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Naïve Learning in Social Networks and the Wisdom of Crowds[†]

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We study learning in a setting where agents receive independent noisy signals about the true value of a variable and then communicate in a network. They naïvely update beliefs by repeatedly taking weighted averages of neighbors' opinions. We show that all opinions in a large society converge to the truth if and only if the influence of the most influential agent vanishes as the society grows. We also identify obstructions to this, including prominent groups, and provide structural conditions on the network ensuring efficient learning. Whether agents converge to the truth is unrelated to how quickly consensus is approached. (*JEL* D83, D85, Z13)

Social networks are primary conduits of information, opinions, and behaviors. They carry news about products, jobs, and various social programs; influence decisions to become educated, to smoke, and to commit crimes; and drive political opinions and attitudes toward other groups. In view of this, it is important to understand how beliefs and behaviors evolve over time, how this depends on the network structure, and whether or not the resulting outcomes are efficient. In this paper, we examine one aspect of this broad theme: for which social network structures will a society of agents who communicate and update naïvely come to aggregate decentralized information completely and correctly?

Given the complex forms that social networks often take, it can be difficult for the agents involved (or even for a modeler with full knowledge of the network) to update beliefs properly. For example, Syngjoo Choi, Douglas Gale, and Shachar Kariv (2005, 2008) find that although subjects in simple three-person networks update fairly well in some circumstances, they do not do so well in evaluating repeated observations and judging indirect information for which the origin is uncertain. Given that social communication often involves repeated transfers of information among large numbers of individuals in complex networks, fully rational learning becomes infeasible. Nonetheless, it is possible that agents using fairly simple updating rules will arrive at outcomes like those achieved through fully rational learning. We identify social networks for which naïve individuals converge to fully rational

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beliefs despite using simple and decentralized updating rules and we also identify social networks for which beliefs fail to converge to the rational limit under the same updating.

We base our study on an important model of network influence largely due to Morris H. DeGroot (1974). The social structure of a society is described by a weighted and possibly directed network. Agents have beliefs about some common question of interest—for instance, the probability of some event. At each date, agents communicate with their neighbors in the social network and update their beliefs. The updating process is simple. An agent's new belief is the (weighted) average of his or her neighbors' beliefs from the previous period. Over time, provided the network is strongly connected (so there is a directed path from any agent to any other) and satisfies a weak aperiodicity condition, beliefs converge to a consensus. This is easy to understand. At least one agent with the lowest belief must have a neighbor who has a higher belief, and similarly, some agent with the highest belief must have a neighbor with a lower belief. So, distance between highest and lowest beliefs decays over time.

We focus on situations where there is some true state of nature that agents are trying to learn and each agent's initial belief is equal to the true state of nature plus some idiosyncratic zero-mean noise. An outside observer who could aggregate all of the decentralized initial beliefs could develop an estimate of the true state that would be arbitrarily accurate in a large enough society. Agents using the DeGroot rule will converge to a consensus estimate. Our question is: for which social networks will agents using the simple and naïve updating process all converge to an accurate estimate of the true state?

The repeated updating model we use is simple, tractable, and captures some of the basic aspects of social learning, so it is unsurprising that it has a long history. Its roots go back to sociological measures of centrality and prestige that were introduced by Leo Katz (1953) and further developed by Phillip Bonacich (1987). There are precursors, reincarnations, and cousins of the framework discussed by John R. P. French, Jr. (1956); Frank Harary (1959); Noah E. Friedkin and Eugene C. Johnsen (1997); and Peter M. DeMarzo, Dimitri Vayanos, and Jeffrey Zwiebel (2003), among others. In the DeGroot (1974) version of the model that we study, agents update their beliefs or attitudes in each period simply by taking weighted averages of their neighbors' opinions from the previous period, possibly placing some weight on their own previous beliefs. The agents in this scenario are boundedly rational, failing to adjust correctly for repetitions and dependencies in information that they hear multiple times.[1] While this model captures the fact that agents repeatedly communicate with each other and incorporate indirect information in a boundedly rational way, it is rigid in that agents do not adjust the weights they place on others' opinions over time. Nonetheless, it is a useful and tractable first approximation that serves as a benchmark. In fact, the main results of the paper show that even this rigid and naïve

[1] For more discussion and background on the form of the updating, there are several sources. For Bayesian foundations under some normality assumptions, see DeGroot (1974, 416–17). Behavioral explanations are discussed in Friedkin and Johnsen (1997) and DeMarzo, Vayanos, and Zwiebel (2003). For additional results from other versions of the model, see Jackson (2008).

process can lead agents to converge jointly to fully accurate beliefs in the limit as society grows large in a variety of social networks. Moreover, the limiting properties of this process are useful not only for understanding belief evolution, but also as a basis for analyzing the influence or power of the different individuals in a network.[2]

Our contributions are outlined as follows, in the order in which they appear in the paper.

Section I introduces the model, discusses the updating rule, and establishes some definitions. Then, to lay the groundwork for our study of convergence to true beliefs, we briefly review issues of convergence itself in Section II. Specifically, for strongly connected networks, we state the necessary and sufficient condition for all agents' beliefs to converge as opposed to oscillating indefinitely; the condition is based on the well-known characterization of Markov chain convergence. When beliefs do converge, they converge to a consensus. In Section A of the Appendix, we provide a full characterization of convergence even for networks that are not strongly connected, based on straightforward extensions of known results from linear algebra.

When convergence obtains, the consensus belief is a weighted average of agents' initial beliefs and the weights provide a measure of social influence or importance. Those weights are given by a principal eigenvector of the social network matrix. This is what makes the DeGroot model so tractable, and we take advantage of this known feature to trace how influential different agents are as a function of the structure of the social network. This leads us to the novel theoretical results of the paper. In Sections III and IV, we ask for which social networks will a large society of naïve DeGroot updaters converge to beliefs such that all agents learn the true state of nature, assuming that they all start with independent (but not necessarily identically distributed), noisy signals about the state. For example, if all agents listen to just one particular agent, then their beliefs converge, but they converge to that agent's initial information, and thus the beliefs are not accurate, in the sense that they have a substantial probability of deviating substantially from the truth. In contrast, if all agents place equal weight on all agents in their communication, then clearly they immediately converge to an average of all of the signals in the society, and then, by a law of large numbers, agents in large societies all hold beliefs close to the true value of the variable. We call networked societies that converge to this accurate limit "wise." The question is what happens for large societies that are more complex than those two extremes.

Our main results begin with a simple but complete characterization of wisdom in terms of influence weights in Section III. A society is wise if and only if the influence of the most influential agent is vanishing as the society grows. Building on this characterization, we focus on the relationship between social structure and wisdom in Section IV. First, in a setting where all ties are reciprocal and agents pay equal attention to all their neighbors, wisdom can fail if and only if there is an agent whose degree (number of neighbors) is a nonvanishing fraction of the total number of links in the network, no matter how large the network grows. Thus, in this setting, disproportionate popularity is the sole obstacle to wisdom. Moving to more general results,

[2] The model can also be applied to study a myopic best-response dynamic of a game in which agents care about matching the behavior of those in their social network (possibly placing some weight on themselves).

we show that having a bounded number of agents who are *prominent* (receiving a nonvanishing amount of possibly indirect attention from everyone in the network) causes learning to fail, since their influence on the limiting beliefs is excessive. This result is a fairly direct elaboration of the characterization of wisdom given above, but it is stated in terms of the geometry of the network as opposed to the influence weights. Next, we provide examples of types of network patterns that prevent a society from being wise. One is a lack of balance, where some groups get much more attention than they give out, and the other is a lack of dispersion, where small groups do not pay sufficient attention to the rest of the world. Based on these examples, we formulate structural conditions that are sufficient for wisdom. The sufficient conditions formally capture the intuition that societies with balance and dispersion in their communication structures will have accurate learning.

In Section V, we discuss some of what is known about the speed and dynamics of the updating process studied here. Understanding the relationship between communication structures and the persistence of disagreement is independently interesting, and also sheds light on when steady-state analysis is relevant. We note that the speed of convergence is not related to wisdom.

The proofs of all results appear in Section B of the Appendix; some additional results, along with their proofs, appear in Sections A and C of the Appendix.

Our work relates to several lines of research other than the ones already discussed. There is a large theoretical literature on social learning, both fully and boundedly rational. Herding models (e.g., Abhijit V. Banerjee 1992; Sushil Bikhchandani, David Hirshleifer, and Ivo Welch 1992; Glenn Ellison and Drew Fudenberg 1993, 1995; Gale and Kariv 2003, Boğaçhan Çelen and Kariv 2004; and Banerjee and Fudenberg 2004) are prime examples, and there agents converge to holding the same belief or at least the same judgment as to an optimal action. These conclusions generally apply to observational learning, where agents are observing choices and/or payoffs over time and updating accordingly.[3] In such models, the structure determining which agents observe which others when making decisions is typically constrained, and the learning results do not depend sensitively on the precise structure of the social network. Our results are quite different from these. In contrast to the observational learning models, convergence and the efficiency of learning in our model depend critically on the details of the network architecture and on the influences of various agents.

The work of Venkatesh Bala and Sanjeev Goyal (1998) is closer to the spirit of our work, as they allow for richer network structures. Their approach is different from ours in that they examine observational learning where agents take repeated actions and can observe each other's payoffs. There, consensus within connected components generally obtains because all agents can observe whether their neighbors are earning payoffs different from their own.[4] They also examine the question of whether agents might converge to taking the wrong actions, which is a sort of wisdom question,

[3] For a general version of the observational learning approach, see Dinah Rosenberg, Eilon Solan, and Nicolas Vieille (2009).

[4] Bala and Goyal (2001) shows that heterogeneity in preferences in the society can cause similar individuals to converge to different actions if they are not connected.

and the answer depends on whether some agents are too influential, which has some similar intuition to the prominence results that we find in the DeGroot model. Bala and Goyal also provide sufficient conditions for convergence to the correct action. Roughly speaking, these require some agent to be arbitrarily confident in each action, so that each action gets chosen enough to reveal its value; and the existence of paths of agents observing each such agent, so that the information diffuses. While the questions are similar, the analysis and conclusions are quite different in two important ways. First, the pure communication we study is different from observational learning, and changes the sorts of conditions that are needed for wisdom. Second, the DeGroot model allows for precise calculations of the influence of every agent in any network, which is not seen in the observational learning literature. The second point is obvious, so let us explain the first aspect of the difference, which is especially useful to discuss since it highlights fundamental differences between issues of learning through repeated observation and actions, and updating via repeated communication. In the observational learning setting, if some agent is sufficiently stubborn in pursuing a given action, then, through repeated observation of that action's payoffs, the agent's neighbors learn that action's value if it is superior. That leads them to take the action, and then their neighbors learn, and so forth. Thus, to be arbitrarily sure of converging to the best action, all that is needed is for each action to have a player who has a prior that places sufficiently high weight on that action so that its payoff will be sufficiently accurately assessed. And, if it turns out to be the highest payoff action, it will eventually diffuse throughout the component regardless of network structure. In contrast, in the updating setting of the DeGroot model, every agent starts with just one noisy signal, and the question is how that decentralized information is aggregated through repeated communication. Generally, we do not require any agent to have an arbitrarily accurate signal, nor would this circumstance be sufficient for wisdom except for some very specific network structures. In this repeated communication setting, signals can quickly become mixed with other signals, and the network structure is critical to determining what the ultimate mixing of signals is. So, the models, basic structure, and conclusions are quite different between the two settings even though there are some superficial similarities.

Closer in terms of the formulation, but less so in terms of the questions asked, is the study by DeMarzo, Vayanos, and Zwiebel (2003), which focuses mainly on a network-based explanation for the "unidimensionality" of political opinions. Nevertheless, they do present some results on the correctness of learning. Our results on sufficient conditions for wisdom may be compared with their Theorem 2, where they conclude that consensus beliefs (for a fixed population of n agents) optimally aggregate information if and only if a knife-edge restriction on the weights holds. Our results show that under much less restrictive conditions, aggregation can be asymptotically accurate even if it is not optimal in finite societies. More generally, our conclusions differ from a long line of previous work which suggests that sufficient conditions for naïve learning are hopelessly strong.[5] We show that beliefs can be correct in the large-society limit for a fairly broad collection of networks.

[5] See Joel Sobel (2000) for a survey.

The most recent work on this subject of which we are aware is a paper (following the first version of this paper) by Daron Acemoglu et al. (2008), which is in the rational observational learning paradigm but relates to our work in terms of the questions asked and the spirit of the main results. The paper both complements and contrasts with ours. In that model, each agent makes a decision once in a predetermined order and observes previous agents' decisions according to a random process for which the distribution is common knowledge. The main result of the paper is that if agents have priors that allow signals to be arbitrarily informative, then the absence of agents who are excessively influential is enough to guarantee convergence to the correct action. The definition of excessive influence is demanding. To be excessively influential, a group must be finite and must provide *all* of the information to an infinite group of other agents. Conversely, an excessively influential group, in this sense, destroys social learning. The structure of the model is quite different from ours. The agents of Acemoglu et al. (2008) take one action as opposed to updating constantly, and the learning there is observational. Nevertheless, these results are interesting to compare with our main theorems. As we mentioned, prominent groups can also destroy learning in our model, and ruling them out is a first step in guaranteeing wisdom. However, our notion of prominence is different from and, intuitively speaking, not as strong as the notion of excessive influence. To be prominent, in our setting, a group must only get some attention from everyone, as opposed to providing all the information to a very large group. Thus, our agents are more easily misled, and the errors that can happen depend more sensitively on the details of the network structure. This is natural. Since they are more naïve, social structure matters more in determining the outcome. We view the approaches of Acemoglu et al. (2008) and our work as being quite complementary in the sense that some of these differences are driven by differences in agents' rationality. However, there are also more basic differences between the models in terms of what information represents, as well as the repetition, timing, and patterns of communication.

In addition, there are literatures in physics and computer science on the DeGroot model, and variations on it.[6] There, the focus has generally been on consensus rather than on wisdom. In sociology, since the work of Katz (1953), French (1956), and Bonacich (1987), eigenvector-like notions of centrality and prestige have been analyzed.[7] As some such models are based on convergence of iterated influence relationships, our results provide insight into the structure of the influence vectors in those models, especially in the large-society limit. Finally, there is an enormous empirical literature about the influence of social acquaintances on behavior and outcomes that we will not attempt to survey here,[8] but simply point out that our model provides testable predictions about the relationships between social structure and social learning.

[6] See Jackson (2008, Section 8.3) for an overview and more references.

[7] See, also, Stanley Wasserman and Katherine Faust (1994), Bonacich and Paulette Lloyd (2001), and Jackson (2008) for more recent elaborations.

[8] The *Handbook of Social Economics* (Jess Benhabib, Alberto Bisin, and Jackson (forthcoming) provides overviews of various aspects of this.

I. The DeGroot Model

A. *Agents and Interaction*

A finite set $N = \{1, 2, \ldots, n\}$ of *agents* or *nodes* interact according to a social network. The interaction patterns are captured through an $n \times n$ nonnegative matrix $\mathbf{T}$, where $T_{ij} > 0$ indicates that i pays attention to j. The matrix $\mathbf{T}$ may be asymmetric, and the interactions can be one-sided, so that $T_{ij} > 0$ while $T_{ji} = 0$. We refer to $\mathbf{T}$ as the *interaction matrix*. This matrix is stochastic, so that its entries across each row are normalized to sum to one.

B. *Updating*

Agents update beliefs by repeatedly taking weighted averages of their neighbors' beliefs with T_{ij} being the weight or trust that agent i places on the current belief of agent j in forming his or her belief for the next period. In particular, each agent has a belief $p_i^{(t)} \in \mathbb{R}$ at time $t \in \{0, 1, 2, \ldots\}$. For convenience, we take $p_i^{(t)}$ to lie in $[0, 1]$, although it could lie in a multidimensional Euclidean space without affecting the results below. The vector of beliefs at time t is written $\mathbf{p}^{(t)}$. The updating rule is

$$\mathbf{p}^{(t)} = \mathbf{T}\mathbf{p}^{(t-1)},$$

and so

$$\mathbf{p}^{(t)} = \mathbf{T}^t\mathbf{p}^{(0)}. \tag{1}$$

The evolution of beliefs can be motivated by the following Bayesian setup discussed by DeMarzo, Vayanos, and Zwiebel (2003). At time $t = 0$, each agent receives a noisy signal $p_i^{(0)} = \mu + e_i$, where $e_i \in \mathbb{R}$ is a noise term with expectation zero and μ is some state of nature. Agent i hears the opinions of the agents with whom he interacts, and assigns precision π_{ij} to agent j. These subjective estimates may, but need not, coincide with the true precisions of their signals. If agent i does not listen to agent j, then agent i gives j precision $\pi_{ij} = 0$. In the case where the signals are normal, Bayesian updating from independent signals at $t = 1$ entails the rule (1) with $T_{ij} = \pi_{ij} / \sum_{k=1}^{n} \pi_{ik}$. As agents may only be able to communicate directly with a subset of agents due to some exogenous constraints or costs, they will generally wish to continue to communicate and update based on their neighbors' evolving beliefs, since that allows them to incorporate information from those whom they do not observe directly. The key behavioral assumption is that the agents continue using the same updating rule throughout the evolution. That is, they do not account for the possible repetition of information and for the "cross-contamination" of their neighbors' signals. This bounded rationality arising from persuasion bias is discussed at length by DeMarzo, Vayanos, and Zwiebel (2003), and so we do not reiterate that discussion here.

It is important to note that other applications also have the same form as that analyzed here. What we refer to as "beliefs" could also be some behavior that people adjust in response to their neighbors' behaviors, either through some desire to match behaviors or through other social pressures favoring conformity. As another example, Google's "PageRank" system is based on a measure related to the influence vectors derived below, where the **T** matrix is the normalized link matrix.[9] Other citation and influence measures also have similar eigenvector foundations (e.g., see Ignacio Palacios-Huerta and Oscar Volij 2004). Finally, we also see iterated interaction matrices in studies of recursive utility (e.g., Brian W. Rogers 2006) and in strategic games played by agents on networks where influence measures turn out to be important (e.g., Coralio Ballester, Antoni Calvó-Armengol, and Yves Zenou 2006). In such applications, understanding the properties of $\mathbf{T}^t$ and related matrices is critical.

C. *Walks, Paths, and Cycles*

The following are standard graph-theoretic definitions applied to the directed graph of connections induced by the interaction matrix **T**.

A *walk* in **T** is a sequence of nodes $i_1, i_2, \ldots, i_K$, not necessarily distinct, such that $T_{i_k i_{k+1}} > 0$ for each $k \in \{1, \ldots, K-1\}$. The length of the walk is defined to be $K-1$. A *path* in **T** is a walk consisting of distinct nodes.

A *cycle* is a walk $i_1, i_2, \ldots, i_K$ such that $i_1 = i_K$. The *length* of a cycle with K (not necessarily distinct) entries is defined to be $K-1$. A cycle is *simple* if the only node appearing twice in the sequence is the starting (and ending) node.

The matrix **T** is *strongly connected* if there is path in **T** from any node to any other node. Similarly, we say that $B \subset N$ is strongly connected if **T** restricted to B is strongly connected. This is true if and only if the nodes in B all lie on a cycle that involves only nodes in B. If **T** is undirected in the sense that $T_{ij} > 0$, if and only if $T_{ji} > 0$, then we simply say the matrix is *connected*.

II. Convergence of Beliefs Under Naïve Updating

We begin with the question of when the beliefs of all agents in a network converge to well-defined limits as opposed to oscillating forever. Without such convergence, it is clear that wisdom could not be obtained.

DEFINITION 1: *A matrix* **T** *is* convergent *if* $\lim_{t\to\infty} \mathbf{T}^t \mathbf{p}$ *exists for all vectors* $\mathbf{p} \in [0,1]^n$.

This definition of convergence requires that beliefs converge for *all* initial vectors of beliefs. Clearly, any network will have convergence for some initial vectors, since, if we start all agents with the same beliefs, then no nontrivial updating will ever occur. It turns out that if convergence fails for some initial vector, then there will be cycles or oscillations in the updating of beliefs and convergence will fail for whole classes of initial vectors.

A condition ensuring convergence in strongly connected stochastic matrices is aperiodicity.

[9] So, $T_{ij} = 1/\ell_i$ if page i has a link to page j, where ℓ_i is the number of links that page i has to other pages. From this basic form **T** is perturbed for technical reasons. See Amy N. Langville and Carl D. Meyer (2006) for details.

DEFINITION 2: *The matrix* $\mathbf{T}$ *is* aperiodic *if the greatest common divisor of the lengths of its simple cycles is* 1.

A. *Examples*

The following very simple and standard example illustrates a failure of aperiodicity.

EXAMPLE 1:

$$\mathbf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly,

$$\mathbf{T}^t = \begin{cases} \mathbf{T} & \textit{if } t \textit{ is odd} \\ \mathbf{I} & \textit{if } t \textit{ is even.} \end{cases}$$

In particular, if $p_1(0) \neq p_2(0)$, *then the belief vector never reaches a steady state, and the two agents keep switching beliefs.*

Here, each agent ignores his own current belief in updating. Requiring at least one agent to weight his current belief ensures convergence. This is a special case of Proposition 1. However, it is not necessary to have $T_{ii} > 0$ for even a single i in order to ensure convergence.

EXAMPLE 2: *Consider,*

$$\mathbf{T} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here,

$$\mathbf{T}^t \rightarrow \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{pmatrix}.$$

Even though $\mathbf{T}$ *has only* 0 *along its diagonal, it is aperiodic and converges. If we change the matrix to*

$$\mathbf{T} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then $\mathbf{T}$ *is periodic as all of its cycles are of even lengths and* $\mathbf{T}$ *is no longer convergent.*

B. *A Characterization of Convergence and Limiting Beliefs*

It is well-known that aperiodicity is necessary and sufficient for convergence in the case where $\mathbf{T}$ is strongly connected (John G. Kemeny and J. Laurie Snell 1960). We summarize this in the following statement.

PROPOSITION 1: *If* $\mathbf{T}$ *is a strongly connected matrix, the following are equivalent:*

(i) $\mathbf{T}$ *is convergent.*

(ii) $\mathbf{T}$ *is aperiodic.*

(iii) *There is a unique left eigenvector* $\mathbf{s}$ *of* $\mathbf{T}$ *corresponding to eigenvalue* 1 *whose entries sum to* 1 *such that, for every* $\mathbf{p} \in [0,1]^n$,

$$\left(\lim_{t\to\infty} \mathbf{T}^t \mathbf{p}\right)_i = \mathbf{sp}$$

for every i.

In addition to characterizing convergence, this fact also establishes what beliefs converge to when they do converge. The limiting beliefs are all equal to a weighted average of initial beliefs, with agent i's weight being s_i. We refer to s_i as the *influence weight* or the *influence* of agent i.

To see why there is an eigenvector involved, let us suppose that we would like to find a vector $\mathbf{s} = (s_1, \ldots, s_n) \in [0,1]^n$ which would measure how much each agent influences the limiting belief. In particular, let us look for a nonnegative vector, normalized so that its entries sum to 1, such that for any vector of initial beliefs $\mathbf{p} \in [0,1]^n$, we have

$$\left(\lim_{t\to\infty} \mathbf{T}^t \mathbf{p}\right)_j = \sum_i s_i p_i(0).$$

Noting that $\lim_{t\to\infty} \mathbf{T}^t \mathbf{p} = \lim_{t\to\infty} \mathbf{T}^t(\mathbf{Tp})$, it must be that

$$\mathbf{sp} = \mathbf{sTp},$$

for *every* $\mathbf{p} \in [0,1]^n$. This implies that $\mathbf{s} = \mathbf{sT}$, and so $\mathbf{s}$ is simply a unit (left-hand or row) eigenvector of $\mathbf{T}$, provided that such an $\mathbf{s}$ can be found.

The eigenvector property, of course, is just saying that $s_i = \sum_{j\in N} T_{ji} s_j$ for all i, so that the influence of i is a weighted sum of the influences of various agents j who pay attention to i, with the influence s_j weighted by T_{ji}, which is the trust that j places on i. This is a very natural property for a measure of influence to have and entails that influential people are those who are trusted by other influential people.

As mentioned in the introduction, the result can be generalized to situations without strong connectedness, which are relevant for many applications. This is discussed in Section A of the Appendix. Much of the structure previously discussed remains in that case, with some modifications, but some aspects of the characterization, such as the equality of everyone's limiting beliefs, do not hold in general settings.

C. *Undirected Networks with Equal Weights*

A particularly tractable special case of the model arises when $\mathbf{T}$ is derived from having each agent equally split attention among his or her neighbors in an undirected network. Suppose that we start with a symmetric, connected adjacency matrix $\mathbf{G}$ of an undirected network, where $G_{ij} = 1$ indicates that i and j have an undirected link between them, and $G_{ij} = 0$ otherwise. Let $d_i(\mathbf{G}) = \sum_{j=1}^{n} G_{ij}$ be the degree, or number of neighbors, of agent i. Then, if we define $\mathbf{T}(\mathbf{G})$ by $T_{ij} = G_{ij}/d_i(\mathbf{G})$, we obtain a stochastic matrix. The interpretation is that $\mathbf{G}$ gives a social network of undirected connections, and everyone puts equal weight on all his neighbors in that network.[10] It is impossible, in this setting, for i to pay attention to j and not vice versa, and it is not possible for someone to pay different amounts of attention to different sources that he or she listens to. Thus, this setting places some real restrictions on the structure of the interaction matrix, but, in return, yields a very intuitive characterization of influence weights. Indeed, as pointed out in DeMarzo, Vayanos, and Zwiebel (2003), the vector $\mathbf{s}$ has a simple structure:

$$s_i = \frac{d_i(\mathbf{G})}{\sum_{i=1}^{n} d_i(\mathbf{G})},$$

as can be verified by a direct calculation, using Proposition 1 (iii). Thus, in this special case, influence is directly proportional to degree.

III. The Wisdom of Crowds: Definition and Characterization

With the preliminaries out of the way, we now turn to the central question of the paper. Under what circumstances does the decentralized DeGroot process of communication correctly aggregate the diverse information initially held by the different agents? In particular, we are interested in large societies. The large-society limit is relevant in many applications of the theory of social learning. Moreover, a large number of agents is necessary for there to be enough diversity of opinion for a society, even in the best case, to be able to wash out idiosyncratic errors and discover the truth.

To capture the idea of a "large" society, we examine sequences of networks in which we let the number of agents n grow and work with limiting statements. In discussing wisdom, we are taking a double limit. First, for any fixed network, we ask what its beliefs converge to in the long run. Next, we study limits of these long-run beliefs as the networks grow. The second limit is taken across a sequence of networks.

[10] In Markov chain language, $\mathbf{T}(\mathbf{G})$ corresponds to a symmetric random walk on an undirected graph, and the Markov chain is reversible (Persi Diaconis and Daniel Stroock 1991).

The sequence of networks is captured by a sequence of n-by-n interaction matrices. We say that a *society* is a sequence $(\mathbf{T}(n))_{n=1}^{\infty}$ indexed by n, the number of agents in each network. We will denote the (i,j) entry of interaction matrix n by $\mathbf{T}_{ij}(n)$, and, more generally, all scalars, vectors, and matrices associated to network n will be indicated by an argument n in parentheses.

Throughout this section and the next, we maintain the assumption that each network is convergent for each n. It does not make sense to talk about wisdom if the networks do not even have convergent beliefs, and so convergence is an a priori necessary condition for wisdom.[11] Let us now specify the underlying probability space and give a formal definition of a wise society.

A. *Defining Wisdom*

There is a true state of nature $\mu \in [0,1]$.[12] We do not need to specify anything regarding the distribution from which this true state is drawn; we treat the truth as fixed. If it is actually the realization of some random process, then all of the analysis is conditional on its realization.

At time $t = 0$, agent i in network n sees a signal $p_i^{(0)}(n)$ that lies in a bounded set, normalized without loss of generality to be in $[0,1]$. The signal is distributed with mean μ and a variance of at least $\sigma^2 > 0$, and the signals $p_1^{(0)}(n), \dots, p_n^{(0)}(n)$ are independent for each n. No further assumptions are made about the joint distribution of the variables $p_i^{(0)}(n)$ as n and i range over their possible values. The common lower bound on variance ensures that convergence to truth is not occurring simply because there are arbitrarily well informed agents in the society.[13]

Let $\mathbf{s}(n)$ be the influence vector corresponding to $\mathbf{T}(n)$, as defined in Proposition 1 (or, more generally, Theorem 3). We write the belief of agent i in network n at time t as $p_i^{(t)}(n)$.

For any given n and realization of $\mathbf{p}^{(0)}(n)$, the belief of each agent i in network n approaches a limit which we denote by $p_i^{(\infty)}(n)$. The limits are characterized in Proposition 1 (or, more generally, Theorem 3). Each of these limiting beliefs is a random variable that depends on the initial signals. We say the sequence of networks is *wise* when the limiting beliefs converge jointly in probability to the true state μ.

DEFINITION 3: *The sequence* $(\mathbf{T}(n))_{n=1}^{\infty}$ *is* wise *if,*

$$\operatorname*{plim}_{n\to\infty} \max_{i\le n} |p_i^{(\infty)}(n) - \mu| = 0.$$

While this definition is given with a specific distribution of signals in the background, it follows from Proposition 2 that a sequence of networks will be wise for all

[11] We do not, however, require strong connectedness. All the results go through for general convergent networks. Thus, some of the proofs use results in Section A of the Appendix.

[12] This is easily extended to allow the true state to lie in any finite-dimensional Euclidean space, as long as the signals that agents observe have a bounded support.

[13] The lower bound on variance is only needed for one part of one result, which is the "only if" statement in Lemma 1. Otherwise, one can dispose of this assumption.

such distributions or for none. Thus, the specifics of the distribution are irrelevant for determining whether a society is wise, provided the signals are independent, have mean μ, and have variances bounded away from zero. If these conditions are satisfied, the network structure alone determines wisdom.

B. *Wisdom in Terms of Influence: A Law of Large Numbers*

To investigate the question of which societies are wise, we first state a simple law of large numbers that is helpful in our setting, as we are working with weighted averages of potentially nonidentically distributed random variables. The following result will be used to completely characterize wisdom in terms of influence weights.

Without loss of generality, label the agents so that $s_i(n) \geq s_{i+1}(n) \geq 0$ for each i and n. That is, the agents are arranged by influence in decreasing order.

LEMMA 1: [*A Law of Large Numbers*] *If* $(\mathbf{s}(n))_{n=1}^{\infty}$ *is any sequence of influence vectors, then*

$$\operatorname*{plim}_{n\to\infty} \mathbf{s}(n)\mathbf{p}^{(0)}(n) = \mu$$

if and only if $s_1(n) \to 0$.[14]

Thus, in strongly connected networks, the limiting belief of all agents,

$$p^{(\infty)}(n) = \sum_{i\leq n} s_i(n)\, p_i^{(0)}(n),$$

will converge to the truth as $n \to \infty$ if and only if the most important agent's influence tends to zero (recall that we labeled agents so that $s_1(n)$ is maximal among the $s_i(n)$). With slightly more careful analysis, it can be shown that the same result holds whether or not the networks are strongly connected, which is the content of the following proposition.

PROPOSITION 2: *If* $(\mathbf{T}(n))_{n=1}^{\infty}$ *is a sequence of convergent stochastic matrices, then it is wise if and only if the associated influence vectors are such that* $s_1(n) \to 0$.

This result is natural in view of the examples in Section IVC below, which show that a society can be led astray if the leader has too much influence. Indeed, the proofs of both results follow a very simple intuition. For the idiosyncratic errors to wash out and for the limiting beliefs—which are weighted averages of initial beliefs—to converge to the truth, nobody's idiosyncratic error should be getting positive weight in the large-society limit.

[14] Since $\sum_{i\leq n} s_i(n)p_i^{(0)}(n)$ is bounded due to our assumption that $p_i^{(0)}(n) \in [0,1]$ for each n and i, the statement $\operatorname{plim}_{n\to\infty}\mathbf{s}(n)\mathbf{p}^{(0)}(n) = \mu$ is equivalent to having $\operatorname{plim}_{n\to\infty}(|\mathbf{s}(n)\mathbf{p}^{(0)}(n) - \mu|^r) = 0$ for all $r > 0$.

IV. Wisdom in Terms of Social Structure

The characterization in Section III is still abstract in that it applies to influence vectors and not directly to the structure of the social network. It is interesting to see how wisdom is determined by the geometry of the network, which structures prevent wisdom, and which ones ensure it. That is the focus of this section.

We begin with a simple characterization in the special case of undirected networks with equal weights discussed in Section IIC. After that, we state a general necessary condition for wisdom—the absence of prominent groups that receive attention from everyone in society. However, simple examples show that when wisdom fails, it is not always possible to identify an obvious prominent group. Ensuring wisdom is thus fairly subtle. Some sufficient conditions are given in Section IVD.

A. *Wisdom in Undirected Networks with Equal Weights*

A particularly simple characterization is obtained in the setting of Section IIC, where agents weight their neighbors equally and communication is reciprocal. It is stated in the following corollary of Proposition 2.

COROLLARY 1: *Let* $(\mathbf{G}(n))_{n=1}^{\infty}$ *be a sequence of symmetric, connected adjacency matrices. The sequence* $(\mathbf{T}(\mathbf{G}(n)))_{n=1}^{\infty}$ *is wise if and only if*

$$\max_{1 \le i \le n} \frac{d_i(\mathbf{G}(n))}{\sum_{i=1}^{n} d_i(\mathbf{G}(n))} \xrightarrow{n} 0.$$

That is, a necessary and sufficient condition for wisdom in this setting is that the maximum degree becomes vanishingly small relative to the sum of degrees. In other words, disproportionate popularity of some agent is the only obstacle to wisdom.

While this characterization is very intuitive, it also depends on the special structure of reciprocal attention and equal weights, as the examples in Section IVC show.

B. *Prominent Families as an Obstacle to Wisdom*

We now discuss a general obstacle to wisdom in arbitrary networks, namely, the existence of prominent groups that receive a disproportionate share of attention and lead society astray. This is reminiscent of the discussion in Bala and Goyal (1998) of what can go wrong when there is a commonly observed "royal family" under a different model of observational learning. However, as noted in the introduction, the way in which this works, and the implications for wisdom, are quite different.[15]

[15] The similarity is that in both observational learning and in the repeated updating discussed here, having all agents concentrate their attention on a few agents can lead to societal errors if those few are in error. The difference is in the way that this is avoided. In the observational learning setting, the sufficient condition for complete learning of Bala and Goyal (1998) is for each action to be associated with some very optimistic agent, and then to have every other agent have a path to every action's corresponding optimistic agent. Thus, the payoff to every action will be correctly figured out by its optimistic agent, and then society will eventually see which is the best of those actions. The only property of the network that is needed for this conclusion is connectedness. In our context, the analogue of this condition would be to have some agent, who observes the true state of nature with

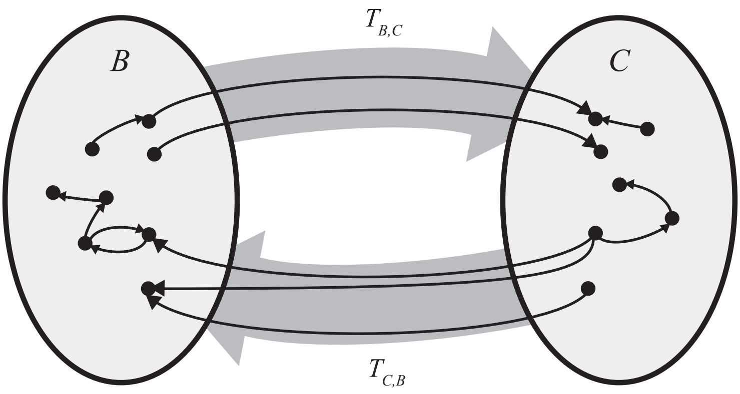

Figure 1

Note: The large arrows illustrate the concept of the weight of one group on another.

To introduce this concept, we need some definitions and notation. It is often useful to consider the weight of groups on other groups. To this end, we define

$$T_{B,C} = \sum_{\substack{i \in B \\ j \in C}} T_{ij}$$

which is the weight that group B places on group C. The concept is illustrated in Figure 1.

Returning to the setting of a fixed network of n agents for a moment, we begin by making a natural definition of what it means for a group to be observed by everyone.

DEFINITION 4: *The group B is* prominent in t steps *relative to* $\mathbf{T}$ *if* $(\mathbf{T}^t)_{i,B} > 0$ *for each* $i \notin B$.

Call $\pi_B(\mathbf{T};t) := \min_{i \notin B}(\mathbf{T}^t)_{i,B}$ *the* t-step prominence *of B relative to* $\mathbf{T}$.

Thus, a group that is prominent in t steps is one such that each agent outside of it is influenced by at least someone in that group in t steps of updating. Note that the way in which the weight is distributed among the agents in the prominent group is left arbitrary, and some agents in the prominent group may be ignored altogether. If $t = 1$, then everyone outside the prominent group is paying attention to somebody in the prominent group directly, i.e., not through someone else in several rounds of updating.

This definition is given relative to a single matrix $\mathbf{T}$. While this is useful in deriving explicit bounds on influence (see Section B of the Appendix), we also define

very high accuracy, and then does not weight anyone else's opinion. However, in keeping with our theme of starting with noisy information, we are instead interested in when the network structure correctly aggregates many noisy signals, none of which is accurate or persistent. Thus, our results do depend critically on network structure.

a notion of prominence in the asymptotic setting. First, we define a *family* to be a sequence of groups (B_n) such that $B_n \subset \{1, \ldots, n\}$ for each n. A family should be thought of as a collection of agents that may be changing and growing as we expand the society. In applications, the families could be agents of a certain type, but a priori there is no restriction on the agents in the group B_n. Now, we can extend the notion of prominence to families.

DEFINITION 5: *The family* (B_n) *is* uniformly prominent *relative to* $(\mathbf{T}(n))_{n=1}^{\infty}$ *if there exists a constant* $\alpha > 0$ *such that for each n there is a t so that the group* B_n *is prominent in t steps relative to* $\mathbf{T}(n)$ *with* $\pi_{B_n}(\mathbf{T}(n); t) \geq \alpha$.

For the family (B_n) to be uniformly prominent, we must have that for each n, the group B_n is prominent relative to $\mathbf{T}(n)$ in some number of steps without the prominence growing too small (hence, the word "uniformly"). Note that at least one uniformly prominent family always exists, namely $\{1, \ldots, n\}$.

We also define a notion of finiteness for families. A family is finite if it stops growing eventually.

DEFINITION 6: *The family* (B_n) *is* finite *if there is a q such that* $\sup_n |B_n| \leq q$.

With these definitions in hand, we can state a first necessary condition for wisdom in terms of prominence—wisdom rules out finite, uniformly prominent families. This result, and the other facts in this section, rely on bounds on various influences, as shown in Section B of the Appendix.

PROPOSITION 3: *If there is a finite, uniformly prominent family with respect to* $(\mathbf{T}(n))$, *then the sequence is not wise.*

To see the intuition behind this result, consider a special but illuminating example. Let (B_n) be a finite, uniformly prominent family so that, in the definition of uniform prominence, $t = 1$ for each n—that is, the family is always prominent in one step. Further, consider the strongly connected case, with agent i in network n getting weight $s_i(n)$. Normalize the true state of the world to be $\mu = 0$, and, for the purposes of exposition, suppose that everyone in B_n starts with belief 1, and that everyone outside starts with belief 0. Let α be a lower bound on the prominence of B_n. Then after one round of updating, everyone outside B_n has belief at least α. So, for a large society, the vast majority of agents have beliefs that differ by at least α from the truth. The only way they could conceivably be led back to the truth is if, after one round of updating, at least some agents in B_n have beliefs equal to zero and can lead society back to the truth. Now we may forget what happened in the past and just view the current beliefs as new starting beliefs. If the agents in B_n have enough influence to lead everyone back to zero forever when the other agents are α away from it, then they also have enough influence to lead everyone away from zero forever at the very start. So, at best, they can only lead the group part of the way back. Thus, we conclude that starting B_n with incorrect beliefs and everyone else with correct beliefs can lead the entire network to incorrect beliefs.

C. *Other Obstructions to Wisdom: Examples*

While prominence is a simple and important obstruction to wisdom, not all examples where wisdom fails have a group that is prominent in a few steps. The following example illustrates Proposition 3 and demonstrates its limitations.

EXAMPLE 3: *Consider the following network, defined for arbitrary n. Fix* $\delta, \varepsilon \in (0,1)$ *and define, for each* $n \geq 1$*, an n-by-n interaction matrix*

$$\mathbf{T}(n) := \begin{bmatrix} 1-\delta & \frac{\delta}{n-1} & \frac{\delta}{n-1} & \cdots & \frac{\delta}{n-1} \\ 1-\varepsilon & \varepsilon & 0 & \cdots & 0 \\ 1-\varepsilon & & \varepsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-\varepsilon & 0 & 0 & \cdots & \varepsilon \end{bmatrix}.$$

The network is shown in Figure 2 for $n = 6$ *agents.*

We find that

$$s_i(n) = \begin{cases} \dfrac{1-\varepsilon}{1-\varepsilon+\delta} & \text{if } i = 1 \\ \dfrac{\delta}{(n-1)(1-\varepsilon+\delta)} & \text{if } i > 1 \end{cases}.$$

This network will not converge to the truth. Observe that in society n, the limiting belief of each agent is $s_1(n)p_1^{(0)}(n)$ *plus some other independent random variables that have mean* μ*. As* $s_1(n)$ *is constant and independent of n, the variance of the limiting belief remains bounded away from zero for all n. So beliefs will deviate from the truth by a substantial amount with positive probability. The intuition is simply that the leader's information, even when it is far from the mean, is observed by everyone and weighted heavily enough that it biases the final belief, and the followers' signals cannot do much to correct it. Indeed, Proposition 2 above establishes the lack of wisdom due to the nonvanishing influence of the central agent. If* δ *and* ε *are fixed constants, then the central agent (due to his or her position) is prominent in one step, making this an illustration of Proposition 3.*

However, note that even if we let $1 - \varepsilon$ *approach 0 at any rate we like, so that people are not weighting the center very much, the center has nonvanishing influence as long as* $1 - \varepsilon$ *is of at least the order*[16] *of* δ*. Thus, it is not simply the total weight on a given individual that matters, but the relative weights coming in and out of particular nodes (and groups of nodes). In particular, if the weight on the center decays (so that nobody is prominent in one step), wisdom may still fail.*

[16] Formally, suppose we have a sequence $\varepsilon(n)$ and $\delta(n)$ with $(1 - \varepsilon(n))/\delta(n) \geq c > 0$ for all n.

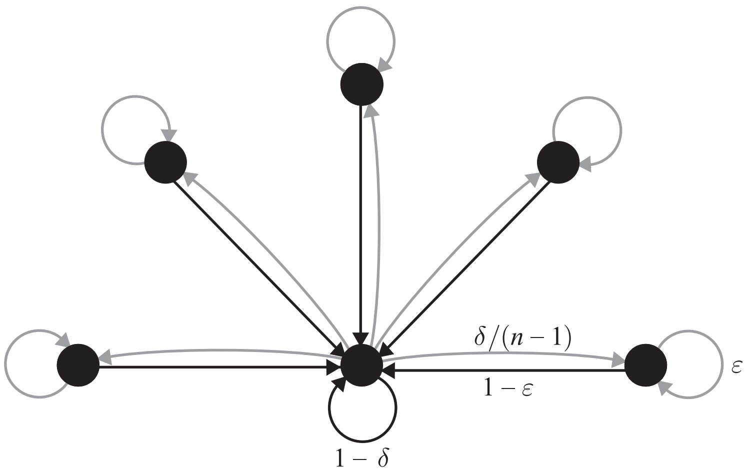

FIGURE 2

Note: The unbalanced star network (shown here for $n = 6$ agents) is an example demonstrating that the limiting belief is not always accurate.

On the other hand, if $1 - \varepsilon$ *becomes small relative to* δ *as society grows, then we can obtain wisdom despite the seemingly unbalanced social structure. This demonstrates that the result of Section IVA is sensitive to the assumption that agents must place equal amounts of weight on each of their neighbors including themselves.*

One thing that goes wrong in this example is that the central agent receives a high amount of trust relative to the amount given back to others, making him or her unduly influential. However, this is not the only obstruction to wisdom. There are examples in which the weight coming into any node is bounded relative to the weight going out, and there is still an extremely influential agent who can keep society's beliefs away from the truth. The next example shows how indirect weight can matter.

EXAMPLE 4: *Fix* $\delta \in (0, 1/2)$ *and define, for each* $n \geq 1$, *an* n*-by-*n *interaction matrix by*

$$
\begin{aligned}
T_{11}(n) &= 1 - \delta && \\
T_{i,i-1}(n) &= 1 - \delta && \text{if } i \in \{2, \ldots, n\} \\
T_{i,i+1}(n) &= \delta && \text{if } i \in \{1, \ldots, n-1\} \\
T_{nn}(n) &= \delta && \\
T_{ij}(n) &= 0 && \text{otherwise.}
\end{aligned}
$$

The network is shown in Figure 3.

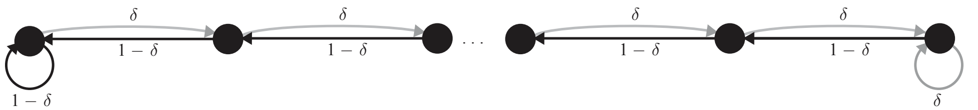

FIGURE 3

Notes: The unbalanced line demonstrates that the beliefs in a society may not converge to truth even if the ratio of incoming to outgoing weight is bounded for each agent. Agents are numbered from left to right.

It is simple to verify that

$$s_i(n) = \left(\frac{\delta}{1-\delta}\right)^{i-1} \cdot \frac{1-\left(\frac{\delta}{1-\delta}\right)}{1-\left(\frac{\delta}{1-\delta}\right)^n}.$$

In particular, $\lim_{n\to\infty} s_1(n)$ *can be made as close to* 1 *as desired by choosing a small* δ*, and then Proposition* 2 *shows that wisdom does not obtain. The reason for the leader's undue influence here is somewhat more subtle than in Example* 3. *It is not the weight agent* 1 *directly receives, but indirect weight due to this agent's privileged position in the network. Thus, while agent* 1 *is not prominent in any number of steps less than* $n-1$*, the agent's influence can exceed the sum of all other influences by a huge factor for small* δ*. This shows that it can be misleading to measure agents' influence based on direct incoming weight or even indirect weight at a few levels. Instead, the entire structure of the network is relevant.*

D. *Ensuring Wisdom: Structural Sufficient Conditions*

We now provide structural sufficient conditions for a society to be wise. The examples of Section IVC make it clear that wisdom is, in general, a subtle property. Thus, formulating the sufficient conditions requires defining some new concepts, which can be used to rule out obstructions to wisdom.

Property 1 (**Balance**)**:** There exists a sequence $j(n) \to \infty$ such that if $|B_n| \le j(n)$ then

$$\sup_n \frac{T_{B_n^c, B_n}(n)}{T_{B_n, B_n^c}(n)} < \infty.$$

The balance condition says that no family below a certain size limit captured by $j(n)$ can be getting infinitely more weight from the remaining agents than it gives to the remaining agents. The sequence $j(n) \to \infty$ can grow very slowly, which makes the condition reasonably weak.

Balance rules out, among other things, the obstruction to wisdom identified by Proposition 3, since a finite prominent family will be receiving an infinite amount of weight but can only give finitely much back (since it is finite). The condition also rules out situations like Example 3, in which there is a single agent who gets much more weight than he or she gives out.

The basic intuition of the condition is that in order to ensure wisdom, one not only has to worry about single agents getting infinitely more weight than they give out, but also about finite groups being in this position. And one needs not only to rule out this problem for groups of some given finite size, but for any finite size. This accounts for the sequence $j(n)$ tending to infinity in the definition. The sequence could grow arbitrarily slowly, but it must eventually get large enough to catch any particular finite size. This is a tight condition in the sense that if one, instead, requires $j(n)$ to be below some finite bound for all n, then one can always find an example that satisfies the condition and yet does not exhibit wisdom.

We know, from Example 4, that it is not enough simply to rule out situations where there is infinitely more direct weight into some family of agents than out. One also has to worry about large-scale asymmetries of a different sort, which can be viewed as small groups focusing their attention too narrowly. The next condition deals with this.

Property 2 (**Minimal Out-Dispersion**)**:** There is a $q \in \mathbb{N}$ and $r > 0$ such that if B_n is finite, $|B_n| \geq q$, and $|C_n|/n \to 1$, then $T_{B_n,C_n}(n) > r$ for all large enough n.

The minimal out-dispersion condition requires that any large enough finite family must give at least some minimal weight to any family which makes up almost all of society. This rules out situations like Example 4, in which there are agents who ignore the vast majority of society. Thus, this ensures that no large group's attention is narrowly focused.

Having stated these two conditions, we can give the main result of this section, which states that the conditions are sufficient for wisdom.

THEOREM 1: *If* $(\mathbf{T}(n))_{n=1}^{\infty}$ *is a sequence of convergent stochastic matrices satisfying balance and minimal out-dispersion, then it is wise.*

Note, however, that neither condition is sufficient on its own. Example 4 satisfies the first property but not the second. The square of the matrix in Example 3 satisfies the second property but not the first. In both examples, the society fails to be wise.[17]

Theorem 1 suggests that there are two important ingredients in wisdom: a lack of extreme imbalances in the interaction matrix and an absence of small families that interact with a very narrow slice of the outside world. To explore this idea further, we formulate another dispersion condition—one that focuses on the weight into small families rather than out of them and is also sufficient, when combined with balance, to guarantee wisdom. This is discussed in Section C of the Appendix.

The proof of Theorem 1 is technical, but the intuition behind it is not difficult. Suppose, by way of contradiction, that the wisdom conclusion does not hold. Then there must be a family of agents that have positive influence as $n \to \infty$, and a

[17] Since the left eigenvector of eigenvalue 1 is the same for $\mathbf{T}(n)^2$ as for $\mathbf{T}(n)$, the fact that the sequence of Example 3 is not wise also shows that the same is true when we replace each $\mathbf{T}(n)$ by its square. A generalization of this simple observation is Proposition 4 in Section B of the Appendix.

remaining uninfluential family. Since the sum of influences must add up to one, having some very influential agents requires having a great number of uninfluential agents. In particular, the influential family must be fairly small. As a result, it can only give out a limited amount of trust, and thus can only have a similarly limited amount of trust coming in, using the balance condition. Recall that the influence of an agent is a trust-weighted sum of the influences of those who trust him. Now, the uninfluential family does not have enough influence to support the high influence of the influential family, since it can give this family only a limited amount of trust. The influential family cannot get all its support from inside itself because the minimal out-dispersion condition requires it to send a nontrivial amount of its trust outside.

It turns out that this informal argument is challenging to convert to a formal one because the array of influence weights $s_i(n)$ as n and i range over all possible values has some surprising and difficult properties. Nevertheless, the basic ideas outlined above can be carried through successfully.

V. The Speed of Convergence

Our analysis has focused on long-run consensus beliefs. Given that disagreement is often observed in practice, even within a community, there seem to be many situations where convergence—if it obtains eventually—is slow relative to the rate at which the environment (the true parameter μ in our model) changes. Understanding how the speed of convergence depends on social structure can be crucial in judging when the steady-state results are relevant. In mathematical terms, this question can be translated via (1) into the question of how long it takes $\mathbf{T}^t$ to approach its limit, when that limit exists. There is a large literature on convergence of iterated stochastic matrices, some of which we informally describe in this section, without any effort to be comprehensive. The interested reader is referred to the papers discussed below for more complete discussions and references.

A key insight is that the convergence time of an iterated stochastic matrix is related to its second largest eigenvalue in magnitude, which we denote by $\lambda_2(\mathbf{T})$. Indeed, convergence time is essentially proportional to $-1/\log(|\lambda_2(\mathbf{T})|)$ under many measures of convergence. While a characterization in terms of eigenvalues is mathematically enlightening and useful for computations, more concrete insight is often needed.[18] To this end, a variety of techniques have been developed to characterize convergence times in terms of the structure of $\mathbf{T}$. One such method relies on conductance, which is a measure of how inward-looking various sets of nodes or states are. Loosely speaking, if there is a set that is not most of society and that keeps most of its weight inside, then convergence can take a long time.[19] Another approach, which is similar in some intuitions but differs in its mathematics, uses

[18] There is intuition as to the role of the second eignenvalue and why it captures convergence speed. See the explanation in Jackson (2008).

[19] The famous Cheeger inequality (see the Ravi Montenegro and Prasad Tetali (2005, Section 6.3) survey) is the seminal example of this technique. A paper by D. J. Hartfiel and Carl D. Meyer (1998) also focuses on a related notion of insularity and shows that an extremely large second eigenvalue corresponds to a society split into inward-looking factions.

Poincaré inequalities to relate convergence to the presence of bottlenecks. The basic notion is that if there are segments of society connected only by narrow bridges, then convergence will be slow.[20]

A technique for understanding rates of convergence that is particularly relevant to the setting of social networks has recently been developed in Benjamin Golub and Matthew O. Jackson (2008). There, we focus on the important structural feature of many social networks called homophily, which is the tendency of agents to associate with others who are somehow "similar" to themselves. In the setting of Section IIC, homophily provides general lower bounds on the convergence time. With some additional (probabilistic) structure, it is also possible to prove that these bounds are essentially tight, so that homophily is an exact proxy for convergence time.[21] A common thread running through all these results is that societies that are split up, or insular in some way, have slow convergence, while societies that are cohesive have fast convergence. The speed of convergence can thus be essentially orthogonal to whether or not the network exhibits wisdom, as we now discuss.

Speed of Convergence and Wisdom.—The lack of any necessary relationship between convergence and wisdom can easily be seen via some examples.

- First, consider the case where all agents weight each other equally. This society is wise and has immediate convergence.
- Second, consider a society where all agents weight just one agent. Here, we have immediate convergence but no wisdom.
- Third, consider a setting where all agents place $1 - \varepsilon$ weight on themselves and distribute the rest equally. This society is wise but can have arbitrarily slow convergence if ε is small enough.
- Lastly, suppose all agents place $1 - \varepsilon$ weight on themselves and the rest on one particular agent. Then there is neither wisdom nor fast convergence.

Thus, in general, convergence speed is independent of wisdom. One can have both, neither, or either one without the other.

VI. Conclusion

The main topic of this paper concerns whether large societies, whose agents get noisy estimates of the true value of some variable, are able to aggregate dispersed information in an approximately efficient way despite their naïve and decentralized

[20] These techniques are discussed extensively and compared with other approaches in Diaconis and Stroock (1991), which has a wealth of references. The results there are developed in the context of reversible Markov chains (i.e., the types of networks discussed in Section IIC), but extensions to more general settings are also possible (Montenegro and Tetali (2005)). Beyond this, there is a large literature on expander graphs. An introduction is provided by Shlomo Hoory, Nathan Linial, and Avi Wigderson (2006). These are networks that are designed to have extremely small second eigenvalues as the graph grows large; DeGroot communication on such networks converges very quickly.

[21] Beyond the interest in tying convergence speed to some intuitive attributes of the society, this approach also sometimes gives bounds that are stronger than those obtained from previous techniques based on the spectrum of the matrix, such as Cheeger inequalities.

updating. We show, on the one hand, that naïve agents can often be misled. The existence of small prominent groups of opinion leaders, who receive a substantial amount of direct or indirect attention from everyone in society, destroys efficient learning. The reason is clear. Due to the attention it receives, the prominent group's information is overweighted, and its idiosyncratic errors lead everyone astray. While this may seem like a pessimistic result, the existence of such a small but prominent group in a very large society is a fairly strong condition. If there are many different segments of society, each with different leaders, then it is possible for wisdom to obtain as long as the segments have some interconnection. Thus, in addition to the negative results about prominent groups, we also provide structural sufficient conditions for wisdom. The flavor of the first condition of balance is that no group of agents (unless it is large) should get arbitrarily more weight than it gives back. The second condition requires that small groups not be too narrow in distributing their attention, as, otherwise, their beliefs will be too slow to update and will end up dominating the eventual limit. Under these conditions, we show that sufficiently large societies come arbitrarily close to the truth.

These results suggest two insights. First, excessive attention to small groups of pundits or opinion makers is bad for social learning, unless those individuals have information that dominates that of the rest of society. On the other hand, there are natural forms of networks such that even very naïve agents will learn well. There is room for further work along the lines of structural sufficient conditions. The ones that we give here can be hard to check for given sequences of networks. Nevertheless, they provide insight into the types of structural features that are important for efficient learning in this type of naïve society. Perhaps most importantly, these results demonstrate that, in contrast to much of the previous literature, the efficiency of learning can depend, in sensitive ways, on the way the social network is organized. From a technical perspective, the results also show that the DeGroot model provides an unusually tractable framework for characterizing the relationship between structure and learning and should be a useful benchmark.

More broadly, our work can be seen as providing an answer, in one context, to a question asked by Sobel (2000): can large societies whose agents are naïve individually be smart in the aggregate? In this model, they can, if there is enough dispersion in the people to whom they listen, and if they avoid concentrating too much on any small group of agents. In this sense, there seems to be more hope for boundedly rational social learning than has previously been believed. On the other hand, our sufficient conditions can fail if there is just one group that receives too much weight or is too insular. This raises a natural question: which processes of network formation produce societies that satisfy the sufficient conditions we have set forth (or different sufficient conditions)? In a setting where agents decide on weights, how must they allocate those weights to ensure that no group obtains an excessive share of influence in the long run? If most agents begin to ignore stubborn or insular groups over time, then the society could learn quite efficiently. These are potential directions for future work.

The results that we surveyed regarding convergence rates provide some insight into the relationship between social structure and the formation of consensus. A theme which seems fairly robust is that insular or balkanized societies will converge slowly, while cohesive ones can converge very quickly. However, the proper way to measure insularity depends heavily on the setting, and many different approaches have been useful for various purposes.

To finish, we mention some other extensions of the project. First, the theory can be applied to a variety of strategic situations in which social networks play a role. For instance, consider an election in which two political candidates are trying to convince voters. While the voters remain nonstrategic about their communications, the politicians (who may be viewed as being outside the network) can be quite strategic about how they attempt to shape beliefs. A salient question is whom the candidates would choose to target. The social network would clearly be an important ingredient. A related application would consider firms competitively selling similar products (such as Coke and Pepsi).[22] Here, there would be some benefits to one firm of the other firms' advertising. These complementarities, along with the complexity added by the social network, would make for an interesting study of marketing. Second, it would be interesting to involve heterogeneous agents in the network. In this paper, we have focused on nonstrategic agents who are all boundedly rational in essentially the same way. We might consider how the theory changes if the bounded rationality takes a more general form (perhaps with full rationality being a limiting case). Can a small admixture of different agents significantly change the group's behavior? Such extensions would be a step toward connecting fully rational and boundedly rational models, and would open the door to a more robust understanding of social learning.

Mathematical Appendix

A. *Convergence in the Absence of Strong Connectedness*

In this section, we rely on known results about Markov chains to give a full characterization of when individual beliefs converge (as opposed to oscillating forever) and what the limiting beliefs are. Mathematically, we state a necessary and sufficient condition for the existence of $\lim_{t\to\infty} \mathbf{T}^t$, where $\mathbf{T}$ is an arbitrary stochastic matrix, and characterize the limit. The full characterization that we state on this point is in terms of the geometric structure of the network. It does not assume strong connectedness, and is slightly more general than what has previously been stated in the literature on the DeGroot model. Most of this literature—even when it allows for the absence of strong connectedness—works under a technical assumption that at least some agents always place some weight on their own opinions when updating, which guarantees convergence of beliefs via an application of some basic results about the spectrum of a stochastic matrix. While we might expect the assumption to be satisfied in many situations, there are applications where agents start without information, or believe that others may be better informed, and thus defer to their opinions. The theory we develop in the paper goes through even in settings where the usual self-trust assumption does not apply, but where a weaker condition given below does hold.

To state the condition, we need a few further definitions.

[22] See Andrea Galeotti and Sanjeev Goyal (2007) and Arthur Campbell (2009) for one-firm models of optimal advertising on a network.

A group of nodes $B \subset N$ is *closed* relative to $\mathbf{T}$ if $i \in B$ and $T_{ij} > 0$ imply that $j \in B$. A closed group of nodes is a *minimal* closed group relative to $\mathbf{T}$ (or *minimally closed*) if it is closed, and no nonempty strict subset is closed. Observe that $\mathbf{T}$ restricted to any minimal closed group is strongly connected.[23]

With these notions in hand, we can define a strengthening of aperiodicity which will characterize convergence.

DEFINITION 7: *The matrix* $\mathbf{T}$ *is* strongly aperiodic *if it is aperiodic when restricted to every closed group of nodes.*

The following result is an immediate application of a theorem of Peter Perkins (1961) and standard facts from the Perron-Frobenius theory of nonnegative matrices. The details of how they are combined to yield the theorem are given in the proofs at the end of this section.

THEOREM 2: A *stochastic matrix* $\mathbf{T}$ *is convergent if and only if it is strongly aperiodic.*

Beyond knowing whether or not beliefs converge, we are also interested in characterizing what beliefs converge to when they do converge. The following simple extension of Theorem 10 in DeMarzo, Vayanos, and Zwiebel (2003) answers this question. They consider a case where $\mathbf{T}$ has positive entries on the diagonal, but their proof is easily extended to the case with 0 entries on the diagonal.

To understand what beliefs converge to, let us discuss the structure of the groups of agents and who pays attention to whom.

Let $\mathcal{M}$ be the collection of minimal closed groups of agents and set $M = \bigcup_{B \in \mathcal{M}} B$. The set of agents N is partitioned into the groups of agents $B_1, \ldots, B_m$ which compose $\mathcal{M}$, and then a remaining set of agents C. The agents in any minimal closed group B_k will be weighting each other's beliefs (directly or indirectly), and only each other's beliefs. Provided $\mathbf{T}$ is convergent, each such group will converge to a consensus belief. However, different minimal closed groups can converge to different limiting beliefs. The remaining group—call it C—must be paying attention collectively to some agents in M, or else some subset of C would be a minimal closed group, contrary to the construction. The beliefs of agents in C will then converge to some weighted averages of the limiting beliefs of the various minimal closed groups B_k, depending on the precise interaction structure.

To understand the limit of beliefs inside the minimal closed group B_k, without loss of generality consider the case in which this set is all of N, so that $\mathbf{T}$ is strongly connected. This is legitimate because B_k is not influenced by anyone outside it. Section IIB treated this case in detail. From the results there, it follows that the influence of

[23] In the language of Markov chains, strongly connected matrices are referred to as irreducible, and minimal closed groups are also called communication classes. We use some terminology from graph theory rather than from Markov processes since our process is not a Markov chain. Nodes here are not states, and $\mathbf{T}$ is not a transition matrix. We emphasize that even though many mathematical results from Markov processes are useful in the context of the DeGroot model, the DeGroot model is very different from a Markov chain in its interpretation.

any agent in a minimal closed group corresponds to his or her weight in an associated eigenvector of $\mathbf{T}$ restricted to that group.

These observations can be combined to yield the following characterization of limiting beliefs.

Some additional notation: a subscript B indicates restriction of vectors or operators to the subspace of $[0,1]^n$ corresponding to the set of agents in B, and we write $\mathbf{v} > 0$ when each entry of the vector $\mathbf{v}$ is positive.

THEOREM 3: *A stochastic matrix* $\mathbf{T}$ *is convergent if and only if there is a nonnegative row vector* $\mathbf{s} \in [0,1]^n$, *and for each* $j \notin M$ *a vector* $\mathbf{w}^j \geq 0$ *with* $|\mathcal{M}|$ *entries that sum to* 1 *such that*

(i) $\sum_{i \in B} s_i = 1$ *for any minimal closed group* B,

(ii) $s_i = 0$ *if* i *is not in a minimal closed group,*

(iii) $\mathbf{s}_B > 0$ *and is the left eigenvector of* $\mathbf{T}_B$ *corresponding to the eigenvalue* 1,

(iv) *for any minimal closed group* B *and any vector* $\mathbf{p} \in [0,1]^n$, *we have*

$$\left(\lim_{t\to\infty} \mathbf{T}^t \mathbf{p}\right)_j = \mathbf{s}_B \mathbf{p}_B$$

for each $j \in B$,

(v) *for any* $j \notin M$, $(\lim_{t\to\infty} \mathbf{T}^t \mathbf{p})_j = \sum_{B \in \mathcal{M}} w_B^j \mathbf{s}_B \mathbf{p}_B$.

PROOFS:

Theorems 2 and 3 are proved via several lemmas. First, we introduce one more definition.

DEFINITION 8: *A nonnegative matrix is said to be* primitive *if* $\mathbf{T}^t$ *has only positive entries for some* $t \geq 1$.

The following lemma establishes a relationship between primitivity and aperiodicity.

LEMMA 2: *Assume* $\mathbf{T}$ *is strongly connected and stochastic. It is aperiodic if and only if it is primitive.*

The lemma (along with much more) is proved in Theorems 1 and 2 of Peter Perkins (1961).

In the case in which $\mathbf{T}$ is primitive, we can directly give a full characterization of what $\lim_{t\to\infty} \mathbf{T}^t \mathbf{p}$ is.

LEMMA 3: *If* $\mathbf{T}$ *is stochastic and primitive, then there is a row vector* $\mathbf{s} > 0$ *with entries summing to* 1 *such that for any* $\mathbf{p}$,

$$\lim_{t\to\infty} \mathbf{T}^t \mathbf{p} = \mathbf{s}\mathbf{p}.$$

This vector is the unique (up to scale) left eigenvector of $\mathbf{T}$ *corresponding to the eigenvalue* 1. *In particular, all entries of the limit are the same.*

PROOF OF LEMMA 3:

Under the assumption that $\mathbf{T}$ is primitive, it follows from equation (8.4.3) of Carl D. Meyer (2000) that

$$\lim_{t\to\infty} \mathbf{T}^t \mathbf{p} = \mathbf{esp},$$

where $\mathbf{s}$ is as described in the statement of the lemma. The right side is $\mathbf{e}$, which is a vector of all ones, times a 1-by-1 matrix, so all its entries are the same—namely $\mathbf{sp}$.

The next lemma provides a converse to Lemma 3.

LEMMA 4: *Assume* $\mathbf{T}$ *is strongly connected and stochastic. If it is convergent, then it is primitive.*

PROOF OF LEMMA 4:

Since $\mathbf{S} := \lim_{t\to\infty} \mathbf{T}^t$ exists, we have

$$\mathbf{ST} = \left(\lim_{t\to\infty} \mathbf{T}^t\right)\mathbf{T} = \lim_{t\to\infty} \mathbf{T}^t = \mathbf{S}.$$

So each row of $\mathbf{S}$ is a left eigenvector of $\mathbf{T}$ corresponding to the eigenvalue 1. Such eigenvectors have no 0 entries by the Perron-Frobenius theorem. Thus, $\mathbf{S}$ has strictly positive entries, and so all entries of $\mathbf{T}^t$ must simultaneously be strictly positive for all high enough t.

With these lemmas in hand, we can prove the two theorems.

PROOF OF THEOREM 2:

By permuting agents, $\mathbf{T}$ can be transformed into

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ 0 & \mathbf{T}_{22} \end{bmatrix}, \tag{2}$$

where the bottom right block corresponds to all agents in M, i.e., all agents in any minimal closed group, and the rows above it correspond to the agents (if any) who are in no minimal closed group. We may further decompose

$$\mathbf{T}_{22} = \begin{bmatrix} \mathbf{T}_{B_1} & & \\ & \ddots & \\ & & \mathbf{T}_{B_m} \end{bmatrix},$$

with 0 elsewhere, where each B_k is minimally closed.

If $\mathbf{T}$ is not strongly aperiodic, then some $\mathbf{T}_{B_k}$ will fail to be aperiodic (by definition), and then Lemmas 2 and 4 show that $\mathbf{T}_{B_k}^t$ has no limit as $t \to \infty$. Since the corresponding block of $\mathbf{T}^t$ is $\mathbf{T}_{B_k}^t$, the entire matrix also does not converge. This proves the "only if" direction of Theorem 2.

Conversely, if $\mathbf{T}$ is strongly aperiodic, then each $\mathbf{T}_{B_k}$ is aperiodic, and hence, primitive by Lemma 2. Lemma 3 then shows that for each k,

$$\lim_{t \to \infty} \mathbf{T}_{B_k}^t \mathbf{p}_{B_k} = \mathbf{s}_{B_k} \mathbf{p}_{B_k}, \tag{3}$$

where $\mathbf{s}_{B_k}$ is the unique left eigenvector of $\mathbf{T}_{B_k}$ corresponding to eigenvalue 1, scaled so that its entries sum to 1.

To complete the proof, we note by Meyer (2000, Section 8.4) that the decomposition in (2) entails

$$\lim_{t \to \infty} \mathbf{T}^t = \begin{bmatrix} 0 & \mathbf{Z} \\ 0 & \mathbf{E} \end{bmatrix}, \tag{4}$$

where $\mathbf{Z}$ is some matrix and

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_{B_1} \mathbf{s}_{B_1} & & \\ & \ddots & \\ & & \mathbf{e}_{B_m} \mathbf{s}_{B_m} \end{bmatrix}. \tag{5}$$

(Here, $\mathbf{e}_{B_k}$ is a $|B_k|$-by-1 vector of ones.) This shows that $\mathbf{T}$ is convergent.

PROOF OF THEOREM 3:

The "if" direction is trivial, since conditions (iv) and (v) in the statement of the theorem imply convergence directly.

To prove the "only if" direction, we assume that $\mathbf{T}$ is convergent. Then, using the block decomposition at the beginning of the previous proof, each $\mathbf{T}_{B_k}$ is convergent. We now proceed to show conditions (i–v) in the statement of the theorem.

Lemma 4 shows that for each k, the matrix $\mathbf{T}_{B_k}$ is primitive. Lemma 3 then shows that for each k, equation (3) holds. Define

$$\mathbf{s} = \mathbf{0} \oplus \mathbf{s}_{B_1} \oplus \cdots \oplus \mathbf{s}_{B_m},$$

where $\mathbf{0}$ is a zero row vector such that $\mathbf{s} \in \mathbb{R}^n$ and the $\oplus$ symbol denotes concatenation. This vector satisfies (i–iii) of Theorem 3.

Next, note that equation (4) and the block-diagonal form of $\mathbf{E}$ in (5) immediately imply condition (iv) of Theorem 3.

To finish the proof, we use equation (4). Since powers of stochastic matrices are stochastic, $\mathbf{Z}$ has rows summing to 1. For each $j \notin M$, define $\mathbf{w}^j \in \mathbb{R}^{|\mathcal{M}|}$ by $w_k^j = \sum_{i \in B_k} Z_{ji}$. Then $\sum_{k=1}^{m} w_k^j = 1$. Note that

$$\lim_{t\to\infty} \mathbf{T}^t = \mathbf{T}\Big(\lim_{t\to\infty} \mathbf{T}^t\Big)$$

so that

$$\lim_{t\to\infty} \mathbf{T}^t = \Big(\lim_{r\to\infty} \mathbf{T}^r\Big)\Big(\lim_{t\to\infty} \mathbf{T}^t\Big),$$

and so the matrix on the right-hand side of (4) is idempotent. Then (4) can be written as

$$\lim_{t\to\infty} \mathbf{T}^t\mathbf{p} = \begin{bmatrix} 0 & \mathbf{Z} \\ 0 & \mathbf{E} \end{bmatrix}\mathbf{q}, \tag{6}$$

where

$$\mathbf{q} = \begin{bmatrix} 0 & \mathbf{Z} \\ 0 & \mathbf{E} \end{bmatrix}\mathbf{p}.$$

Since $\mathbf{E}_{B_k}\mathbf{p}_{B_k} = \mathbf{s}_{B_k}\mathbf{p}_{B_k}$, it follows that $q_i = \mathbf{s}_{B_k}\mathbf{p}_{B_k}$ if $i \in B_k$. From this, we deduce that for each $j \notin M$, we have

$$\Big(\lim_{t\to\infty} \mathbf{T}^t\mathbf{p}\Big)_j = \sum_{i \in M} Z_{ji}q_i = \sum_{k=1}^{m} w_k^j \mathbf{s}_{B_k}\mathbf{p}_{B_k}$$

by definition of $\mathbf{q}$ and $\mathbf{w}^j$. This completes the proof of (v) in Theorem 3.

B. *Proofs of Results on Wisdom*

PROOF OF LEMMA 1:

We know that the variance of each $p_i^{(0)}(n)$ lies between σ^2 and 1, the latter being true because $p_i^{(0)}(n) \in [0,1]$ for all n and i.

Let $X(n) = \sum_i s_i(n)p_i^{(0)}(n)$. Then $\operatorname{var}(X(n)) \le \overline{\sigma}^2 \sum_i s_i(n)^2$.

First, suppose $s_1(n) \to 0$. Since $s^i(n) \ge s_{i+1}(n) \ge 0$ for all i and n, it follows that

$$\operatorname{var}(X(n)) \le \overline{\sigma}^2 \sum_i s_i(n)^2 \le \overline{\sigma}^2 s_1(n) \sum_i s_i(n) = \overline{\sigma}^2 s_1(n) \to 0.$$

By Chebychev's inequality, fixing any $\varepsilon > 0$,

$$\mathbb{P}\left[\left|\sum_i s_i(n)p_i^{(0)}(n) - \mu\right| > \varepsilon\right] \le \frac{\operatorname{var}(X(n))}{\varepsilon^2} \to 0.$$

For the converse, suppose (taking a subsequence if necessary) $s_1(n) \to s > 0$. Since each $p_i^0(n)$ has a variance bounded below, it then follows that there exists $\delta > 0$ such that $\operatorname{var}(X(n)) > \delta$ for all n. It is well-known that for uniformly bounded random variables, convergence in probability to 0 implies that the same holds in L^2, which means that the $X(n)$ cannot converge to 0 in probability.

PROOF OF PROPOSITION 2:

First, we prove that if the condition $s_1(n) \to 0$ holds, then convergence to truth occurs. By Theorem 3, agents with no influence converge to weighted averages of limiting beliefs of agents with influence, so it suffices to show that if $i_n \leq n$ is any sequence of agents in minimal closed groups, then $\operatorname{plim}_{n\to\infty} p_{i_n}^{(\infty)}(n) = \mu$. Let B_n be the minimal closed group of i_n. Without loss of generality, we may replace $\mathbf{T}(n)$ with induced interaction matrix on the agents in B_n. Now, by Lemma 1, all that is required for every agent in B_n to converge to true beliefs is that the most influential agent in B_n have influence converging to 0. But this condition holds, because the most influential agent in $\{1, \dots, n\}$ has influence converging to 0, and a fortiori the same must hold for the leader in B_n.

Conversely, if the influence of some agent remains bounded above zero, then we may restrict attention to his or her closed group and conclude from the argument of the above lemma that convergence to truth is not generally guaranteed.

Lastly, the following is a small technical result which is useful in that it allows us to work with whatever powers of the interaction matrices are most convenient in studying wisdom.

PROPOSITION 4: *If for each n there exists a* $k(n)$ *such that*

$$\mathbf{R}(n) = \mathbf{T}(n)^{k(n)},$$

then $(\mathbf{T}(n))_{n=1}^{\infty}$ *is wise if and only if* $(\mathbf{R}(n))_{n=1}^{\infty}$ *is wise.*

PROOF OF PROPOSITION 4:

Note that $\lim_{t\to\infty}\mathbf{T}(n)^t = \lim_{t\to\infty}\mathbf{R}(n)^t$, so that for every n, the influence vectors will be the same for both matrices by an easy application of Theorem 3.

Prominence and Wisdom.—The next results focus on how prominence rules out wisdom. We start in the finite setting and then apply the results to the asymptotic context. We write $\kappa(\mathbf{T})$ for the number of closed and strongly connected groups relative to $\mathbf{T}$, and we let $s_B = \sum_{i\in B} s_i$. Also, we write $\mathbf{T}_{ij}^{(t)}$ for the (i,j) entry of $\mathbf{T}^t$. The following fact is a direct consequence of Theorem 3.

PROPOSITION 5: *The entries of* $\mathbf{s}$ *sum to* $\kappa(\mathbf{T})$.

With this property in hand, we can proceed to prove the following lemma.

LEMMA 5: *For any $B \subseteq N$ and natural number t,*

$$s_B \geq \frac{\kappa(\mathbf{T})\pi_B(\mathbf{T};t)}{1+\pi_B(\mathbf{T};t)} \tag{7}$$

and

$$\max_{i \in N} s_i \geq \frac{\kappa(\mathbf{T})\pi_B(\mathbf{T};t)}{|B|(1+\pi_B(\mathbf{T};t))}. \tag{8}$$

PROOF OF LEMMA 5:

Since $\mathbf{s}$ is a row unit eigenvector of $\mathbf{T}^t$, it follows that

$$\begin{aligned}\sum_{i \in B} s_i &\geq \sum_{i \in B} \sum_{j \notin B} T_{ji}^{(t)} s_j \\ &= \sum_{j \notin B} s_j \sum_{i \in B} T_{ji}^{(t)} \\ &\geq \pi_B(\mathbf{T};t) \sum_{j \notin B} s_j.\end{aligned}$$

Then, since the sum of $\mathbf{s}$ is $\kappa(\mathbf{T})$ by Proposition 5, we know that

$$\sum_{j \notin B} s_j = \kappa(\mathbf{T}) - s_B.$$

After substituting this into the inequality above, it follows that

$$s_B \geq \pi_B(\mathbf{T};t)(\kappa(\mathbf{T}) - s_B),$$

which yields the first claim of the lemma. The second claim follows directly.

PROOF OF PROPOSITION 3:

The fact that $s_1(n)$ does not converge to 0 as $n \to \infty$ follows immediately upon applying Lemma 5 to each matrix in the sequence. We use the finiteness of (B_n) to prevent the denominator in equation (8) in the lemma from exploding, and the uniform lower bound on the prominence of each B_n relative to $\mathbf{T}(n)$ to keep the numerator from going to 0.

PROOF OF THEOREM 1:

Recall that we have ordered the agents so that $s_i(n) \geq s_{i+1}(n)$ for all i. Take q and r guaranteed by the minimal out-dispersion property. We will first show that $\lim_{n\to\infty} s_q(n) = 0$, which will reduce the argument to a simple calculation.

To this end, let us first argue that there exists a sequence $k(n)$ such that three properties hold: $k(n) \geq q$ for large enough n, $k(n)s_{k(n)}(n) \to 0$, and $k(n)/n \to 0$.

In order to verify this, consider first the sequence $j(n)$ guaranteed by the balance condition. We may assume not only that $j(n) \to \infty$ and the inequality in the balance condition holds, but also, by reducing the $j(n)$ if necessary, that $j(n)/n \to 0$. Next, we argue that for each $x > 0$ there is at most a finite set of n such that $is_i(n) \geq x$ for all i satisfying $q \leq i \leq j(n)$. Suppose to the contrary that there exists $x > 0$ such that, for an infinite set of n, we have $is_i(n) \geq x$ for all i satisfying $q \leq i \leq j(n)$. Thus, for these n,

$$\sum_{i=q}^{j(n)} s_i(n) \geq \sum_{i=q}^{j(n)} \frac{x}{i} \to \infty,$$

which is a contradiction. It follows that for each x there is a smallest natural number n_x, such that for every $n \geq n_x$, the set $Z_{x,n} = \{i : is_i(n) < x, q \leq i \leq j(n)\}$ is nonempty. For all $n \leq n_1$, define $k(n) = 1$. For all other n, select $k(n)$ by choosing an arbitrary element from $Z_{y_n,n}$, where $y_n = \inf_{n_x \leq n} x + (1/2)^n$. Of course, we should verify that this set is nonempty. To this end, note that as x increases, n_x is weakly decreasing. Since there exists an $x < y_n$ with $n \geq n_x$, it follows by this monotonicity that $n \geq n_{y_n}$, and $Z_{y_n,n}$ is nonempty. Additionally, since n_x is a well-defined integer for all x, we see that $\inf_{n_x \leq n} x \to 0$ as $n \to \infty$, and hence the same is true for y_n. It follows by construction of $Z_{y_n,n}$ and the fact that $j(n)/n \to \infty$ that all three properties claimed at the start of the paragraph hold.

For each n, let $H_n = \{1, \ldots, k(n)\}$ and $L_n = H_n^c$. Observe that since $\mathbf{s}(n)$ is a left hand eigenvector of $\mathbf{T}_n$, we have

$$\sum_{j \in H_n} s_j(n) = \sum_{i \in H_n} \sum_{j \in H_n} T_{ij}(n) s_i(n) + \sum_{i \in L_n} \sum_{j \in H_n} T_{ij}(n) s_i(n).$$

Rewrite this as

$$\sum_{j \in H_n} s_j(n) \left(1 - \sum_{i \in H_n} T_{ji}(n)\right) = \sum_{i \in L_n} \sum_{j \in H_n} T_{ij}(n) s_i(n)$$

or

$$\sum_{j \in H_n} s_j(n) \left(\sum_{i \in L_n} T_{ji}(n)\right) = \sum_{i \in L_n} \left(\sum_{j \in H_n} T_{ij}(n) s_i(n)\right). \tag{9}$$

Let

$$S_H(n) = \sum_{j \in H_n} s_j(n) \cdot \frac{\sum_{i \in L_n} T_{ji}(n)}{T_{H_n,L_n}(n)} \tag{10}$$

and

$$S_L(n) = \sum_{i \in L_n} s_i(n) \cdot \frac{\sum_{j \in H_n} T_{ij}(n)}{T_{L_n,H_n}(n)}.$$

We rewrite (9) as

$$S_H(n)T_{H_n,L_n}(n) = S_L(n)T_{L_n,H_n}(n). \tag{11}$$

Now, taking $B_n = \{1, \ldots, q\}$ and $C_n = L_n$ in the statement of the minimal dispersion condition, we have that $\mathrm{T}_{B_n,L_n} > r$ eventually. (We showed at the beginning of the proof that $k(n)/n \to 0$, so that $|L_n|/n \to 1$, and therefore the condition applies.) By construction of the $k(n)$, we know that $k(n) \geq q$ eventually, so that $B_n \subseteq H_n$ eventually. By (10), we deduce that eventually

$$S_H(n) \geq \sum_{j \in B_n} s_j(n) \cdot \frac{\sum_{i \in L_n} T_{ji}(n)}{T_{H_n,L_n}(n)} \geq s_q(n) \cdot \frac{\sum_{j \in B_n} \sum_{i \in L_n} T_{ji}(n)}{T_{H_n,L_n}(n)}.$$

As the numerator of the fraction on the right-hand side is at least r and the denominator is at most $k(n)$, we conclude that

$$S_H(n) \geq \frac{s_q(n)r}{k(n)}$$

for a positive real r.

Also, $S_L(n) \leq s_{k(n)}(n)$. Thus, the above equation with (11) implies that

$$rs_q(n)T_{H_n,L_n}(n) \leq k(n)s_{k(n)}(n)T_{L_n,H_n}(n). \tag{12}$$

Since $T_{L_n,H_n}(n)/T_{H_n,L_n}(n)$ is bounded (by balance) and $k(n)s_{k(n)}(n) \to 0$ (by what we showed at the beginning of the proof), this implies that $\lim_{n\to\infty} s_q(n) = 0$.

So, we are reduced to the case $\lim_{n\to\infty} s_q(n) = 0$. Suppose that, contrary to the theorem's assertion, the sequence $s_1(n)$ does not converge to 0. Let k be the largest i such that $\limsup_n s_i(n) > 0$, which is well-defined and finite by the supposition that $s_1(n)$ does not converge to 0, and the result above that $s_q(n) \to 0$. Let $H_n = \{1, 2, \ldots, k\}$. Then, as above, we have the following facts:

$$\sum_{i \in H_n} s_i(n) \sum_{j \in L_n} T_{ij}(n) = \sum_{i \in L_n} \sum_{j \in H_n} T_{ij}(n) s_i(n)$$

$$s_k(n) \sum_{i \in H_n} \sum_{j \in L_n} T_{ij}(n) \leq s_{k+1}(n) \sum_{i \in L_n} \sum_{j \in H_n} T_{ij}(n) \qquad \text{by the ordering of the } s_i(n)$$

$$\frac{s_k(n)}{s_{k+1}(n)} \leq \frac{T_{H_n^c,H_n}(n)}{T_{H_n,H_n^c}(n)}.$$

The left side will have supremum ∞ over all n because $s_k(n)$ has positive limsup while $s_{k+1}(n) \to 0$. The right side, however, is bounded using the balance property. This is a contradiction, and therefore the proof, is complete.

C. *Alternative Sufficient Conditions for Wisdom*

In this section, we formulate an alternative to the minimal out-dispersion property which, when paired with balance, also ensures wisdom. The difference between this property and minimal out-dispersion is that this one is about links coming into a group rather than ones coming out of it.

Property 3 (**Minimal In-Dispersion**): There is a $q \in \mathbb{N}$ and an $r < 1$ such that if $|B_n| = q$ and $C_n \subseteq B_n^c$ is finite then $T_{C_n,B_n}(n) \leq r T_{B_n,B_n^c}(n)$ for all large enough n.

This condition requires that the source of the weight coming into a finite family not be too concentrated. The finite family B_n cannot have a finite neighborhood which gives B_n as much weight, asymptotically, as B_n gives out. This essentially requires influential families to have a broad base of support, and rules out situations like Example 4. Indeed, along with balance, it is enough to generate wisdom.

THEOREM 4: *If* $(\mathbf{T}(n))_{n=1}^{\infty}$ *is a sequence of convergent stochastic matrices satisfying balance and minimal in-dispersion, then it is wise.*

PROOF OF THEOREM 4:

By Proposition 2 and the ordering we have chosen for $\mathbf{s}(n)$, it suffices to show that

$$\lim_{n \to \infty} s_1(n) = 0 \tag{13}$$

Suppose otherwise.

We proceed by cases. First, assume that there are only finitely many i, such that $\lim_{n\to\infty} s_i(n) > 0$. Then we can proceed as at the end of the proof of Theorem 1 to reach a contradiction. Note that only balance for finite families (B_n) is needed, which is implied by the balance property.

From now on, we may assume that there are infinitely many i such that $\limsup_n s_i(n) > 0$. In particular, if we take the q guaranteed by Property 3 and set $B_n = \{1, 2, \ldots, q\}$, then we know that $\limsup_n s_i(n) > 0$ for each $i \in B_n$. Now, for a function $g : \mathbb{N} \to \mathbb{N}$, whose properties will be discussed below, define $C_n = \{q + 1, \ldots, q + g(n)\}$. Finally, let $D_n = \{q + g(n) + 1, q + g(n) + 2, \ldots, n\}$. That is, $D_n = B_n^c \setminus C_n$.

We claim g can be chosen such that $\lim_{n\to\infty} g(n) = \infty$ and

$$\limsup_n \frac{T_{C_n,B_n}(n)}{T_{B_n,B_n^c}(n)} \leq r,$$

where $r < 1$ is the number provided by Property 3. Let $C_n^k = \{q + 1, q + 2, \ldots, q + k\}$. By Property 3, there exists an n_1, such that for all $n \geq n_1$, we have

$$\frac{T_{C_n^1,B_n}(n)}{T_{B_n,B_n^c}(n)} \leq r.$$

Having chosen $n_1, \ldots, n_{k-1}$, there exists an $n_k > n_{k-1}$, such that for all $n \geq n_k$, we have

$$\frac{T_{C_n^k, B_n}(n)}{T_{B_n, B_n^c}(n)} \leq r.$$

Define

$$g(n) = \max\{k : n_k \leq n\}.$$

Since $n_1, n_2, \ldots$ is an increasing sequence of integers, the set whose maximum is being taken is finite. It is also nonempty for $n \geq n_1$, so g is well defined there. For $n < n_1$, let $g(n) = 1$. Next, note that g is nondecreasing by construction, that $g(n_k) \geq k$, and that $n_k \to \infty$, so that $\lim_{n\to\infty} g(n) = \infty$. Finally, since C_n, defined above, is equal to $C_n^{g(n)}$, and

$$\frac{T_{C_n^{g(n)}, B_n}(n)}{T_{B_n, B_n^c}(n)} \leq r$$

for all $n \geq n_1$ by construction, it follows that

$$\limsup_n \frac{T_{C_n, B_n}(n)}{T_{B_n, B_n^c}(n)} \leq r. \tag{14}$$

This shows our claim about the choice of g.

Now we have the following string of implications:

$$\sum_{i \in B_n} s_i(n) = \sum_{i \in B_n} \sum_{j \in N} T_{ji}(n) s_j(n)$$

$$\sum_{i \in B_n} s_i(n) = \sum_{i \in B_n} \sum_{j \in B_n} T_{ji}(n) s_j(n) + \sum_{i \in B_n} \sum_{j \in C_n} T_{ji}(n) s_j(n) + \sum_{i \in B_n} \sum_{j \in D_n} T_{ji}(n) s_j(n)$$

$$\sum_{i \in B_n} s_i(n) = \sum_{i \in B_n} \sum_{j \in B_n} T_{ij}(n) s_i(n) + \sum_{i \in C_n} \sum_{j \in B_n} T_{ij}(n) s_i(n) + \sum_{i \in D_n} \sum_{j \in B_n} T_{ij}(n) s_i(n)$$

$$\sum_{i \in B_n} s_i(n) \sum_{j \notin B_n} T_{ij}(n) = \sum_{i \in C_n} s_i(n) \sum_{j \in B_n} T_{ij}(n) + \sum_{i \in D_n} s_i(n) \sum_{j \in B_n} T_{ij}(n).$$

Rearranging,

$$\sum_{i \in B_n} s_i(n) \sum_{j \notin B_n} T_{ij}(n) - \sum_{i \in C_n} s_i(n) \sum_{j \in B_n} T_{ij}(n) = \sum_{i \in D_n} s_i(n) \sum_{j \in B_n} T_{ij}(n). \tag{15}$$

Using the ordering of the $s_i(n)$, the first double summation on the left side satisfies

$$\sum_{i \in B_n} s_i(n) \sum_{j \notin B_n} T_{ij}(n) \geq s_q(n) \sum_{i \in B_n} \sum_{j \notin B_n} T_{ij}(n) = s_q(n) T_{B_n, B_n^c}(n).$$

Similarly, the second summation on the left side of (15) satisfies

$$\sum_{i \in C_n} s_i(n) \sum_{j \in B_n} T_{ij}(n) \leq s_{q+1}(n) \sum_{i \in C_n} \sum_{j \in B_n} T_{ij}(n) = s_{q+1}(n) T_{C_n, B_n}(n).$$

Finally, the summation on the right side of (15) satisfies

$$\sum_{i \in D_n} s_i(n) \sum_{j \in B_n} T_{ij}(n) \leq s_{q+g(n)+1}(n) \sum_{i \in D_n} \sum_{j \in B_n} T_{ij}(n) = s_{q+g(n)+1}(n) T_{B_n^c \setminus C_n, B_n}(n).$$

We will write $f(n) = q + g(n) + 1$. Combining the above facts with (15), we find

$$s_q(n) T_{B_n, B_n^c}(n) - s_{q+1}(n) T_{C_n, B_n}(n) \leq s_{f(n)}(n) T_{B_n^c \setminus C_n, B_n}(n).$$

By the ordering of the $s_i(n)$, it follows that

$$s_{q+1}(n) T_{B_n, B_n^c}(n) - s_{q+1}(n) T_{C_n, B_n}(n) \leq s_{f(n)}(n) T_{B_n^c \setminus C_n, B_n}(n). \tag{16}$$

By the argument at the beginning of this proof, there is an $r < 1$, so that for all large enough n, we have

$$T_{C_n, B_n}(n) \leq r T_{B_n, B_n^c}(n).$$

Using this and a trivial bound on the right hand side of (16), we may rewrite (16) as

$$s_{q+1}(n)(1 - r) T_{B_n, B_n^c}(n) \leq s_{f(n)}(n) T_{B_n^c, B_n}(n). \tag{17}$$

To finish the proof, we need two observations. The first is that $s_{f(n)}(n) \to 0$. Suppose not, so that it exceeds some $a > 0$ for infinitely many n. Then for all such n, we use the ordering of the $s_i(n)$ to find

$$\sum_{i=1}^{f(n)} s_i(n) \geq a f(n),$$

and this quantity tends to $+\infty$, contradicting the fact that

$$\sum_{i=1}^{n} s_i(n) = 1.$$

The second observation is that we may, without loss of generality, assume g is a function satisfying all the properties previously discussed and also $g(n) \leq j(n)$, where the $j(n)$ are from the balance condition. For, if we have a g, so that this condition does not hold, it is easy to verify that reducing g to some smaller function tending to $+\infty$, for which the condition does hold, cannot destroy the property in (14).

Now we rewrite (17) as

$$(1-r)\frac{s_{q+1}(n)}{s_{f(n)}(n)} \leq \frac{T_{B_n^c,B_n}(n)}{T_{B_n,B_n^c}(n)}.$$

Arguing as at the end of the proof of Theorem 1, the observations we have just derived along with Property 1 generate the needed contradiction.

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