu$.) Thus

$$P(z) = \frac{2p_0}{2 - \rho z(z + 1)}$$

Since $P(1) = 1 = 2p_0/(2 - 2\rho)$ we have $p_0 = 1 - \rho$. Hence

$$P(z) = \frac{2(1 - \rho)}{2 - \rho z(z + 1)}$$

(d) By Eq. (1.82),

$$P(z) = \frac{\mu(1 - \rho)(1 - z)}{\mu(1 - z) - \lambda z[1 - G(z)]}$$

In the system under consideration, bulks have constant size 2. Thus $G(z) = z^2$ (and $\rho = \lambda G^{(1)}(1)/\mu = 2\lambda/\mu$). Therefore

$$P(z) = \frac{\mu(1 - \rho)(1 - z)}{\mu(1 - z) - \lambda z(1 - z^2)}$$

This simplifies as before to

$$P(z) = \frac{2(1 - \rho)}{2 - \rho z(z + 1)}$$

(e) The mean and variance of the number of customers may be found from the first and second derivatives of $P(z)$. We find that

$$\frac{dP(z)}{dz} = \frac{2(1 - \rho)\rho(2z + 1)}{[2 - \rho z(z + 1)]^2}$$

$$\overline{N} = \left.\frac{dP(z)}{dz}\right|_{z=1} = \frac{2(1 - \rho)\rho(3)}{(2 - 2\rho)^2}$$

$$\overline{N} = \frac{3}{2}\frac{\rho}{1 - \rho}$$

After simplification, the second derivative is

$$\frac{d^2P(z)}{dz^2} = 4(1 - \rho)\rho\left[\frac{[2 - \rho z(z + 1)] + \rho(2z + 1)^2}{[2 - \rho z(z + 1)]^3}\right]$$

$$\overline{N^2} - \overline{N} = \left.\frac{d^2P(z)}{dz^2}\right|_{z=1} = 4(1 - \rho)\rho\left[\frac{2 - 2\rho + 9\rho}{(2 - 2\rho)^3}\right]$$

$$= \frac{\rho}{2(1 - \rho)^2}(2 + 7\rho)$$

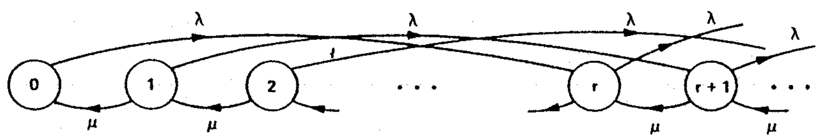
By definition, we may find the variance of N as

$$\sigma_N^2 = \overline{N^2} - (\overline{N})^2 = (\overline{N^2} - \overline{N}) + \overline{N} - (\overline{N})^2$$

$$= \frac{\rho}{2(1-\rho)^2}(2 + 7\rho) + \frac{3}{2}\frac{\rho}{1-\rho} - \frac{9}{4}\frac{\rho^2}{(1-\rho)^2}$$

$$\sigma_N^2 = \frac{\rho(10-\rho)}{4(1-\rho)^2}$$

(f) The state-transition-rate diagram is



The equilibrium equations for p_k are

$$\lambda p_0 = \mu p_1 \qquad k = 0$$

$$(\lambda + \mu)p_k = \mu p_{k+1} \qquad 1 \le k \le r - 1$$

$$(\lambda + \mu)p_k = \lambda p_{k-r} + \mu p_{k+1} \qquad k \ge r$$

Multiply the kth equation by z^k and sum:

$$\lambda \sum_{k=0}^{\infty} p_k z^k + \mu \sum_{k=1}^{\infty} p_k z^k = \lambda \sum_{k=r}^{\infty} p_{k-r} z^k + \mu \sum_{k=0}^{\infty} p_{k+1} z^k$$

$$\lambda P(z) + \mu[P(z) - p_0] = \lambda z^r P(z) + \frac{\mu}{z}[P(z) - p_0]$$

$$P(z) = \frac{\mu p_0(z-1)}{\mu(z-1) - \lambda z(z^r - 1)}$$

$$P(z) = \frac{\mu p_0}{\mu - \lambda z \sum_{k=0}^{r-1} z^k}$$

As $\rho = r\lambda/\mu$ ($\overline{\lambda} = r\lambda$ and so $\rho = \overline{\lambda}\overline{x} = r\lambda/\mu$), we may write

$$P(z) = \frac{r p_0}{r - \rho \sum_{k=1}^{r} z^k}$$

Also $P(1) = 1 = rp_0/(r - r\rho)$ implies that $p_0 = 1 - \rho$. Thus

$$P(z) = \frac{r(1-\rho)}{r - \rho \sum_{k=1}^{r} z^k}$$

To see this in another way, for the bulk arrival system with constant bulk size r, we have $G(z) = z^r$. Substituting this into Eq. (1.82) and simplifying gives

as before

$$P(z) = \frac{r(1-\rho)}{r - \rho \sum_{k=1}^{r} z^k}$$

To find $\overline{N}$ we note that

$$\frac{dP(z)}{dz} = r(1-\rho)\rho \frac{\sum_{k=1}^{r} k z^{k-1}}{\left(r - \rho \sum_{k=1}^{r} z^k\right)^2}$$

so that

$$\overline{N} = \left.\frac{dP(z)}{dz}\right|_{z=1} = r(1-\rho)\rho \frac{r(r+1)/2}{(r - r\rho)^2}$$

Thus

$$\overline{N} = \frac{r+1}{2}\frac{\rho}{1-\rho}$$

To find σ_N^2 we first obtain

$$\overline{N^2} - \overline{N} = \left.\frac{d^2P(z)}{dz^2}\right|_{z=1} = r(1-\rho)\rho\left[\frac{(r - r\rho)\sum_{k=1}^{r} k(k-1) + 2\rho\left(\sum_{k=1}^{r} k\right)^2}{(r - r\rho)^3}\right]$$

Now recall that

$$\sum_{k=1}^{r} k = \frac{r(r+1)}{2} \quad \text{and} \quad \sum_{k=1}^{r} k^2 = \frac{r(r+1)(2r+1)}{6}$$

Therefore

$$\sum_{k=1}^{r} k(k-1) = \frac{(r-1)r(r+1)}{3}$$

and

$$\overline{N^2} - \overline{N} = r(1-\rho)\rho\left[\frac{r(1-\rho)\frac{(r-1)r(r+1)}{3} + 2\rho\left(\frac{r(r+1)}{2}\right)^2}{[r(1-\rho)]^3}\right]$$

$$= \frac{(r+1)\rho}{6(1-\rho)^2}(2r - 2 + \rho r + 5\rho)$$

and so

$$\sigma_N^2 = (\overline{N^2} - \overline{N}) + \overline{N} - (\overline{N})^2$$

$$= \frac{(r+1)\rho}{6(1-\rho)^2}(2r - 2 + \rho r + 5\rho) + \frac{(r+1)\rho}{2(1-\rho)} - \frac{(r+1)^2\rho^2}{4(1-\rho)^2}$$

$$\sigma_N^2 = \frac{(r+1)\rho}{12(1-\rho)^2}(4r + 2 - \rho r + \rho) \qquad \square$$

(b) We first simplify the expression obtained in part (a), and then find the limit as $t \to \infty$.

$$P[N(t)=k] = \sum_{n=k}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\binom{n}{k}\left[\frac{1}{t}\int_0^t [1-B(x)]\,dx\right]^k \left[\frac{1}{t}\int_0^t B(x)\,dx\right]^{n-k}$$

$$= e^{-\lambda t}\sum_{n=k}^{\infty}\frac{\left[\lambda\int_0^t[1-B(x)]\,dx\right]^k}{k!}\cdot\frac{\left[\lambda\int_0^t B(x)\,dx\right]^{n-k}}{(n-k)!}$$

$$= e^{-\lambda t}\frac{\left[\lambda\int_0^t[1-B(x)]\,dx\right]^k}{k!}\sum_{n=0}^{\infty}\frac{\left[\lambda\int_0^t B(x)\,dx\right]^n}{n!}$$

$$P[N(t)=k] = e^{-\lambda t}\frac{\left[\lambda\int_0^t[1-B(x)]\,dx\right]^k}{k!}e^{\lambda\int_0^t B(x)\,dx}$$

$$= e^{-\lambda\int_0^t[1-B(x)]\,dx}\frac{\left[\lambda\int_0^t[1-B(x)]\,dx\right]^k}{k!}$$

Thus, *for every* t, $N(t)$ is Poisson with parameter $\lambda\int_0^t[1-B(x)]\,dx$. Letting $t\to\infty$ and noting that

$$\lim_{t\to\infty}\int_0^t[1-B(x)]\,dx = \int_0^\infty[1-B(x)]\,dx = \overline{x}$$

we see immediately that

$$p_k \triangleq \lim_{t\to\infty}P_k(t) = e^{-\lambda\overline{x}}\frac{(\lambda\overline{x})^k}{k!}$$

Thus as $t\to\infty$, the limiting distribution of number in system is Poisson with parameter $\lambda\overline{x}$, which is independent (except for the mean) of $B(x)$. □

PROBLEM 5.9

Consider M/E$_2$/1.

(a) Find the polynomial for $G^*(s)$.
(b) Solve for $S(y) = P[\text{time in system} \le y]$.

SOLUTION

(a) For the M/E$_2$/1 system, the Laplace transform of the service time density is

$$B^*(s) = \left(\frac{2\mu}{s+2\mu}\right)^2$$

Thus Eq. (1.111) gives

$$G^*(s) = \left[\frac{2\mu}{s+\lambda-\lambda G^*(s)+2\mu}\right]^2$$

Expanding, we get

$$\lambda^2[G^*(s)]^3 - 2\lambda(s+\lambda+2\mu)[G^*(s)]^2 + (s+\lambda+2\mu)^2G^*(s) - 4\mu^2 = 0$$

(b) Equation (1.106) gives

$$S^*(s) = B^*(s)\frac{s(1-\rho)}{s-\lambda+\lambda B^*(s)}$$

Thus

$$S^*(s) = \left(\frac{2\mu}{s+2\mu}\right)^2\frac{s(1-\rho)}{s-\lambda+\lambda\left(\frac{2\mu}{s+2\mu}\right)^2}$$

$$= \frac{4\mu^2(1-\rho)}{s^2+(4\mu-\lambda)s+4\mu(\mu-\lambda)}$$

The denominator $s^2+(4\mu-\lambda)s+4\mu(\mu-\lambda)$ has roots s_1, s_2 (where $\rho=\lambda/\mu$):

$$s_1 = \frac{-\mu(4-\rho)+\mu\sqrt{\rho^2+8\rho}}{2}$$

$$s_2 = \frac{-\mu(4-\rho)-\mu\sqrt{\rho^2+8\rho}}{2}$$

We note that, for $\rho<1$, we have $16\rho<16$ and thus $(4-\rho)^2>\rho^2+8\rho$. Hence $s_2<s_1<0$ for $0<\rho<1$. Factoring,

$$S^*(s) = \frac{4\mu^2(1-\rho)}{(s-s_1)(s-s_2)}$$

$$= \frac{4\mu^2(1-\rho)}{\mu\sqrt{\rho^2+8\rho}}\left(\frac{1}{s-s_1}-\frac{1}{s-s_2}\right)$$

Invert to find the pdf $s(y)$ as

$$s(y) = \frac{4\mu(1-\rho)}{\sqrt{\rho^2+8\rho}}\left(e^{s_1y}-e^{s_2y}\right)$$

Thus the PDF $S(y)$ is

$$S(y) = \frac{4\mu(1-\rho)}{\sqrt{\rho^2+8\rho}}\left[\frac{1}{s_1}\left(e^{s_1y}-1\right)-\frac{1}{s_2}\left(e^{s_2y}-1\right)\right]$$ □

PROBLEM 5.10

Consider an M/D/1 system for which $\bar{x} = 2$ sec.

(a) Show that the residual service time pdf $\hat{b}(x)$ is a rectangular distribution.
(b) For $\rho = 0.25$, show that the result of Eq. (1.108) with four terms may be used as a good approximation to the distribution of queueing time.

SOLUTION

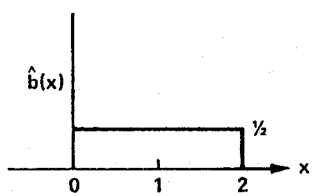
(a) The service time distribution is given by

$$B(x) = \begin{cases} 0 & x < 2 \\ 1 & x \geq 2 \end{cases}$$

The residual service time pdf is

$$\hat{b}(x) = \frac{1 - B(x)}{\bar{x}} = \begin{cases} \frac{1}{2} & x < 2 \\ 0 & x \geq 2 \end{cases}$$

Thus $\hat{b}(x)$ is rectangular.



(b) The first four terms of the series in Eq. (1.108) give

$$w(y) \cong w_{\text{approx}}(y) \triangleq (1-\rho)\left[u_0(y) + \rho\hat{b}(y) + \rho^2\hat{b}_{(2)}(y) + \rho^3\hat{b}_{(3)}(y)\right]$$

As

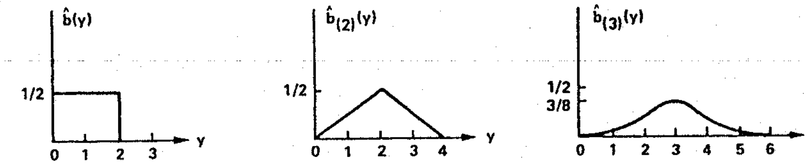
$$\hat{b}(y) = \begin{cases} \frac{1}{2} & y < 2 \\ 0 & y \geq 2 \end{cases}$$

we see that

$$\hat{b}_{(2)}(y) = \begin{cases} \dfrac{y}{4} & 0 \leq y \leq 2 \\ 1 - \dfrac{y}{4} & 2 \leq y \leq 4 \end{cases}$$

and

$$\hat{b}_{(3)}(y) = \begin{cases} \dfrac{y^2}{16} & 0 \leq y \leq 2 \\ \dfrac{-y^2}{8} + \dfrac{3}{4}y - \dfrac{3}{4} & 2 \leq y \leq 4 \\ \dfrac{y^2}{16} - \dfrac{3}{4}y + \dfrac{9}{4} & 4 \leq y \leq 6 \end{cases}$$



We compare $w(y)$ and $w_{\text{approx}}(y)$ in three different ways.

First, the area A under the curve $w(y)$ minus the area A_{approx} under the curve $w_{\text{approx}}(y)$ is

$$A - A_{\text{approx}} = (1-\rho)\sum_{k=4}^{\infty}\rho^k \int_0^{\infty} \hat{b}_{(k)}(y)\,dy = (1-\rho)\sum_{k=4}^{\infty}\rho^k$$

$$= (1-\rho)\rho^4\left(\frac{1}{1-\rho}\right) = \rho^4$$

As $\rho = \frac{1}{4}$,

$$A - A_{\text{approx}} = \frac{1}{256}$$

Thus, in terms of area, we have a "good" approximation.

Second, we note that $w_{\text{approx}}(y) = 0$ for $y \geq 6$. Thus the tail of the density $w(y)$ is *not* approximated very well.

Third, we compare the mean wait W with an approximation W_{approx} calculated from $w_{\text{approx}}(y)$. [Note that $w_{\text{approx}}(y)$ is not a pdf.]

$$W = \int_0^{\infty} yw(y)\,dy = (1-\rho)\sum_{k=1}^{\infty}\rho^k \int_0^{\infty} y\hat{b}_{(k)}(y)\,dy$$

We now observe that $\int_0^{\infty} y\hat{b}_{(k)}(y)\,dy$ has value k, since it represents the mean of a sum of k random variables each having mean 1. Thus

$$W = (1-\rho)\sum_{k=1}^{\infty} k\rho^k = (1-\rho)\rho\frac{\partial}{\partial\rho}\left(\frac{1}{1-\rho}\right)$$

or

$$W = \frac{\rho}{1-\rho}$$

For $\rho = \frac{1}{4}$,

$$W = \tfrac{1}{3}$$

Now

$$W_{\text{approx}} = \int_0^{\infty} y w_{\text{approx}}(y)\,dy = (1-\rho)\sum_{k=1}^{3}\rho^k \int_0^{\infty} y\hat{b}_{(k)}(y)\,dy$$

$$= (1-\rho)\sum_{k=1}^{3} k\rho^k = (1-\rho)(\rho + 2\rho^2 + 3\rho^3)$$

For $\rho = \frac{1}{4}$,

$$W_{\text{approx}} = \tfrac{3}{4}\left(\tfrac{1}{4} + \tfrac{1}{8} + \tfrac{3}{64}\right) = \tfrac{3}{4}\cdot\tfrac{27}{64} = \tfrac{81}{256}$$

or

$$W_{\text{approx}} = 0.31640625$$

Thus

$$\frac{W - W_{\text{approx}}}{W} = \frac{\frac{1}{3} - \left(\frac{3}{4}\right)^4}{\frac{1}{3}} \cong 0.0508$$

and so W_{approx} is within 5% of the mean W. □