1. Let $f_n = n 2^n + 4^n \quad n = 0, 1, 2, ...$

Find $F(z) = \sum_{n=0}^{\infty} f_n z^n$

2. Let $F(z) = \frac{2z-1}{-3z^2 + 4z - 1}$

Find $f_n \Leftrightarrow F(z) \quad n = 0, 1, 2, ...$

3. Given $G(z) = F(z) + z^2 F(z)$

where $F(z)$ is as given as in Problem 2.

Find $g_n \Leftrightarrow G(z) \quad n = 0, 1, 2, ...$

4. Let $f(t) = u_0(t) + e^{-t} + e^{2t} \quad t \geq 0$

Find $F^\ast(s) = \int_0^\infty f(t) e^{-st} dt$

5. Let $F^\ast(s) = \frac{5(s+1)}{(s+2)(s+3)}$

Find $F(t) \Leftrightarrow F^\ast(s) \quad t \geq 0$

6. Let $10f_n - 7f_{n-1} + f_{n-2} = \left(\frac{4}{3}\right)^n \quad n = 2, 3, 4, ...$

and $f_0 = 0, \quad f_1 = 3$

a) Find $f_n \quad n = 0, 1, 2, ...$ without using transforms

b) Find $f_n \quad n = 0, 1, 2, ...$ using transforms
Let \( \frac{d^2 f(t)}{dt^2} + 6 \frac{df(t)}{dt} + 8f(t) = 1 \) \( t \geq 0 \)

a) Find \( f(t) \), \( t \geq 0 \), without using Laplace transform.

b) Find \( f(t) \), \( t \geq 0 \), using Laplace transform.
1. \( f_n = 2^n + 4^n \quad n = 0, 1, 2, 3, \ldots \)  
Find \( f(3) \).

\[
F(3) = \frac{2^0 + 2^2}{n(n+2)} = \frac{4}{9} \left( \frac{8}{8} + \frac{4}{4} \right) = \frac{12}{9} = \frac{4}{3}
\]

2. \( F(3) = \frac{\frac{7}{3} - 1}{1 - \frac{1}{3}} \)  
Find \( f_n \)  
\( n = 0, 1, 2, 3, \ldots \)

\[
F(3) = \frac{\frac{7}{3} - 1}{1 - \frac{1}{3}} = \frac{2}{3} = \frac{9}{2}
\]

3. \( G(3) = F(3) + 2^2 F(2) \)

\[
G(3) = \frac{9}{2} = \frac{1}{2} \left( 1 + 3^n \right) 2^n = \frac{9}{2} \left( 1 + 3^n \right) 2^n
\]

\( \Rightarrow \)

\[
\begin{aligned}
g_0 &= f_0 = 1 \\
g_1 &= f_1 = 2 \\
g_n &= f_n + f_{n-2} = \frac{3^n}{2} + \frac{3^{n-2}}{2} + 1
\end{aligned}
\]

\( \text{or } g_n = \frac{1}{2} \left( 3^{n-2} + n + \frac{3}{2} \right) \)

4. \( f(t) = u(t) + \frac{2}{2} e^t + e^{2t} \quad t \geq 0 \)  
Find \( F^*(s) \)

\[
F^*(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty \left( u(t) + \frac{2}{2} e^t + e^{2t} \right) e^{-st} dt
\]

\[
= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s-2}
\]

5. \( F^*(s) = \frac{5(s+1)}{(s+2)(s+3)} \)  
Find \( f(t) \)

\[
F^*(s) = \frac{5(s+1)}{(s+2)(s+3)} = 1 - \frac{4s+6}{(s+2)(s+3)} = 1 + \frac{2}{s+2} - \frac{6}{s+3}
\]

\[
f(t) = u(t) + \frac{2}{2} e^{-2t} \delta(t) - 6 e^{-3t} \delta(t)
\]
10 \ f_n - 7 f_{n-1} + f_{n-3} = \left( \frac{1}{2} \right)^n \quad n = 3, 4, 5, \ldots \quad \text{and} \quad f_0 = 0, \ f_1 = 3

(c) Find \ f_n \ \text{for} \ n = 0, 1, 2, \ldots \ \text{without using} \ \mathcal{Z}\text{-transform}

1. Find the homogeneous equation.

The characteristic equation is

\[ 10 \lambda^2 - 7 \lambda + 1 = 0 \]

\[ \lambda = \frac{1}{2}, \ \frac{1}{5} \]

\[ f_n^{(h)} = C_1 \left( \frac{1}{5} \right)^n + C_2 \left( \frac{1}{2} \right)^n \]

where \( C_1 \) and \( C_2 \) are constants.

2. Find the particular solution:

Guess \( f_n^{(p)} = C \left( \frac{1}{2} \right)^n \) and plug into \( 10 f_n - 7 f_{n-1} + f_{n-3} = \left( \frac{1}{2} \right)^n \).

\[ C \left( \frac{1}{2} \right)^0 - 7 C \left( \frac{1}{2} \right)^{-1} + C \left( \frac{1}{2} \right)^{-3} = \left( \frac{1}{2} \right)^n \]

\[ C = -\frac{1}{2} \]

\[ f_n^{(p)} = -\frac{1}{2} \left( \frac{1}{2} \right)^n \]

3. \( f_n = f_n^{(h)} + f_n^{(p)} = C_1 \left( \frac{1}{5} \right)^n + C_2 \left( \frac{1}{2} \right)^n - \frac{1}{2} \left( \frac{1}{2} \right)^n \)

with initial conditions \( f_0 = 0 \) and \( f_1 = 3 \)

\[ \begin{align*}
  f_0 &= C_1 + C_2 - \frac{1}{2} = 0 \\
  f_1 &= C_1 \frac{1}{5} + C_2 \frac{1}{2} - \frac{1}{2} = 3 \\
\end{align*} \]

\[ C_1 = \frac{175}{18}, \quad C_2 = \frac{92}{9} \]

\[ \text{The final solution is} \quad f_n = \frac{92}{9} \left( \frac{1}{2} \right)^n - \frac{175}{18} \left( \frac{1}{5} \right)^n - \frac{1}{2} \left( \frac{1}{2} \right)^n \]

(b) Find \( f_n \) for \( n = 0, 1, 2, \ldots \) using \( \mathcal{Z}\)-transform

Let

\[ F(z) = \sum_{n=0}^{\infty} f_n z^n \]

\[ \sum_{n=2}^{\infty} \left( 10 f_n - 7 f_{n-1} + f_{n-3} \right) z^n = \sum_{n=2}^{\infty} \left( \frac{1}{2} \right)^n z^n \]

\[ 10 (F(z) - f_0 - f_1 z) - 7 z (F(z) - f_0) + 3 z^2 F(z) = \frac{2}{9} \left( \frac{1}{2} \right)^z \quad \text{for} \quad z = 0 \text{ and } 0 < z \]

\[ F(z) (10 - 7 z + z^2) = 30 z + \frac{2}{9} \left( \frac{1}{2} \right)^z = \frac{92}{9} \left( \frac{1}{5} \right)^z - \frac{175}{9} \left( \frac{1}{2} \right)^z - \frac{1}{2} \left( \frac{1}{2} \right)^z \]

\[ f_n = \frac{92}{9} \left( \frac{1}{2} \right)^n - \frac{175}{18} \left( \frac{1}{5} \right)^n - \frac{1}{2} \left( \frac{1}{2} \right)^n \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \]
\( \frac{d^2 f(t)}{dt^2} + 6 \frac{df(t)}{dt} + 8 f(t) = 1 \quad t \geq 0 \)

(a) \( \text{Find } f(t) \text{ t.o. without using Laplace transform} \)

1. \( \text{Find the homogeneous equation} \)
   \[ \lambda^2 + 6 \lambda + 8 = 0 \]
   \[ \lambda = -2, -4 \]
   \[ f_h(t) = C_1 e^{-2t} + C_2 e^{-4t} \quad \text{where } C_1 \text{ and } C_2 \text{ are constants} \]

2. \( \text{Find the particular equation} \)
   \[ f_p(t) = \frac{1}{8} \]

3. \( f(t) = f_h(t) + f_p(t) = C_1 e^{-2t} + C_2 e^{-4t} + \frac{1}{8} \quad \text{for } t \geq 0 \)

(b) \( \text{Find } f(t) \text{ t.o. using Laplace transform} \)

Let \( F(s) = \int_0^\infty f(t) e^{-st} dt \)

\[ \int_0^\infty \left[ \frac{d^2 f(t)}{dt^2} + 6 \frac{df(t)}{dt} + 8 f(t) \right] e^{-st} dt = \int_0^\infty e^{-st} dt \]

\[ \frac{d^2 f(t)}{dt^2} \Rightarrow s^2 F(s) - sf(0^-) - f'(0^-) \]

\[ s^2 F(s) - sf(0^-) - f'(0^-) + 6 sF(s) - 6f(0^-) + 8 F(s) = \frac{1}{s} \]

\[ F(s) \left[ s^2 + 6s + 8 \right] = \frac{1}{s} + \left[ 6f(0^-) + f''(0^-) \right] + s f(0^-) \]

\[ F(s) = \frac{1 + 3 \left( 6f(0^-) + f''(0^-) \right) + s^2 f(0^-)}{s \left( s^2 + 6s + 8 \right)} \]

\[ = \frac{1}{5} + \frac{C_1}{s + 2} + \frac{C_2}{s + 4} \]

where \( C_1 = F(s), f(0^-), f''(0^-) \) and \( C_2 = F(s), f(0^-), f''(0^-) \)

the all constants

\[ \Rightarrow f(t) = C_1 e^{-2t} + C_2 e^{-4t} + \frac{1}{8} \]